LONGITUDINAL DYNAMICS

Frank Tecker

based on the course by
Joël Le Duff
Many Thanks!

CAS on Intermediate Level Accelerator Physics Course
Chios, 18-30 September 2011

Summary

- Radio-Frequency Acceleration and Synchronism Condition
- Principle of Phase Stability and Consequences
- The Synchrotron
- Dispersion Effects in Synchrotron
- Energy-Phase Equations
- Longitudinal Phase Space Motion
- Stationary Bucket
- Injection Matching
- From Synchrotron to Linac
- Adiabatic Damping
- Dynamics in the vicinity of transition energy

Bibliography

M. Conte, W.W. Mac Kay **An Introduction to the Physics of particle Accelerators** (World Scientific, 1991)

P. J. Bryant and K. Johnsen **The Principles of Circular Accelerators and Storage Rings** (Cambridge University Press, 1993)

D. A. Edwards, M. J. Syphers **An Introduction to the Physics of High Energy Accelerators** (J. Wiley & sons, Inc, 1993)

H. Wiedemann **Particle Accelerator Physics** (Springer-Verlag, Berlin, 1993)

M. Reiser **Theory and Design of Charged Particles Beams** (J. Wiley & sons, 1994)

A. Chao, M. Tigner **Handbook of Accelerator Physics and Engineering** (World Scientific 1998)

K. Wille **The Physics of Particle Accelerators: An Introduction** (Oxford University Press, 2000)

E.J.N. Wilson **An introduction to Particle Accelerators** (Oxford University Press, 2001)

→ And CERN Accelerator Schools (CAS) Proceedings

Main Characteristics of an Accelerator

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

ACCELERATION is the main job of an accelerator.

- It provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field $\vec{E}$, preferably along the direction of the initial momentum.

$$\frac{dp}{dt} = eE_z$$

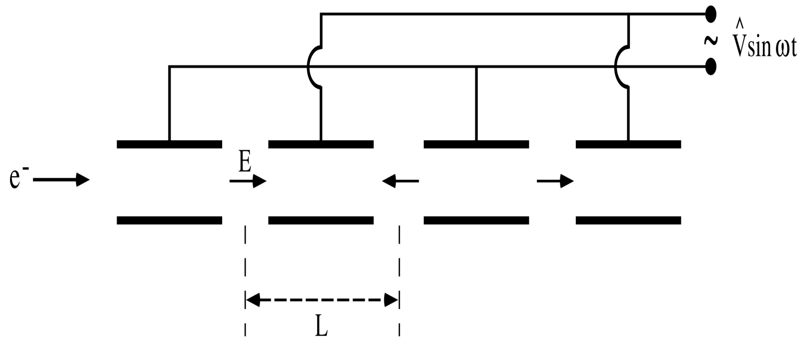
BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\frac{p}{e} = B\rho$$

in practical units: $B\ \rho\ [\mathrm{Tm}] \approx \frac{p\ [\mathrm{GeV/c}]}{0.3}$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

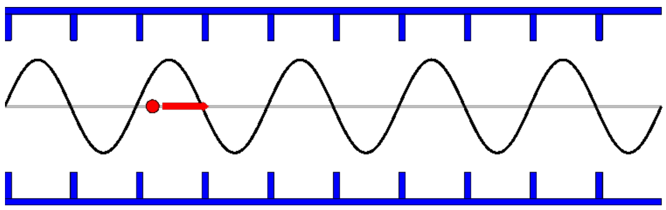
Radio-Frequency Acceleration



Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition ⟶ $L = v\,T/2$

v = particle velocity
T = RF period

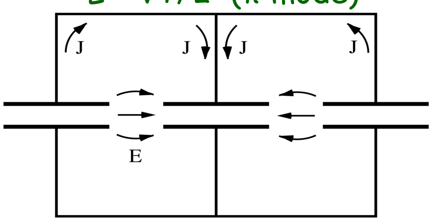


Similar for standing wave cavity as shown

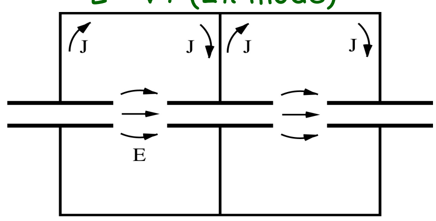
D.Schulte

Radio-Frequency Acceleration (2)

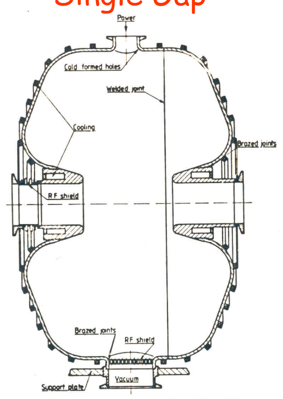
$L = vT/2$ (π mode)



