

Non-Linear

Imperfections

Intermediate Level CAS

Chios October 2011

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Non-Linear Imperfections

equation of motion

→ Hills equation

→ sine and cosine like solutions + one turn map

Poincare section → normalized coordinates

smooth approximation

resonances → tune diagram and fixed points

non-linear resonances

→ driving terms and magnetic multipole expansion

perturbation treatment of non-linear maps

→ amplitude growth and detuning quadrupole

→ fixed points and slow extraction sextupole

→ resonance islands octupole

pendulum model equation of motion and phase space

Hills equations in Cylindrical coordinates

examples → resonance islands

higher order perturbation treatment

Equations of Motion I

● Lorentz Force:

$$\frac{d\vec{p}}{dt} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

● path length as free parameter:

replace time 't' by path length 's': $x' = \frac{d}{ds} x$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \rightarrow x' = \frac{p_x}{p_0}$$

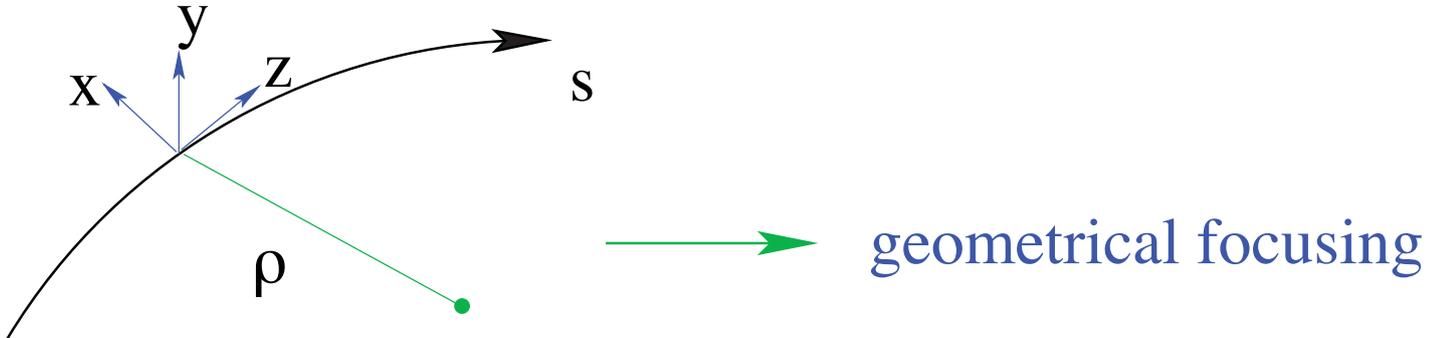


● Equation of motion:

$$\frac{d^2 x}{ds^2} = \frac{F}{v \cdot p_0}$$

Equations of Motion II

Variables in rotating coordinate system:



Hills equation:

$$\frac{d^2 x}{d s^2} + K(s) \cdot x = 0$$

$$K(s) = K(s + L);$$

$$K(s) = \begin{cases} 0 & \text{drift} \\ 1/\rho^2 & \text{dipole} \\ 0.3 \cdot \frac{B[\text{T/m}]}{p[\text{GeV}/c]} & \text{quadrupole} \end{cases}$$

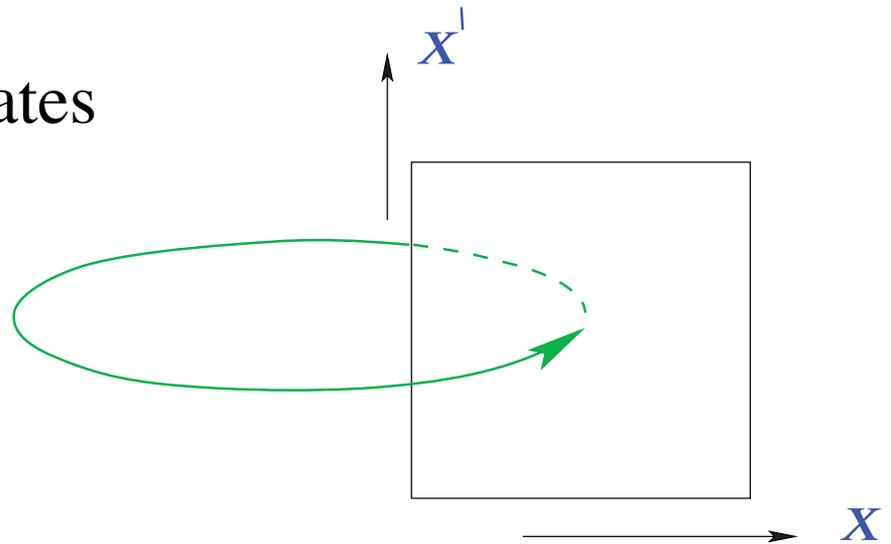
Non-linear equation of motion:

$$\frac{d^2 x}{d s^2} + K(s) \cdot x = \frac{F_x}{v \cdot p}$$

Poincare Section I

Display coordinates

after each turn:

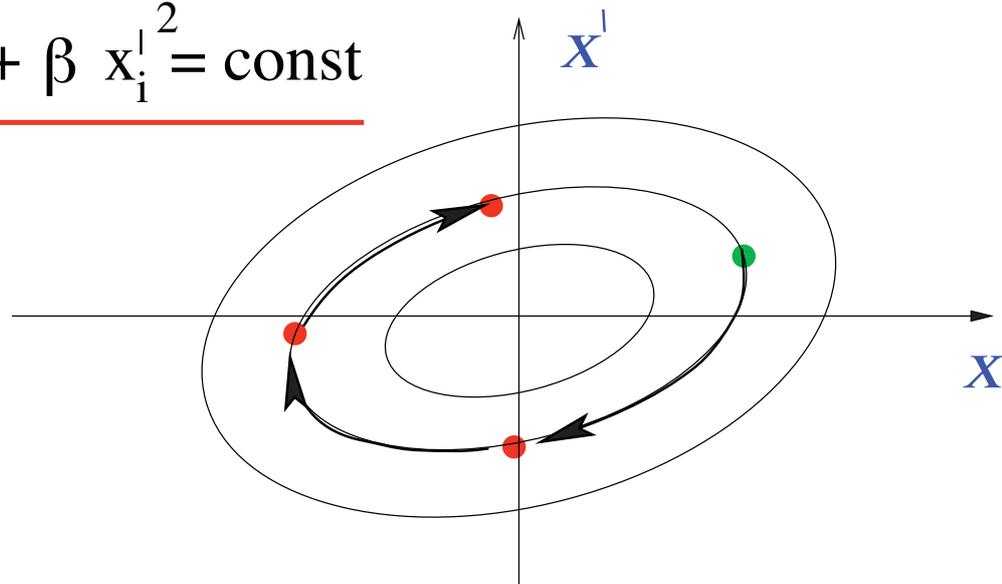


Linear β – motion:

$$x_i = \sqrt{R} \cdot \sqrt{\beta(s)} \cdot \sin(2\pi Q i + \phi_0)$$

$$x'_i = \sqrt{R} \cdot [\cos(2\pi Q i + \phi_0) + \alpha(s) \cdot \sin(2\pi Q i + \phi_0)] / \sqrt{\beta(s)}$$