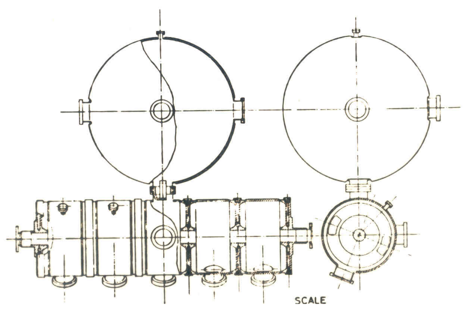
$L = vT$ (2π mode)



Single Gap



Multi-Gap



Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c$$

$$E^2 = E_0^2 + p^2c^2 \longrightarrow dE = v\,dp$$

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

$$dE = dW = eE_z\,dz \rightarrow W = e\int E_z\,dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin\omega_{RF}t = \hat{E}_z \sin\phi(t)$$

$$\int \hat{E}_z\,dz = \hat{V}$$

$$W = e\hat{V}\sin\phi$$

(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Transit time factor

Defined as:

$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \to \infty)}$$

In the general case, the transit time factor is:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r)\cos\left(\omega_{RF}\frac{s}{v}\right) ds}{\int_{-\infty}^{+\infty} E_1(s,r)\,ds}$$

for $E(s,r,t) = E_1(s,r)\cdot E_2(t)$

Simple model uniform field:

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

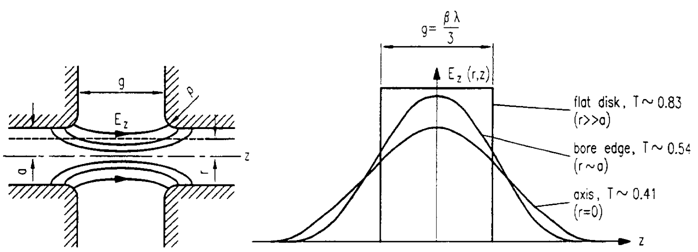
follows:

$$T_a = \sin\frac{\omega_{RF}g}{2v} \Big/ \frac{\omega_{RF}g}{2v}$$

- $T_a < 1$
- $T_a \to 1$ for $g \to 0$, smaller ω_{RF}

Important for low velocities (ions)

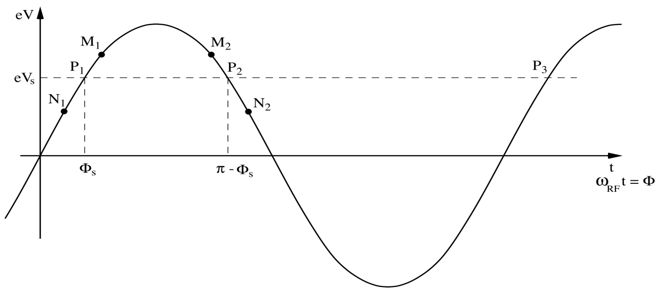
Example for g=βλ/3:



Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.

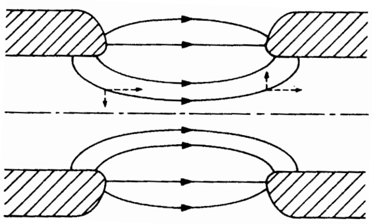


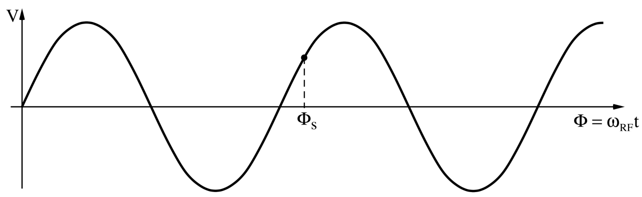
$eV_s = e\hat{V} \sin \Phi_s$ **is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.**

If an energy increase is transferred into a velocity increase =>
M_1 & N_1 will move towards P_1 => stable
M_2 & N_2 will go away from P_2 => unstable

(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability





Transverse focusing fields at the entrance and defocusing at the exit of the cavity.
Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

Longitudinal phase stability means : $\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$

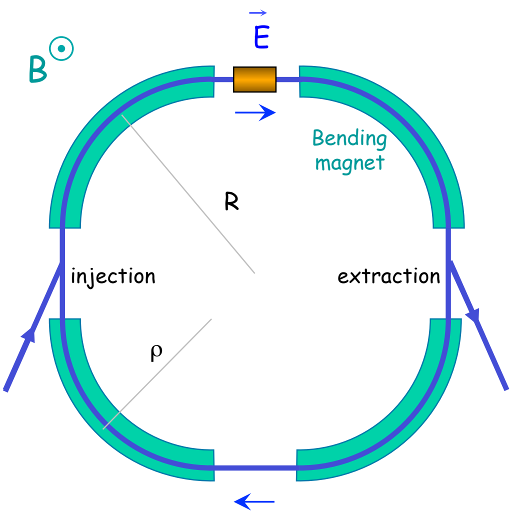
defocusing RF force

The divergence of the field is zero according to Maxwell : $\nabla.\vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} > 0$

External focusing (solenoid, quadrupole) is then necessary

Circular accelerators: The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism (h - harmonic number)

$\rho = cte \quad R = cte \rightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v≈c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field (frequency follows v):

$$p = eB\rho \Rightarrow \frac{dp}{dt} = e\rho\dot{B} \Rightarrow (\Delta p)_{turn} = e\rho\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v}$$

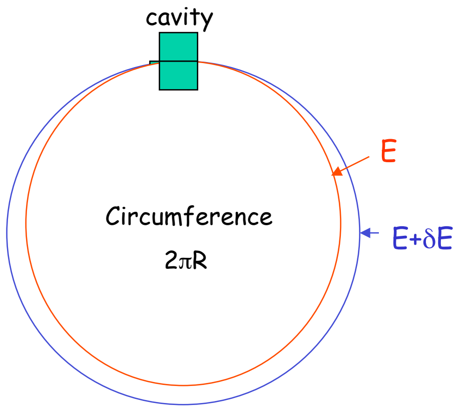
Since: $$E^2 = E_0^2 + p^2c^2 \Rightarrow \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\varphi_s$$

Stable phase φ_s changes during energy ramping

- **The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.**
- **Each synchronous particle satisfies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.**

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit and the length is different.

The "momentum compaction factor" is defined as:

$$\alpha = \frac{dL/L}{dp/p} \quad \Rightarrow \quad \alpha = \frac{p}{L}\frac{dL}{dp}$$

If the particle is shifted in momentum it will have also a different velocity.
As a result of both effects the revolution frequency changes:

$$\eta = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p} \quad \Rightarrow \quad \eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

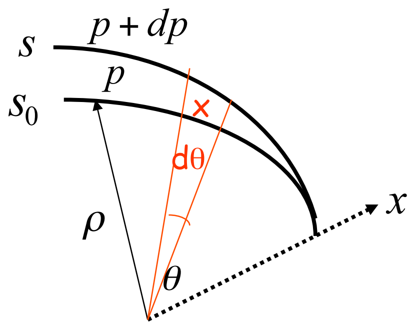
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \int_C dl = \int \frac{x}{\rho} ds_0 = \int \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

$$\alpha = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Dispersion Effects in a Synchrotron (3)

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha \frac{dp}{p}$$

↑ definition of momentum compaction factor

$$p = mv = \beta\gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-1/2}}{\left(1-\beta^2\right)^{-1/2}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right) \frac{dp}{p}$$

$$\frac{df_r}{f_r} = \eta \frac{dp}{p}$$

$$\eta = \frac{1}{\gamma^2} - \alpha$$

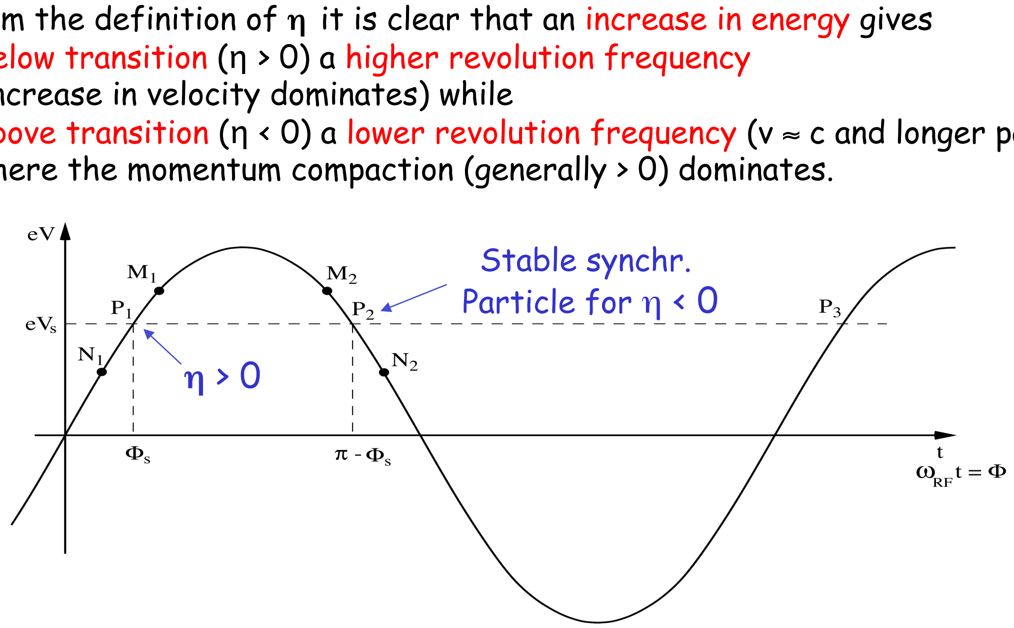
η=0 at the transition energy

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in energy gives

- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while
- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



Crossing transition during acceleration makes the previous stable synchronous phase unstable. RF system needs to make a 'phase jump'.