→ $\gamma x_i^2 + 2\alpha x_i x'_i + \beta x'^2 = \text{const}$



→ *ellipse*

the ellipse orientation and the half axis length vary along the machine

Poincare Section II

for the sake of simplicity assume $\alpha = 0$

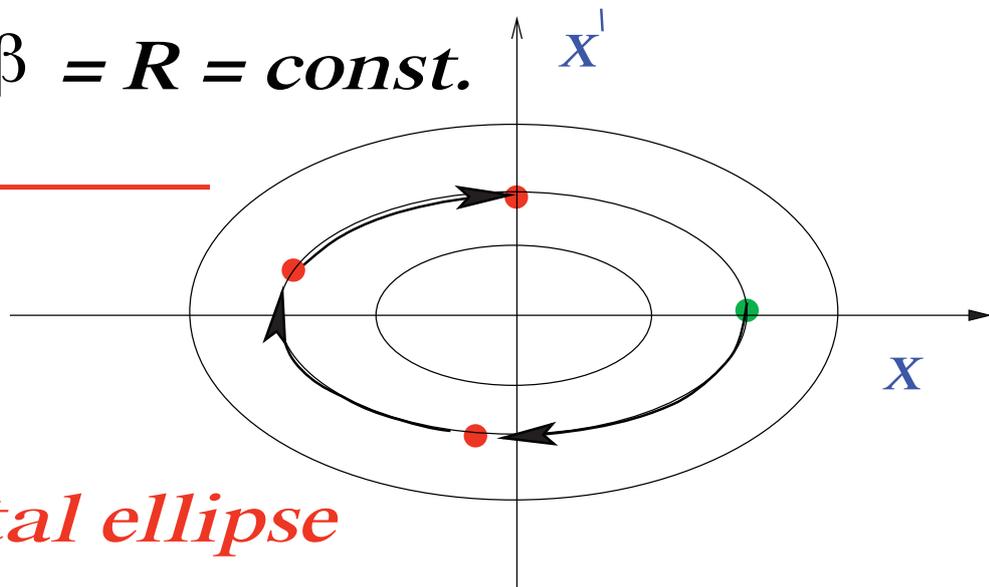
at the location of the Poincare Section



$$x = \sqrt{\beta} \cdot \sqrt{R} \cdot \cos(2\pi Q i + \phi_0)$$

$$x' = \sqrt{R} \cdot \sin(2\pi Q i + \phi_0) / \sqrt{\beta}$$

$$\frac{x^2}{\beta} + x'^2 \cdot \beta = R = \text{const.}$$



horizontal ellipse

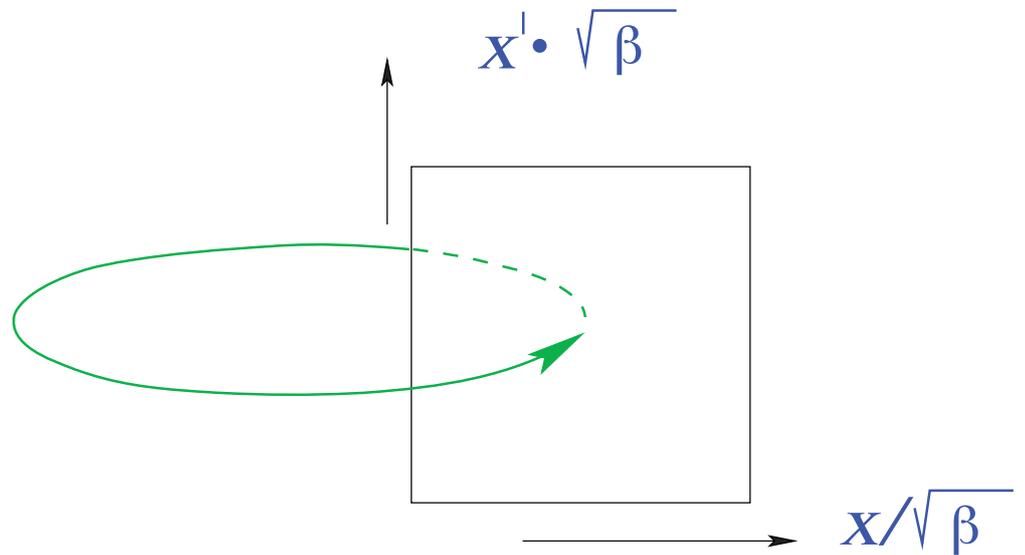
for $\alpha \neq 0$

one can define a new set of coordinates via linear combination of x and x' such

that one axis of the ellipse is parallel to x -axis

Poincare Section III

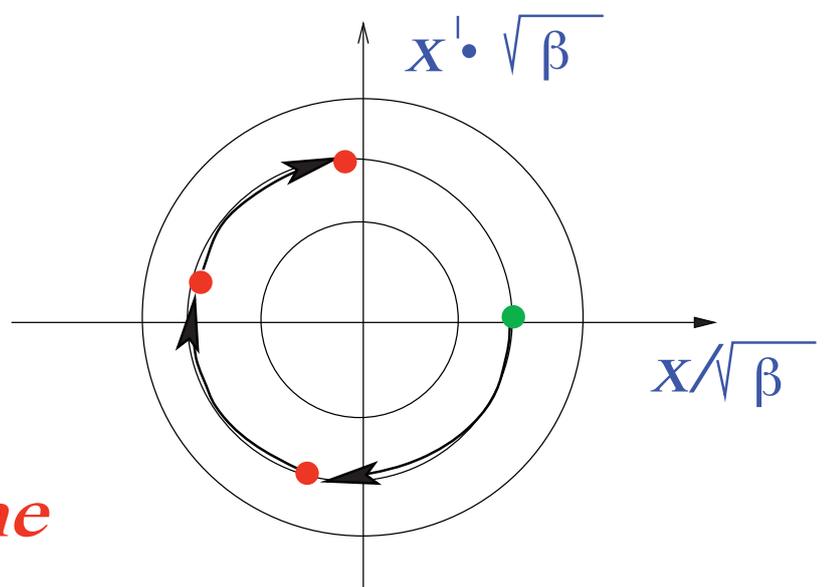
Display normalized coordinates:



normalized coordinates:

$$x/\sqrt{\beta} = \sqrt{R} \cdot \cos(2\pi Q i + \phi_0)$$

$$\sqrt{\beta} \cdot x' = -\sqrt{R} \cdot \sin(2\pi Q i + \phi_0)$$



**→ circles in the
Poincare Section**

Smooth Approximation

assume: $\beta = \text{constant}$

$\rightarrow x = A \cdot \cos[\phi(s)]$ with: $\phi(s) = \int_{s_0}^s \frac{1}{\beta} dt$

$\rightarrow \frac{d\phi}{ds} = \frac{1}{\beta} = \omega = \frac{2\pi Q}{L}$

Linear β - motion: $\beta = \text{const} \rightarrow \alpha = 0$

$$x_i = \sqrt{R} \cdot \sqrt{\beta(s)} \cdot \sin(2\pi Q i + \phi_0)$$

$$x'_i = \sqrt{R} \cdot \cos(2\pi Q i + \phi_0) / \sqrt{\beta(s)}$$

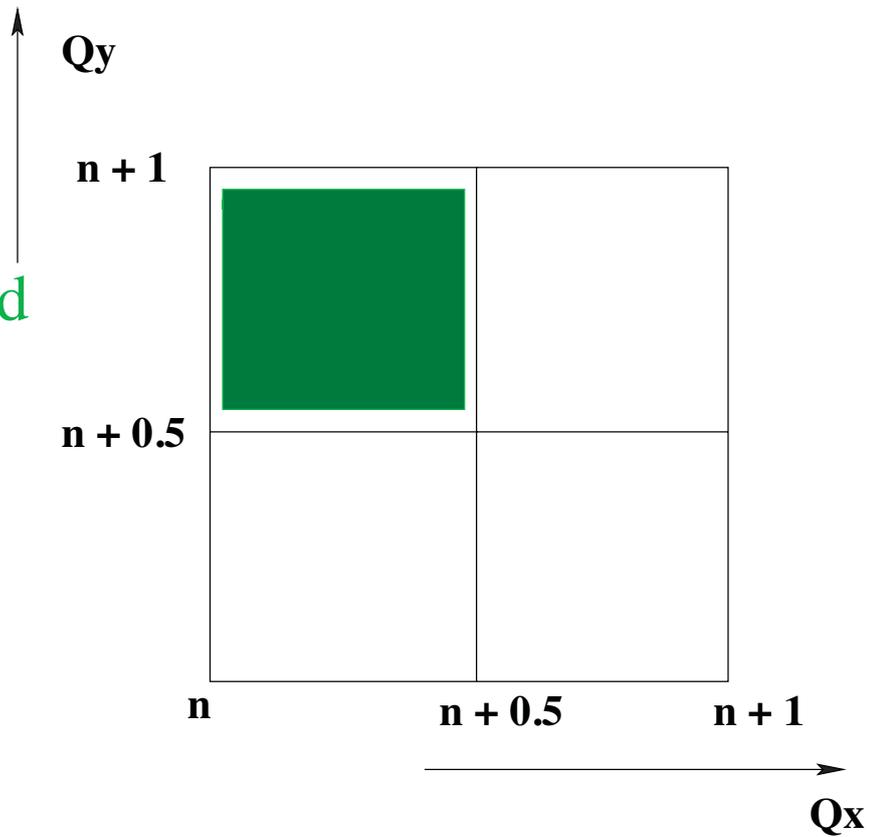
Linear equation of motion:

$$\frac{d^2 x}{ds^2} + \left(\frac{2\pi}{L} \cdot Q \right)^2 x = 0 \rightarrow \text{Harmonic Oscillator}$$

Resonances I

tune diagram with linear resonances:

stability:
avoid integer and
half integer
resonances!