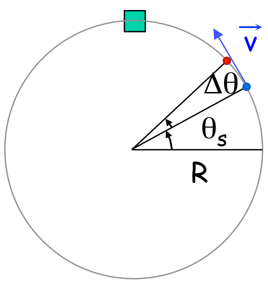
Longitudinal Dynamics

It is also often called "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad with \quad \theta = \int \omega_r\, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}\left(\Delta\theta\right) = -\frac{1}{h}\frac{d}{dt}\left(\Delta\phi\right) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \dfrac{p_s}{\omega_{rs}}\left(\dfrac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$, $\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is: $\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}}\frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}\left(\sin\phi - \sin\phi_s\right) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases later…

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = 2\pi\left(\frac{\Delta E}{\omega_{rs}}\right) = 2\pi R_s \Delta p \quad \longrightarrow \quad \begin{aligned} \frac{d\phi}{dt} &= -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{p_s R_s} W \\ \frac{dW}{dt} &= e\hat{V}(\sin\phi - \sin\phi_s) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W, t) = e\hat{V}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s] - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s} W^2$$

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s) = 0 \quad \text{with} \quad \Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

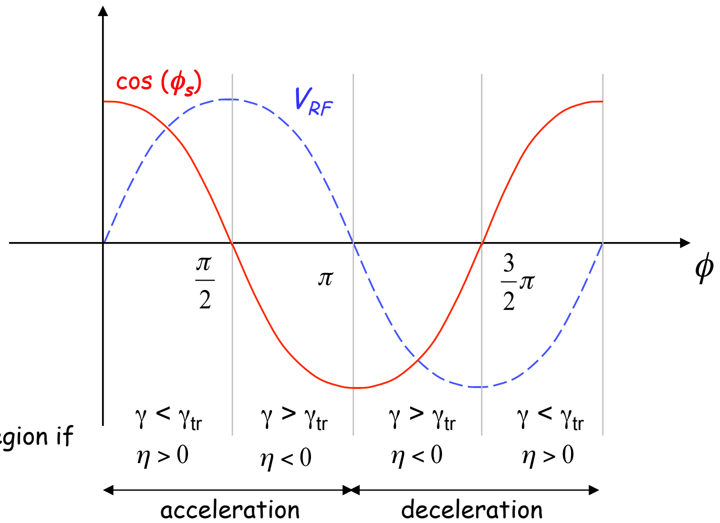
where Ω_s is the synchrotron angular frequency

Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 = \frac{e\hat{V}_{RF}\eta h\omega_s}{2\pi R_s p_s}\cos\phi_s \quad \Rightarrow \quad \Omega_s^2 > 0 \quad \Leftrightarrow \quad \eta\cos\phi_s > 0$$

Stable in the region if



Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

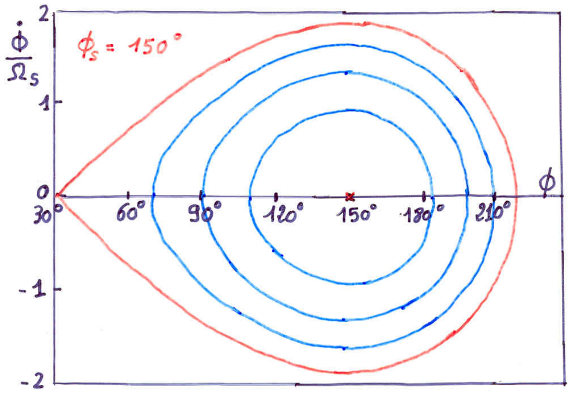
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Area within this separatrix is called "RF bucket".

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2\left\{2 + \left(2\phi_s - \pi\right)\tan\phi_s\right\}$$

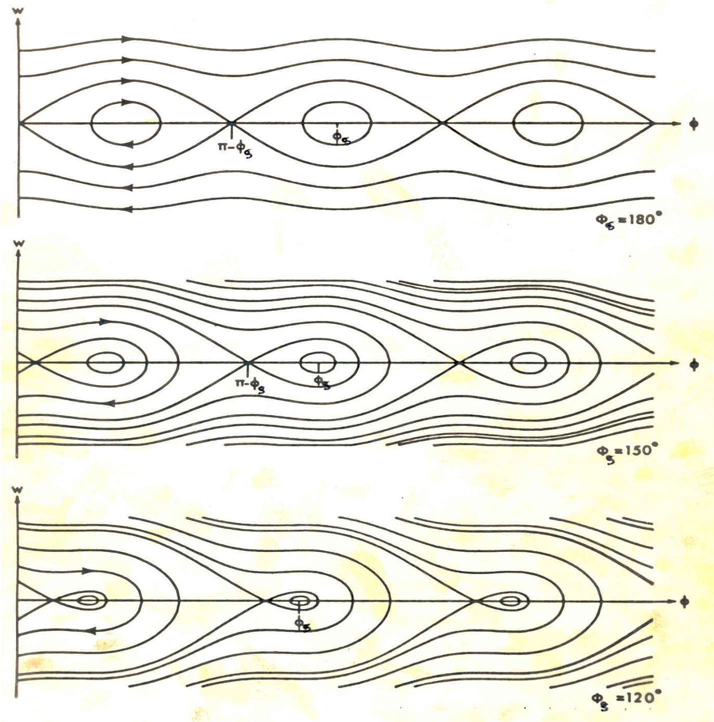
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\beta\sqrt{-\frac{e\hat{V}}{\pi h\eta E_s}G(\phi_s)}$$

$$G(\phi_s) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET".

As the synchronous phase gets closer to 90° the buckets gets smaller.

The number of circulating buckets is equal to "h".

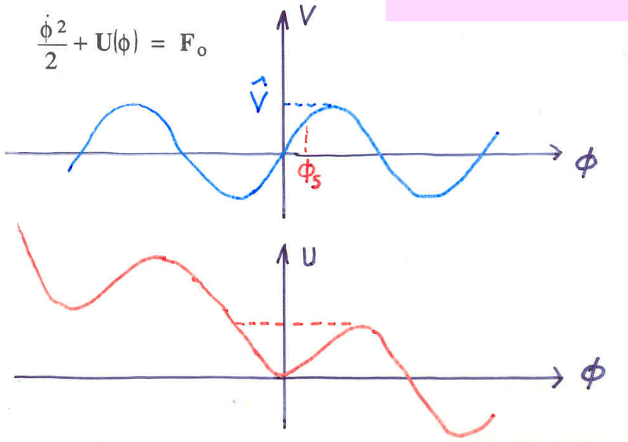
The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^{\phi} F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



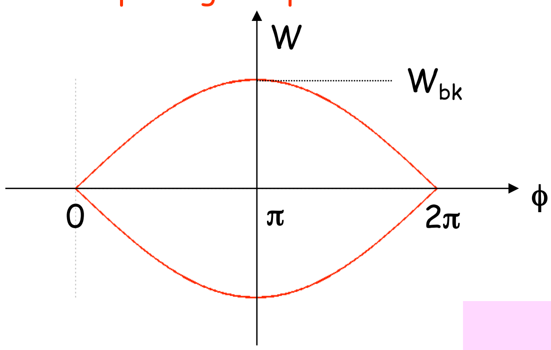
The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2 \qquad \frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the canonical variable W:



$$W=2\pi\frac{\Delta E}{\omega_{rs}}=-2\pi\frac{p_sR_s}{h\eta\omega_{rs}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W=\pm 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2}=\pm W_{bk}\sin\frac{\phi}{2}$$

with $C=2\pi R_s$

Stationnary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk}=2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{max}=\frac{\omega_{rs}}{2\pi}W_{bk}=\beta_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{\pi\eta h}}$$

The area of the bucket is: $A_{bk}=2\int_0^{2\pi}W\,d\phi$

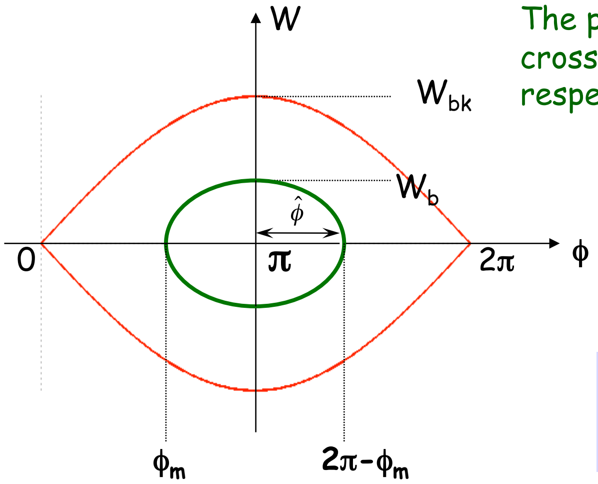
Since: $\int_0^{2\pi}\sin\frac{\phi}{2}d\phi=4$

one gets: $A_{bk}=8W_{bk}=16\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$ → $W_{bk}=\frac{A_{bk}}{8}$

Bunch Matching into a Stationnary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$W = \pm W_{bk}\sqrt{\cos^2\frac{\varphi_m}{2} - \cos^2\frac{\varphi}{2}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationnary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk}\cos\frac{\phi_m}{2} = W_{bk}\sin\frac{\hat{\phi}}{2} \qquad \text{or:} \qquad W_b = \frac{A_{bk}}{8}\cos\frac{\phi_m}{2}$$

$$\rightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF}\cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF}\sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