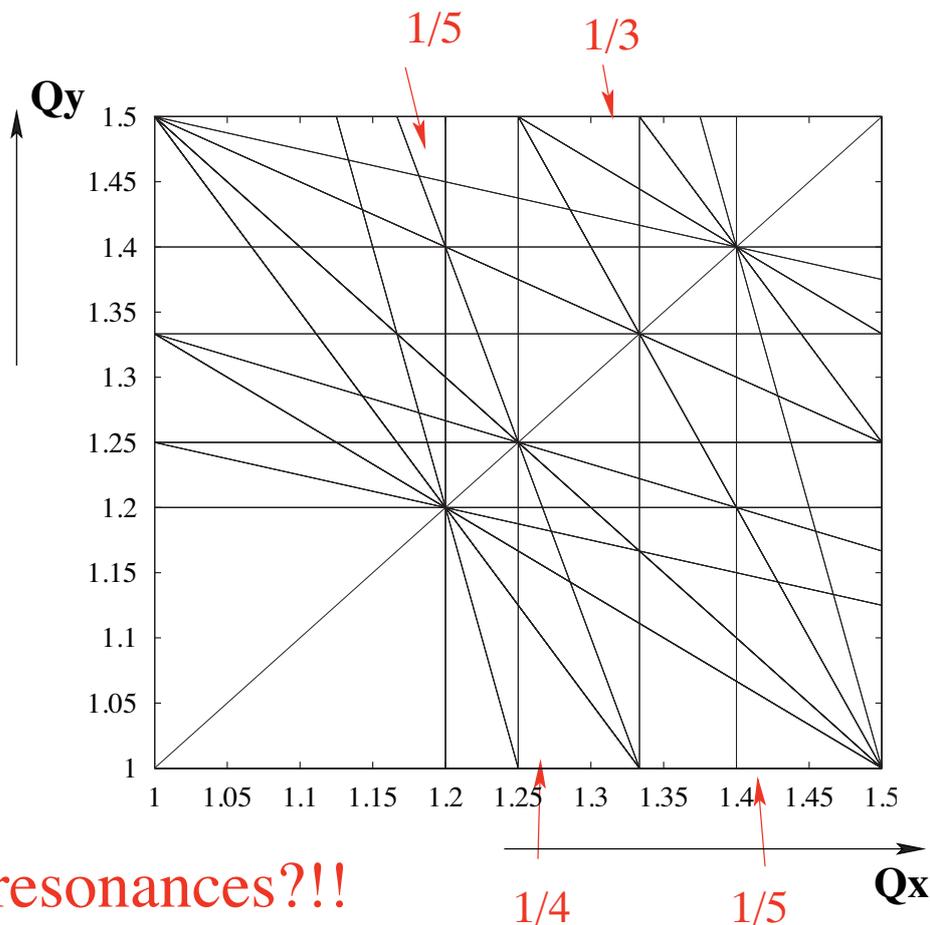


higher order resonances:

$$n Q_x + m Q_y = r$$

the rational numbers
lie 'dense' in the
real numbers

there are resonances
everywhere



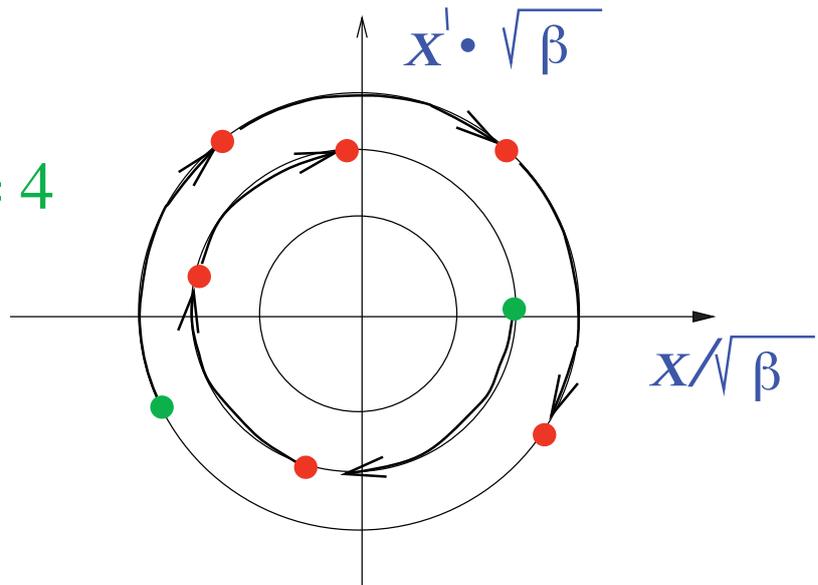
stability of low order resonances?!!

Resonances II

fixed points in the Poincare section:

$$Q = N + 1/n$$

example: $n = 4$



- *every point is mapped on itself after n turns!*
- *every point is a 'fixed point'*
- *motion remains stable if the resonances are not driven*
- *sources for resonance driving terms?*

Non-Linear Resonances I

■ *Sextupoles + octupoles*

■ *Magnet errors:*

pole face accuracy

geometry errors

eddy currents

edge effects

■ *Vacuum chamber:*

LEP I welding

■ *Beam-beam interaction*



*careful analysis of all
components*

Non-Linear Resonances II

Taylor expansion for upright multipoles:

$$B_y + i \cdot B_x = \sum_{n=0} \frac{1}{n!} \cdot f_n \cdot (x + i y)^n$$

with: $f_n = \frac{\partial^n B_y}{\partial x^n}$

multipole	order	B_x	B_y
dipole	0	0	B_0
quadrupole	1	$f_1 \cdot y$	$f_1 \cdot x$
sextupole	2	$f_2 \cdot x \cdot y$	$\frac{1}{2} f_2 \cdot (x^2 - y^2)$
octupole	3	$\frac{1}{6} f_3 \cdot (3y x^2 - y^3)$	$\frac{1}{6} f_3 \cdot (x^3 - 3x y^2)$

convergence:

the Taylor series is normally not convergent for $|x + i y| > 1 \rightarrow$ define 'normalized' coefficients

$$b_n = \frac{f_{n-1}}{(n-1)! \cdot B_0} \cdot R_{\text{ref}}^{n-1}$$

Non-Linear Resonances III

normalized multipole expansion:

$$B_y + i \cdot B_x = B_{main} \sum_{n=1} b_n \cdot \left(\frac{x + i y}{R_{ref}} \right)^{n-1}$$

b_n is the relative field contribution of the n -th multipole at the reference radius

b_1 = dipole; b_2 = quadrupole; b_3 = sextupole; etc

skew multipoles:

rotation of the magnetic field by 1/2 of the

azimuthal magnet symmetry: 90° for dipole

45° for quadrupole

30° for sextupole; etc

general multipole expansion:

$$B_y + i \cdot B_x = B_{main} \sum_{n=1} (b_n - i a_n) \cdot \left(\frac{x + i y}{R_{ref}} \right)^{n-1}$$

Perturbation I

■ perturbed equation of motion:

$$\frac{d^2 x}{d s^2} + \left(\frac{2\pi}{L} \cdot Q_x \right)^2 \cdot x = \frac{F_x(x,y)}{v \cdot p}$$

$$\frac{d^2 y}{d s^2} + \left(\frac{2\pi}{L} \cdot Q_y \right)^2 \cdot y = \frac{F_y(x,y)}{v \cdot p}$$

■ assume motion in one degree only:

$y \equiv 0$ is a solution of the vertical equation of motion

$$\rightarrow B_x \equiv 0; \quad B_y = \frac{1}{n!} \cdot f_n \cdot x^n \quad F_x = -v_s \cdot B_y$$

■ perturbed horizontal equation of motion:

$$\frac{d^2 x}{d s^2} + \left(\frac{2\pi}{L} \cdot Q_x \right)^2 \cdot x = \frac{-1}{n!} \cdot k_n(s) \cdot x^n$$