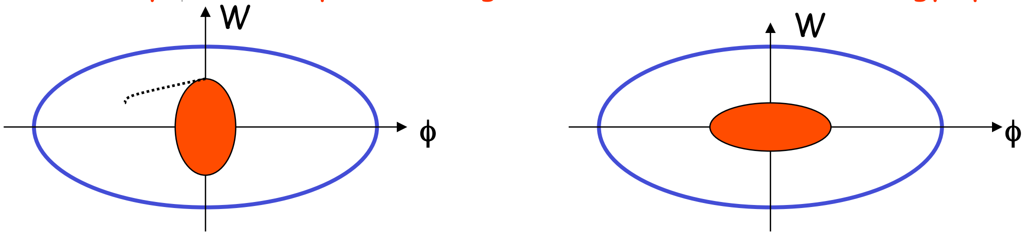
$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

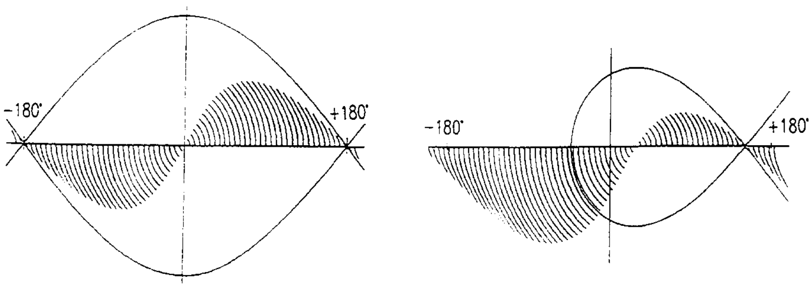
$$A_b = \frac{\pi}{16}A_{bk}\hat{\phi}^2$$

Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.

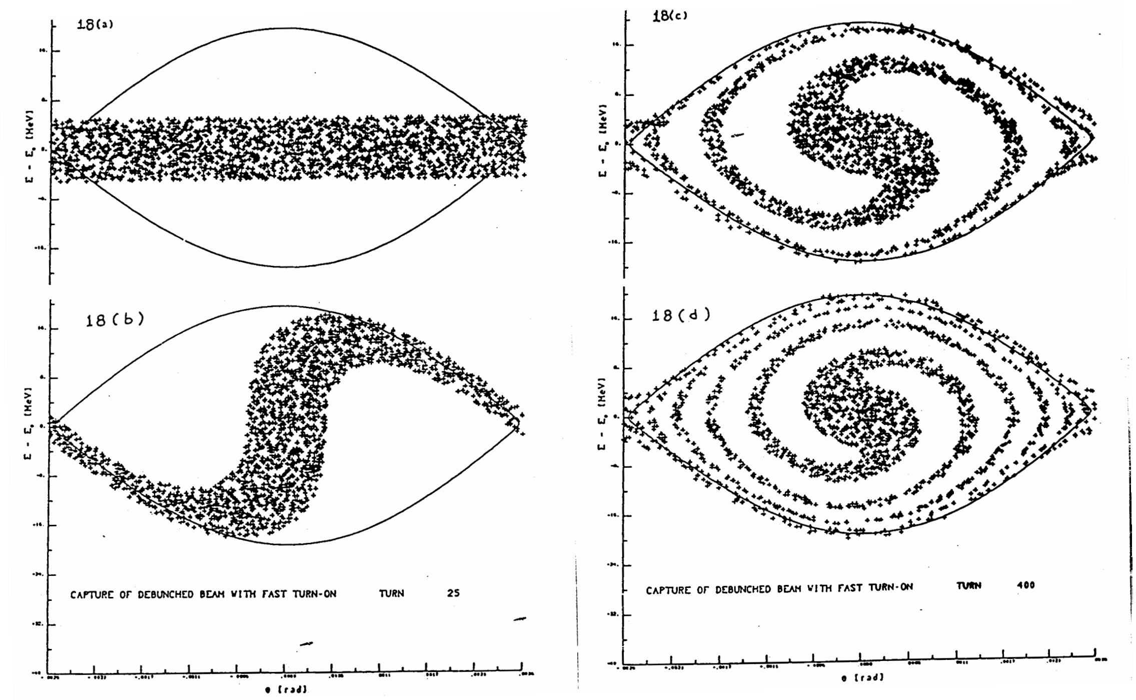


For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below) => can lead to filamentation and emittance growth