■ normalized strength:

$$k_n = 0.3 \cdot \frac{f_n [\text{T/m}^n]}{p [\text{GeV}/c]}; \quad [k_n] = 1 / \text{m}^{n+1}$$

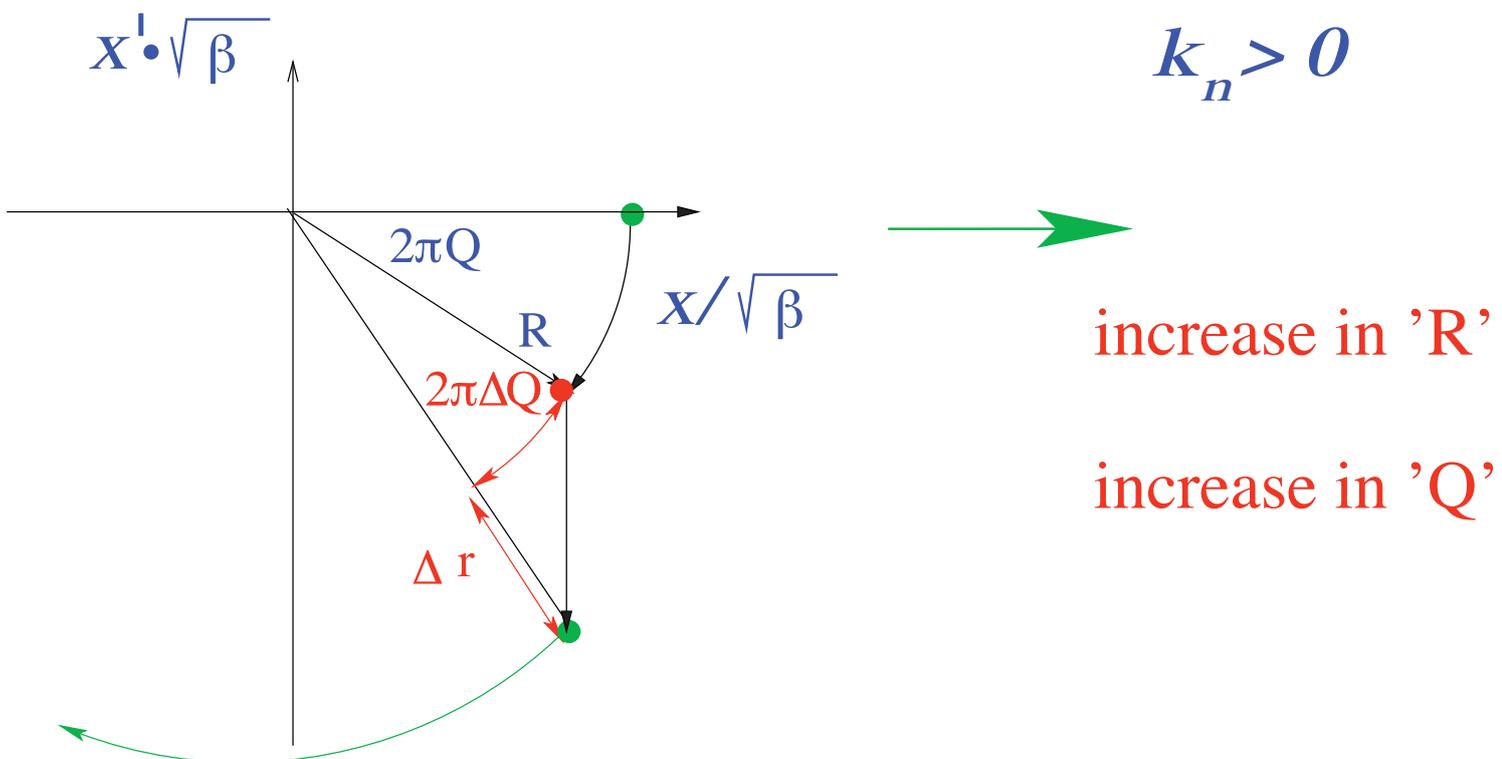
Perturbation II

■ perturbation just in front of Poincare Section:

$$\Delta x' = \int \frac{F_y}{v \cdot p} ds \longrightarrow = \frac{-l}{n!} \cdot k_n \cdot x^n$$

where ' l ' is the length of the perturbation

■ perturbed Poincare Map:



■ stability of particle motion over many turns?

Perturbation III

coordinates after 'i' iteration and before kick:

$$(1) \quad x_i / \sqrt{\beta} = r \cdot \cos(\phi_i) \quad x_i' \cdot \sqrt{\beta} = -r \cdot \sin(\phi_i)$$

$$(2) \quad \text{with: } \phi_i = \phi_{i-1} + 2\pi Q \quad \text{and: } r = \sqrt{R}$$

coordinates after the perturbation kick:

$$(3) \quad x_{i+kick} / \sqrt{\beta} = x_i / \sqrt{\beta}$$

$$(4) \quad x_{i+kick}' \cdot \sqrt{\beta} = x_i' \cdot \sqrt{\beta} - \frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta}$$

write new coordinates in circular coordinates

$$(5) \quad x_{i+kick} / \sqrt{\beta} = (r + \Delta r_i) \cdot \cos(\phi_i + \Delta\phi_i)$$

$$(6) \quad x_{i+kick}' \cdot \sqrt{\beta} = (r + \Delta r_i) \cdot \sin(\phi_i + \Delta\phi_i)$$

Perturbation IV

■ solve for ' Δr_i ' and ' $\Delta\phi_i$ ':

→ substitute (1) and (2) into (3) and (4)

→ set new expression equal to (5) and (6)

→ use: $\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$
 $\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$

and: $\sin(\Delta\phi) = \Delta\phi$; $\cos(\Delta\phi) = 1$

→ solve for ' Δr_i ' and ' $\Delta\phi_i$ ':

$$\Delta r_i = -\Delta x_i^l \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

$$\Delta\phi_i = \frac{-\Delta x_i^l \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta x_i^l \cdot \sqrt{\beta} \cdot \sin(\phi_i)]}$$

■ substitute the kick expression:

$$(7) \quad \Delta r_i = \frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

$$(8) \quad \Delta\phi_i = \frac{\frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta r_i]}$$

Perturbation V

■ quadrupole perturbation:

$$\Delta r_i = l \cdot k_1 \cdot x_i \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

$$\text{with: } x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$$

$$\Delta r_i = l \cdot k_1 \cdot r \cdot \beta \cdot \sin(2\phi_i)$$

sum over many turns with: $\phi_i = 2\pi Q \cdot i$

→ $\sum_i \Delta r_i = 0$ unless: $Q = p/2$

(half integer resonance)

■ tune change (first order in the perturbation):

$$\Delta\phi_i = l \cdot k_1 \cdot \beta \cdot [1 + \cos(2\phi_i)]/2$$

average change per turn: $\phi_i = 2\pi Q \cdot i$

$$\langle \Delta Q \rangle = l \cdot k_1 \cdot \beta / 4\pi$$

→ $Q = Q_0 + \langle \Delta Q \rangle$

Perturbation VI

■ resonance stop band: $Q \neq p/2$

the map perturbation generates a tune oscillation

$$\delta Q_i = l \cdot k_1 \cdot \beta \cdot \cos(4\pi \cdot Q \cdot i + 2\phi_0) / 4\pi$$

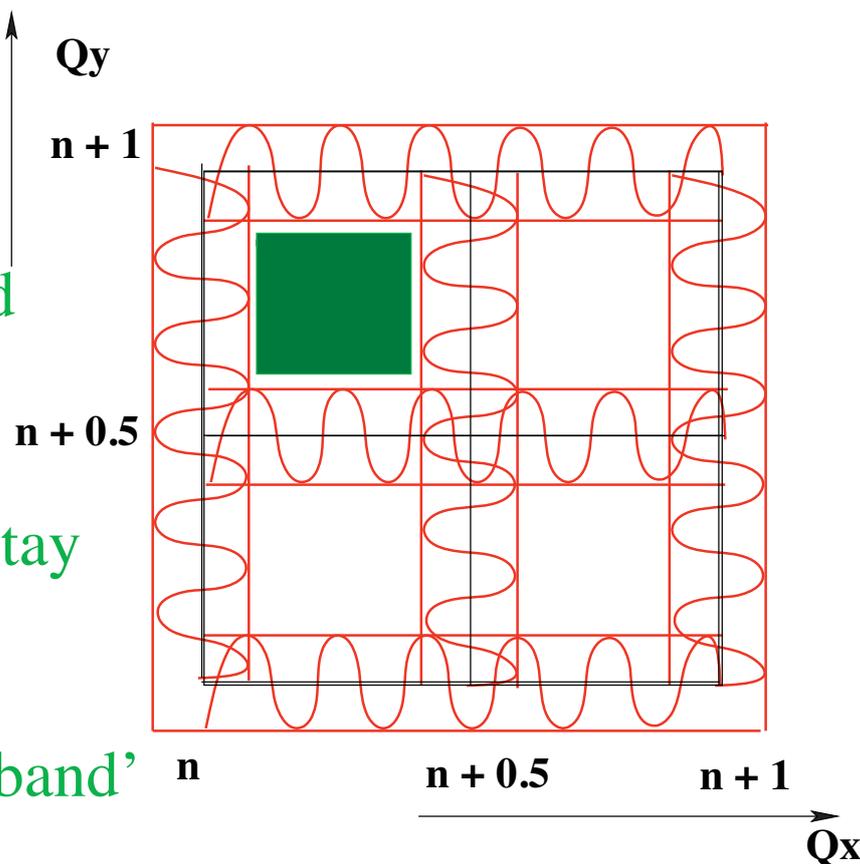
$$= \langle \Delta Q \rangle \cdot \cos(4\pi Q i + 2\phi_0) / 4\pi$$

→ particles will experience the half integer resonance if their tune satisfies:

$$(p/2 - \langle \Delta Q \rangle) < Q < (p/2 + \langle \Delta Q \rangle)$$

■ tune diagram:

avoid integer and half integer resonances and stay away from the resonance 'stop band'



Perturbation VII

■ sextupole perturbation:

$$\Delta r_i = l \cdot k_2 \cdot x_i^2 \sqrt{\beta} \cdot \sin(\phi_i) / 2$$

$$\text{with: } x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$$

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \beta^{3/2} [\sin(\phi_i) + \sin(3\phi_i)] / 8$$

sum over many turns:

$$\phi_i = 2\pi Q \cdot i$$



$$r = 0 \quad \text{unless: } Q = p \text{ or } Q = p/3$$

■ tune change (first order in the perturbation):

$$2\pi \Delta Q_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} [3 \cos(2\pi Q i + \phi_0) + \cos(6\pi Q i + 3\phi_0)] / 8$$

sum over many turns:

(unless: $Q = p$ or $Q = p/3$)

$$\langle \Delta Q \rangle = 0$$



stop band increases with amplitude!

Perturbation VIII

what happens for $Q = p; p/3$?

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \left[\sin(2\pi Q i + \phi_0) + \sin(6\pi Q i + 3\phi_0) \right] / 8$$

constant for each kick

$$2\pi \Delta Q_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} \left[3 \cos(2\pi Q i + \phi_0) + \cos(6\pi Q i + 3\phi_0) \right] / 8$$

amplitude 'r' increases every turn \rightarrow instability

\rightarrow dephasing and tune change

\rightarrow motion moves off resonance

\rightarrow stop of the instability

\rightarrow what happens in the long run?

Perturbation IX

let us assume: $Q = p/3$

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} [\sin(\phi_i) + \sin(3\phi_i)] / 8$$

$$\Delta \phi_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} [3 \cos(\phi_i) + \cos(3\phi_i)] / 8 + 2\pi Q$$

the first terms change rapidly for each turn

→ the contribution of these terms are small and we omit these terms in the following (method of averaging)

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \sin(3\phi_i) / 8$$

$$\Delta \phi_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cos(3\phi_i) / 8 + 2\pi Q$$

Perturbation X

fixed point conditions: $Q_0 \gtrsim p/3; k_2 > 0$

$$\Delta r / \text{turn} = 0 \quad \text{and} \quad \Delta \phi / \text{turn} = 2\pi p / 3$$

with:
$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \sin(3 \phi_i) / 8$$

$$\Delta \phi_i = 2\pi Q_0 + l \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cos(3 \phi_i) / 8$$

→
$$\phi_{\text{fixed point}} = \pi/3; \pi; 5\pi/3;$$

$$r_{\text{fixed point}} = \frac{16\pi (Q_0 - p/3)}{l k_2 \beta^{3/2}}$$

→ $r = 0$ also provides a fixed point in the

$x; x'$ plane

(infinite set in the r, ϕ plane)

Perturbation XI

fixed point stability:

linearize the equation of motion around the fixed points:

Poincare map:
$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{f}(\mathbf{r}_i, \phi_i)$$

$$\phi_{i+1} = \phi_i + \mathbf{g}(\mathbf{r}_i, \phi_i)$$

single sextupole kick:

$$\rightarrow \mathbf{f} = l \cdot k_2 \cdot r_i^2 \beta_i^{3/2} \sin(3\phi_i) / 8$$

$$\mathbf{g} = l \cdot k_2 \cdot r_i \beta_i^{3/2} \cos(3\phi_i) / 8$$

→ linearized map around fixed points:

$$\begin{pmatrix} \mathbf{r}_{i+1} \\ \phi_{i+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{r}_{i+1}}{\partial \mathbf{r}_i} & \frac{\partial \mathbf{r}_{i+1}}{\partial \phi_i} \\ \frac{\partial \phi_{i+1}}{\partial \mathbf{r}_i} & \frac{\partial \phi_{i+1}}{\partial \phi_i} \end{pmatrix} \bigg|_{\text{fixed point}} \cdot \begin{pmatrix} \mathbf{r}_i \\ \phi_i \end{pmatrix}$$

Perturbation XII

Jacobian matrix for single sextupole kick:

Jacobian matrix

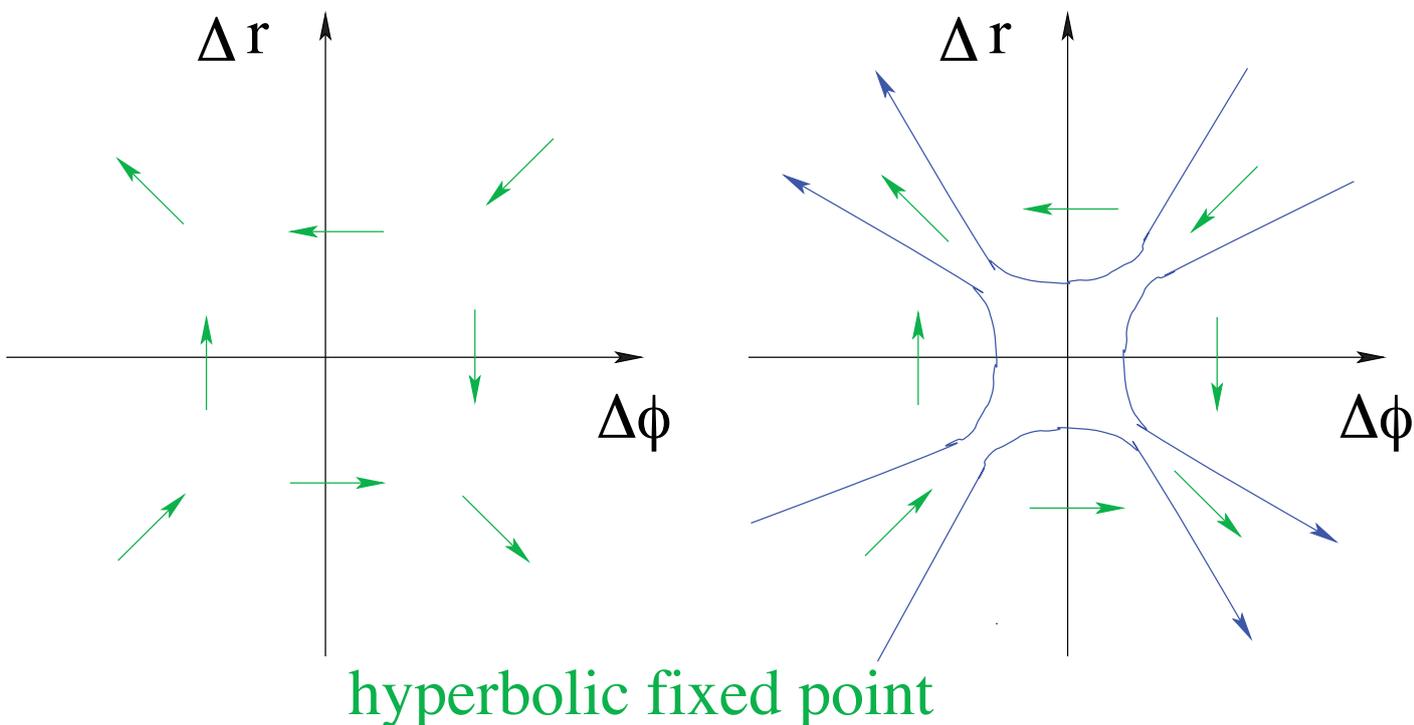
$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \quad \frac{\partial r_{i+1}}{\partial \phi_i} = -3l \cdot k_2 \cdot \beta^{3/2} \cdot r_{\text{fixed point}}^2 / 8$$