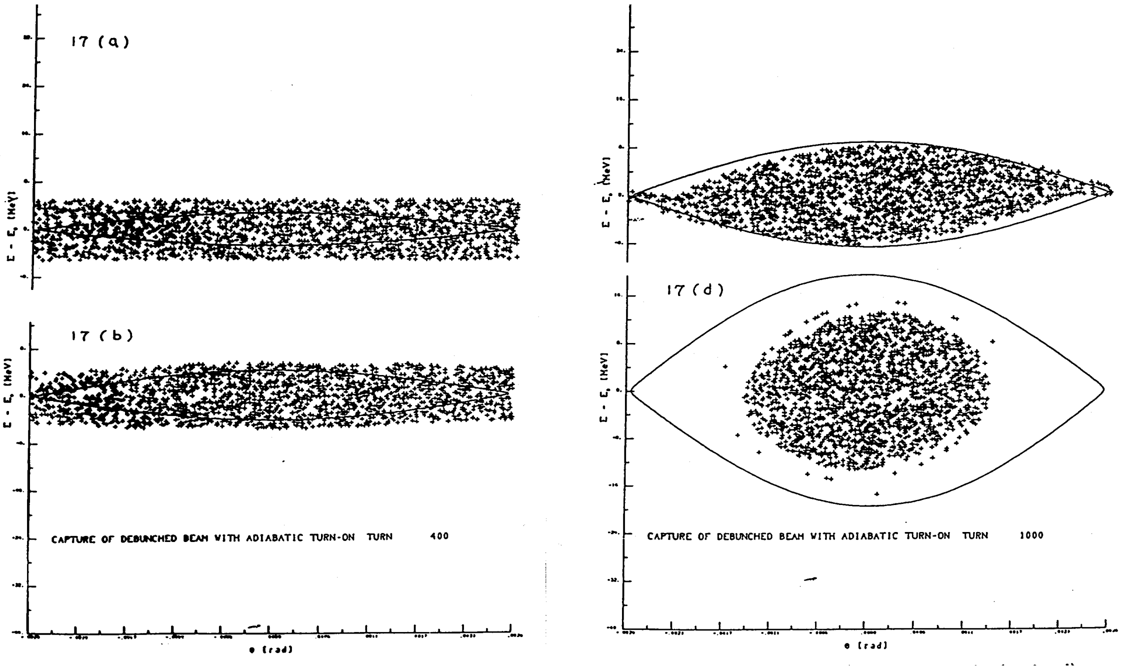


stationary bucket accelerating bucket

Capture of a Debunched Beam with Fast Turn-On

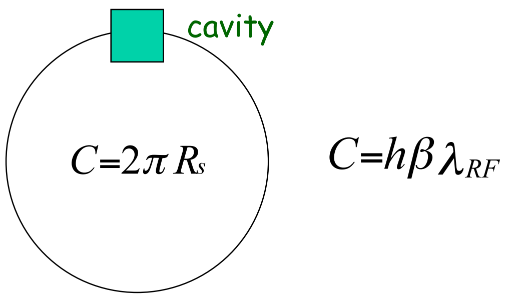


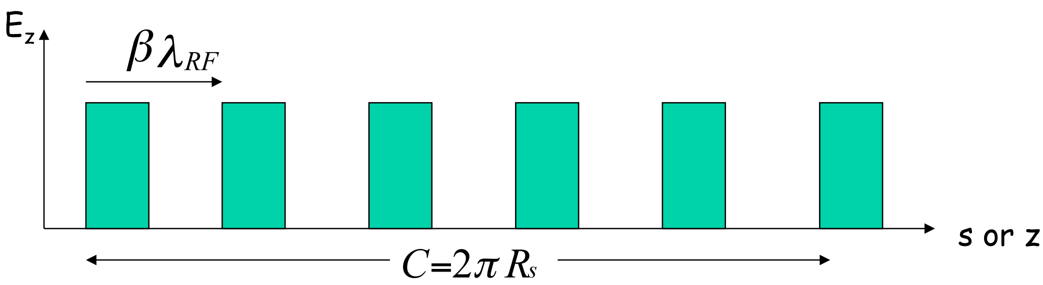
Capture of a Debunched Beam with Adiabatic Turn-On



From Synchrotron to Linac

In the linac there is no bending magnets, hence there is no dispersion effects on the orbit and $\alpha=0$ and $\eta=1/\gamma^2$.





From Synchrotron to Linac (2)

Since in the linac α=0 and η=1/γ^2, the longitudinal frequency becomes:

$$\Omega_s^2 = \frac{h\gamma^{-2}\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Moreover one has:

$$h\omega_s = \omega_{RF} \qquad \hat{V} = 2\pi R_s E_0 \qquad p_s = \gamma m_0 v_s$$

leading to:

$$\Omega_s^2 = \frac{eE_0\omega_{RF}\cos\phi_s}{m_0\gamma^3 v_s}$$

$$\gamma \to \infty \quad \Omega_s \to 0$$

The longitudinal distribution is fixed for higher energy!

Since in a linac the independent variable is z rather than t one gets by using $dz = v_s dt$

in terms of ϕ(z):

$$\left(\frac{2\pi}{\lambda_s}\right)^2 = \frac{eE_0\omega_{RF}\cos\phi_s}{m_0\gamma^3 v_s^3}$$

Adiabatic Damping

Though there are many physical processes that can damp the longitudinal oscillation amplitudes, one is directly generated by the acceleration process itself. It will happen in the synchrotron, even ultra-relativistic, when ramping the energy but not in the ultra-relativistic electron linac which does not show any oscillation.

As a matter of fact, when E_s varies with time, one needs to be more careful in combining the two first order energy-phase equations in one second order equation:

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

$$E_s\ddot{\phi} + \dot{E}_s\dot{\phi} + \Omega_s^2 E_s \Delta\phi = 0$$

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2(E_s)\Delta\phi = 0$$

The damping coefficient is proportional to the rate of energy variation and from the definition of Ω_s one has:

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

Adiabatic Damping (2)

So far it was assumed that parameters related to the acceleration process were constant. Let's consider now that they vary slowly with respect to the period of longitudinal oscillation (adiabaticity).

For small amplitude oscillations the hamiltonian reduces to:

$$H(\phi,W,t) \cong -\frac{e\hat{V}}{2}\cos\phi_s(\Delta\phi)^2 - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W^2$$

with

$$W = \hat{W}\cos\Omega_s t$$

$$\Delta\phi = (\Delta\hat{\phi})\sin\Omega_s t$$

Under adiabatic conditions the Boltzman-Ehrenfest theorem states that the action integral remains constant:

$$I = \oint W\,d\phi = const.$$

(W, ϕ are canonical variables)

Since:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}W$$

the action integral becomes:

$$I = \oint W\frac{d\phi}{dt}dt = -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{R_s p_s}\oint W^2 dt$$

Adiabatic Damping (3)

Previous integral over one period:

$$\oint W^2 dt = \pi\frac{\hat{W}^2}{\Omega_s}$$

leads to:

$$I = -\frac{h\eta\omega_{rs}}{2R_s p_s}\frac{\hat{W}^2}{\Omega_s} = const.$$

From the quadratic form of the hamiltonian one gets the relation:

$$\hat{W} = \frac{2\pi p_s R_s \Omega_s}{h\eta\omega_{rs}}\Delta\hat{\phi}$$

Finally under adiabatic conditions the long term evolution of the oscillation amplitudes is shown to be:

$$\Delta\hat{\phi} \propto \left[\frac{\eta}{E_s R_s^2 \hat{V}\cos\phi_s}\right]^{1/4} \propto E_s^{-1/4}$$

$$\hat{W} \text{ or } \Delta\hat{E} \propto E_s^{1/4}$$

$$\hat{W}\cdot\Delta\hat{\phi} = invariant$$

Dynamics in the Vicinity of Transition Energy

Introducing in the previous expressions: $\eta = \frac{1}{\gamma^2} - \alpha = \gamma^{-2} - \gamma_t^{-2}$

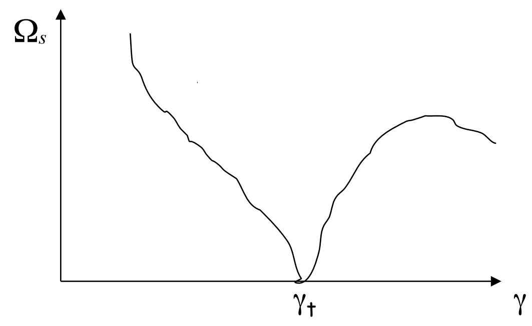
one gets:

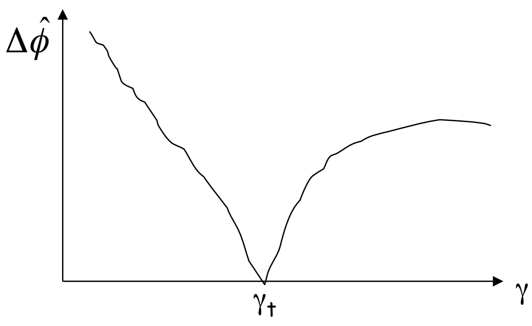
$$\Delta\hat{\phi} \propto \left\{ \frac{1}{\hat{V}|\cos\phi_s|} \frac{|\gamma^{-2} - \gamma_t^{-2}|}{\gamma} \right\}^{1/4}$$

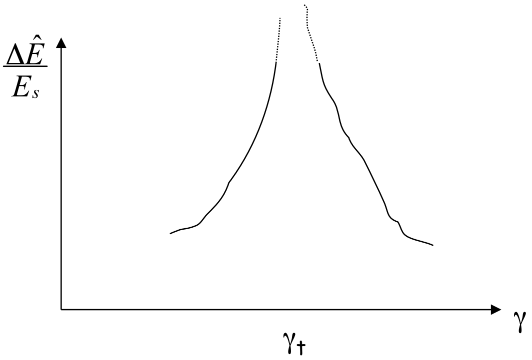
$$\Delta\hat{E} \propto \left\{ \frac{1}{\hat{V}|\cos\phi_s|} \frac{|\gamma^{-2} - \gamma_t^{-2}|}{\gamma} \right\}^{-1/4}$$

$$\Omega_s \propto \left\{ \hat{V}|\cos\phi_s| \frac{|\gamma^{-2} - \gamma_t^{-2}|}{\gamma} \right\}^{1/2}$$

Dynamics in the Vicinity of Transition Energy (2)







In fact close to transition, adiabatic solution are not valid since parameters change too fast. A proper treatment would show that:

- $\Delta\phi$ will not go to zero
- ΔE will not go to infinity