$$\frac{\partial \phi_{i+1}}{\partial r_i} = -l \cdot k_2 \cdot \beta^{3/2} / 8; \quad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

$$\phi_{\text{fixed point}} = \pi/3; \pi; 5\pi/3; \quad \text{and } r_{\text{fixed point}} \neq 0$$

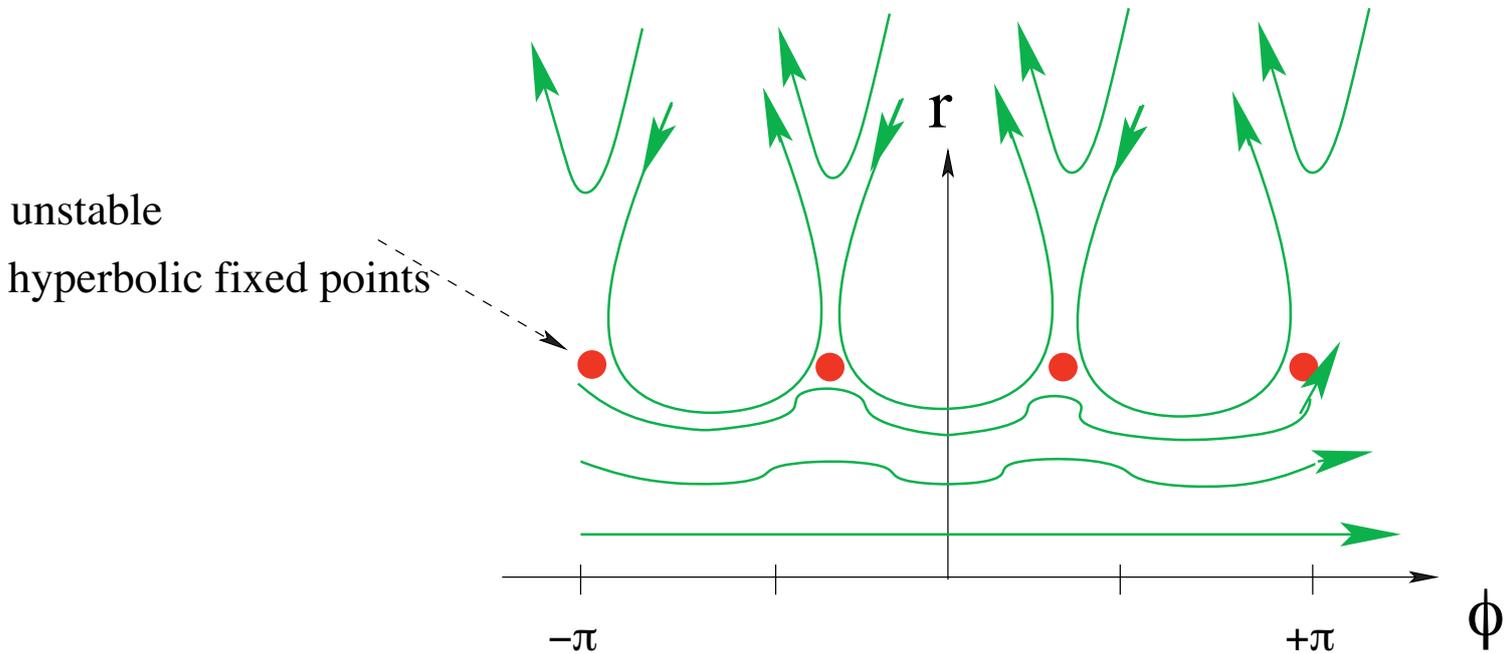
$$\rightarrow \Delta r_{i+1} = -3l \cdot k_2 \cdot \beta^{3/2} \cdot r_{\text{fixed point}}^2 / 8 \cdot \Delta \phi_i$$

$$\Delta \phi_{i+1} = -l \cdot k_2 \cdot \beta^{3/2} / 8 \cdot \Delta r_i \quad \text{stability?}$$

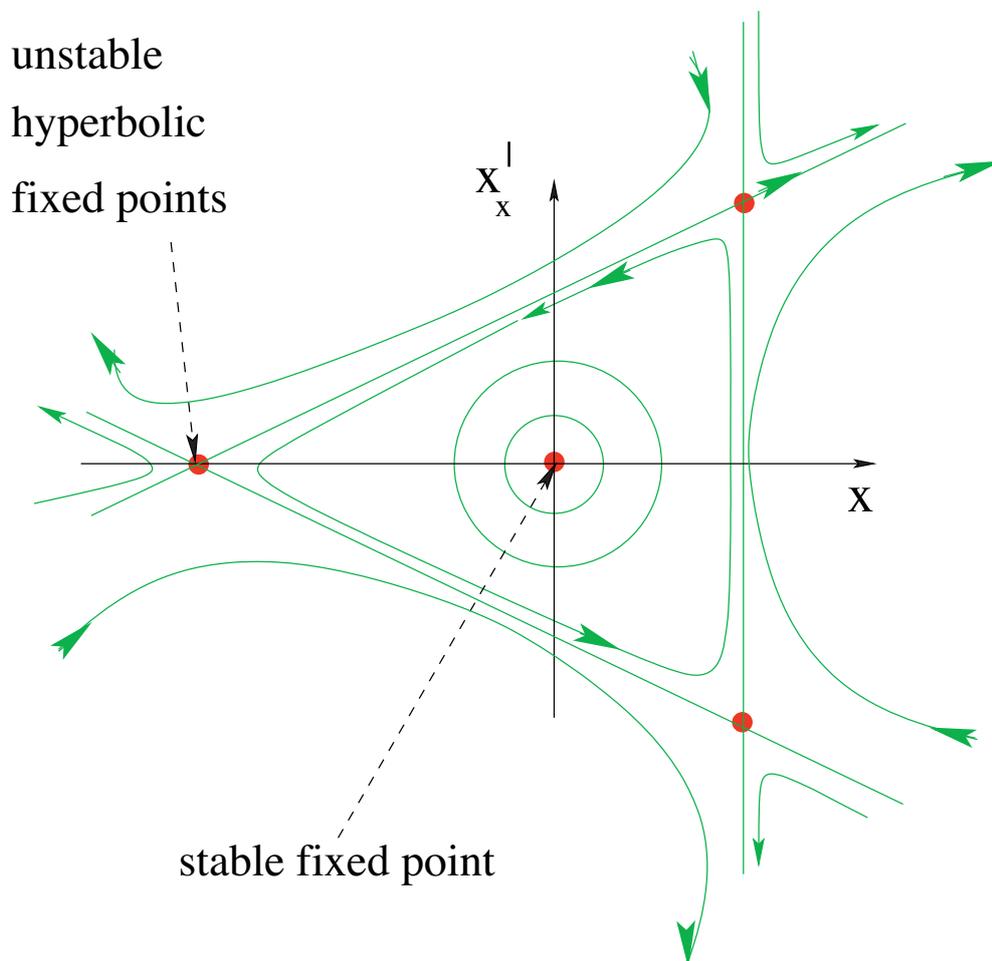


Perturbation XIII

Poincare Section for 'r' and ϕ :



Poincare section in normalized coordinates:

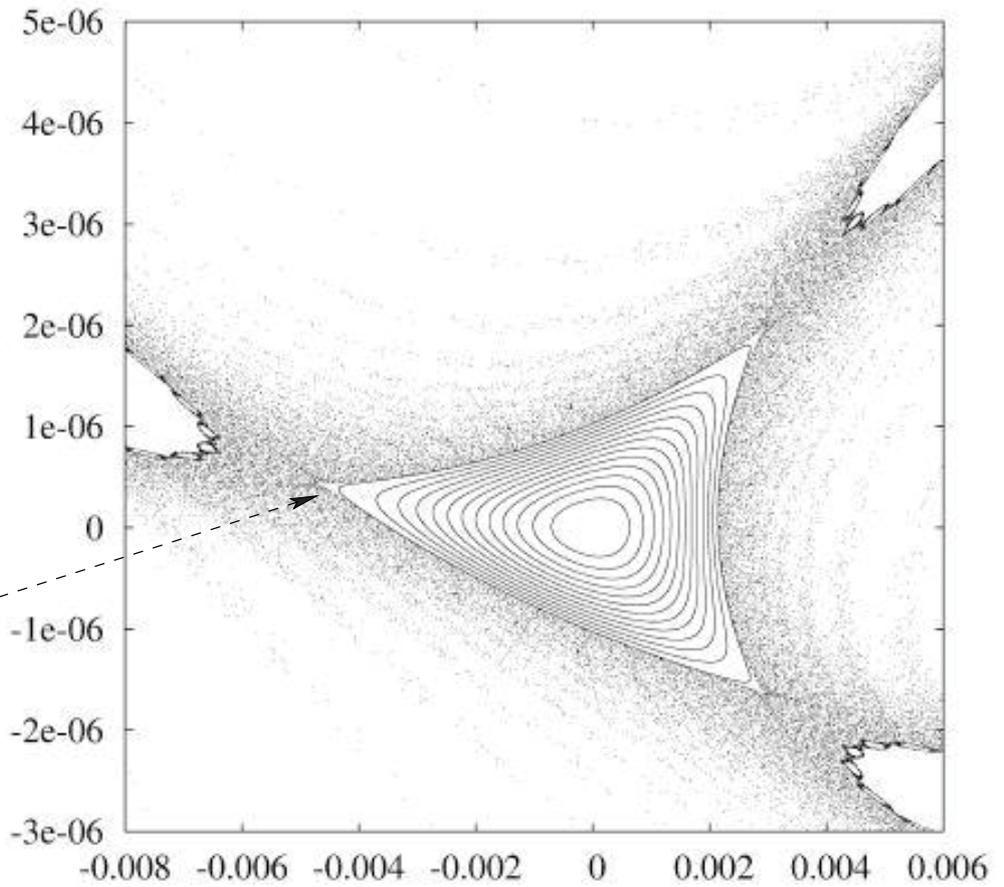


Perturbation XIV

Sextupole x'

Poincare
Section

from
simulations:



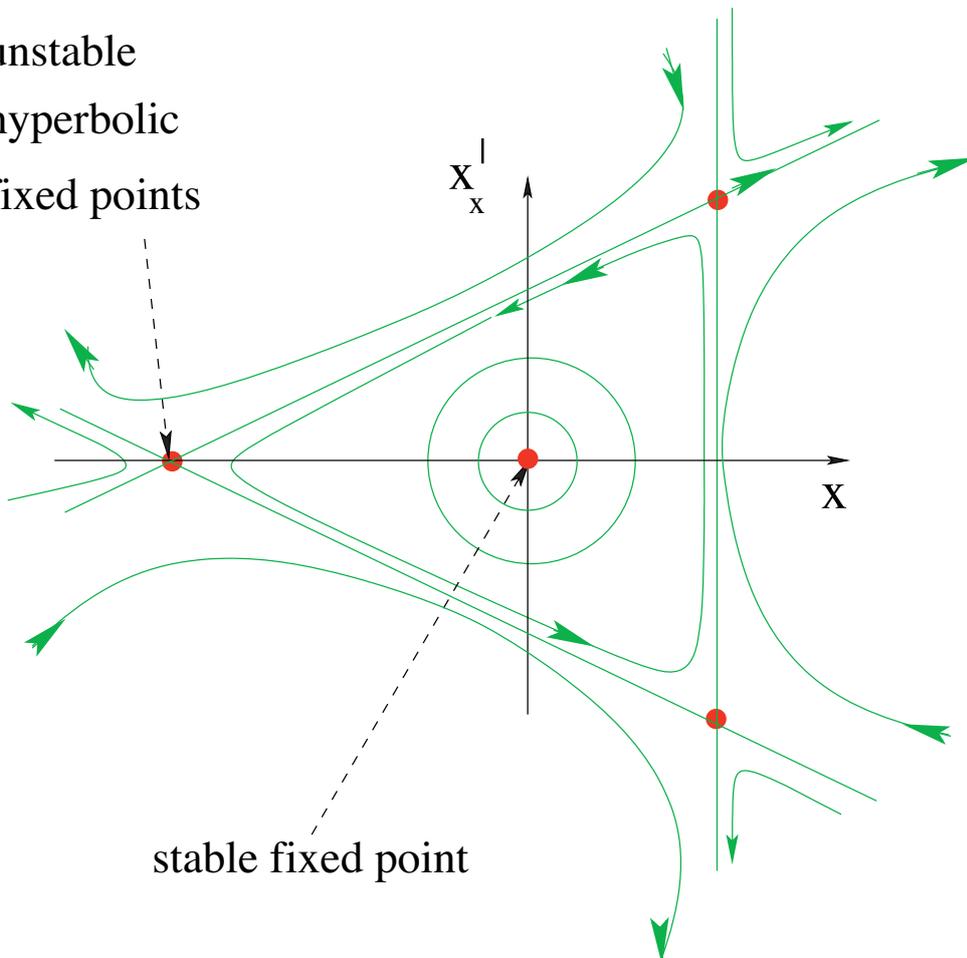
unstable

hyperbolic fixed point

Poincare section in normalized coordinates:

X

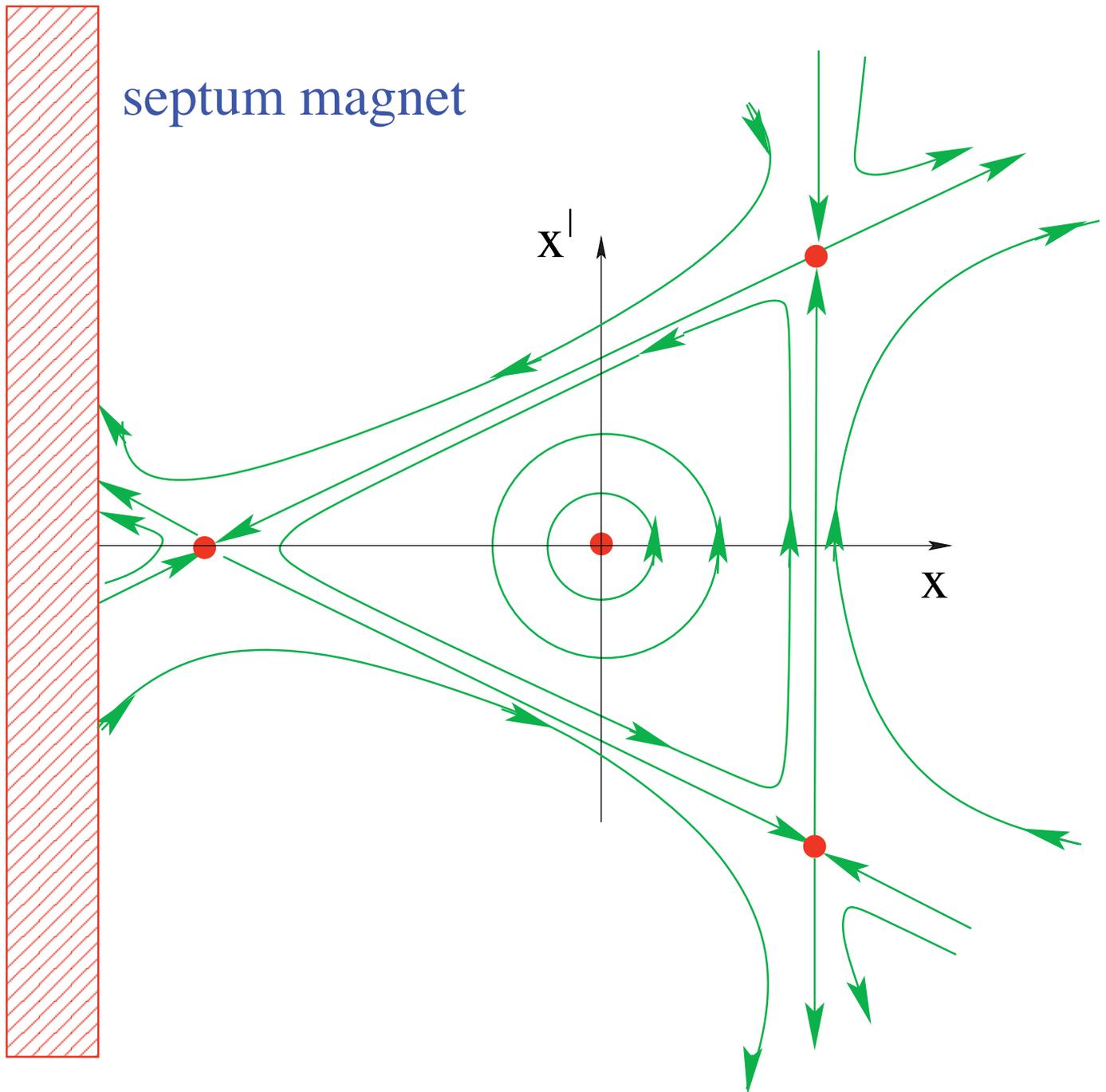
unstable
hyperbolic
fixed points



stable fixed point

Perturbation XVI

slow extraction:



fixed point position:

$$r_{\text{fixed point}} = \frac{16 \pi (Q - \frac{p}{3})}{l \cdot k_2 \cdot \beta^{3/2}}$$

changing the tune
during extraction!

Perturbation XVII

■ octupole perturbation:

$$\Delta r_i = l \cdot k_3 \cdot x_i^3 \sqrt{\beta} \cdot \sin(\phi_i) / 6$$

$$\text{with: } x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$$

$$\Delta r_i = l \cdot k_3 \cdot r_i^3 \beta^2 \cdot [2 \sin(2\phi_i) + \sin(4\phi_i)] / 48$$

sum over many turns: $\phi_i = 2\pi Q \cdot i + \phi_0$



$$r = 0 \quad \text{unless: } Q = p, p/2, p/4$$

■ tune change (first order in the perturbation):

$$2\pi \Delta Q_i = l \cdot k_3 \cdot r_i^2 \beta^2 \cdot [4 \cos(4\pi Q i + 2\phi_0) + 3 + \cos(8\pi Q i + 4\phi_0)] / 48$$

sum over many turns (unless: $Q = p$ or $Q = p/4$):



$$\langle \Delta Q \rangle = l \cdot k_3 \cdot r^2 \cdot \beta^2 / 16 / 2\pi$$

Perturbation XVIII

■ detuning with amplitude:

particle tune depends on particle amplitude

→ tune spread for particle distribution

→ stabilization of collective instabilities

→ install octupoles in the storage ring

→ distribution covers more resonances
in the tune diagram

→ avoid octupoles in the storage ring

→ requires a delicate compromise

■ Poincare section topology:

$Q = p/4$ and apply method of averaging

$$\Delta r_i = l \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot \sin(4\phi_i) / 48$$

$$\Delta\phi_i = l \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [3 + \cos(4\phi_i)] / 48 + 2\pi Q$$

Perturbation XIX

fixed point conditions: $Q_0 \lesssim p/4; k_3 > 0$

$$\Delta r / \text{turn} = 0 \quad \text{and} \quad \Delta \phi / \text{turn} = 2\pi p / 4$$

with:
$$\Delta r_i = l \cdot k_3 \cdot r_i^3 \beta^2 \sin(4 \phi_i) / 48$$

$$\Delta \phi_i = 2\pi Q_0 + l \cdot k_3 \cdot r_i^2 \beta^2 [3 + \cos(4 \phi_i)] / 48$$

→
$$\phi_{\text{fixed point}} = \pi/2; \pi; 3\pi/2; 2\pi$$

$$r_{\text{fixed point}} = \sqrt{\frac{96 \pi (p/4 - Q_0)}{l k_3 \beta^2 (3+1)}}$$

→
$$\phi_{\text{fixed point}} = \pi/4; 3\pi/4; 5\pi/4; 7\pi/4$$

$$r_{\text{fixed point}} = \sqrt{\frac{96 \pi (p/4 - Q_0)}{l k_3 \beta^2 (3-1)}}$$

Perturbation XX

fixed point stability for single octupole kick:

Jacobian matrix

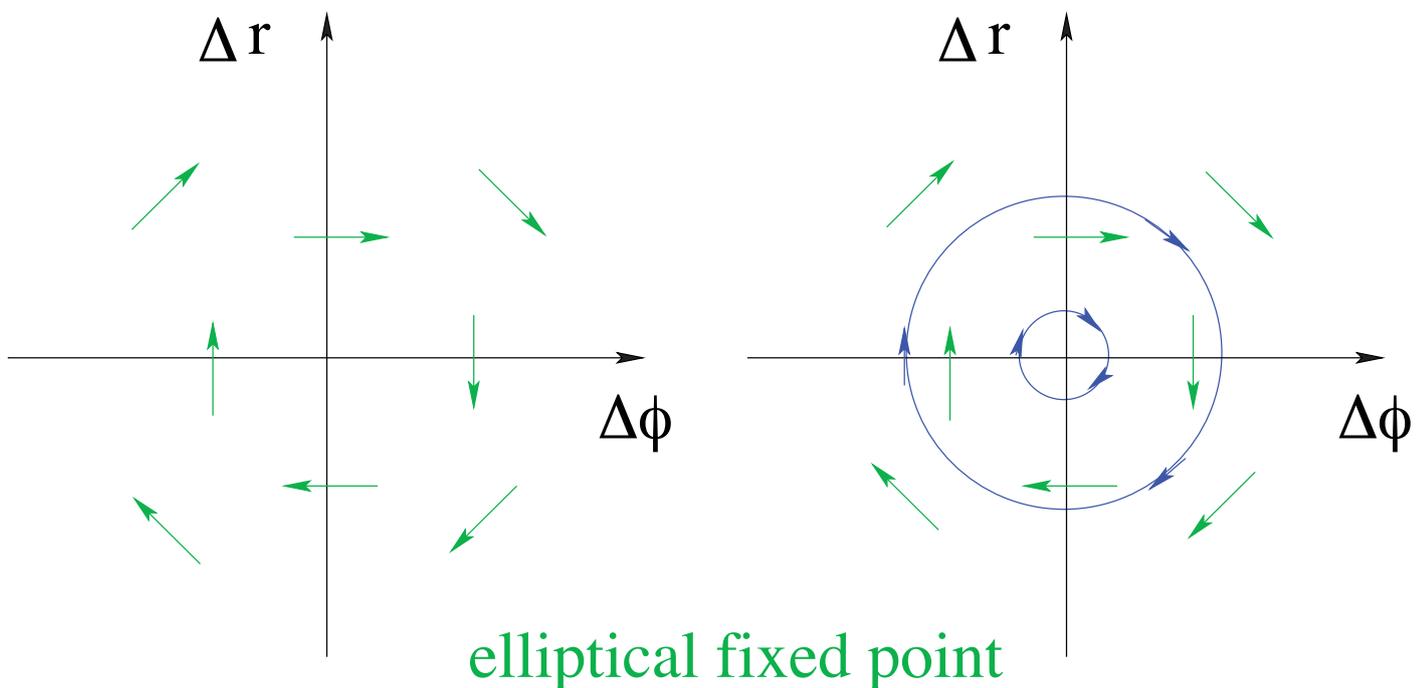
$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \quad \frac{\partial r_{i+1}}{\partial \phi_i} = \pm 4 l \cdot k_3 \cdot \beta^2 \cdot r_{\text{fixed point}}^3 / 48$$

$$\frac{\partial \phi_{i+1}}{\partial r_i} = + l \cdot k_3 \cdot \beta^2 \cdot r (3 \pm 1) / 24; \quad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

→ $\Delta r_{i+1} = \pm 4 l \cdot k_3 \cdot \beta^2 \cdot r_{\text{fixed point}}^3 / 48 \cdot \Delta \phi_i$

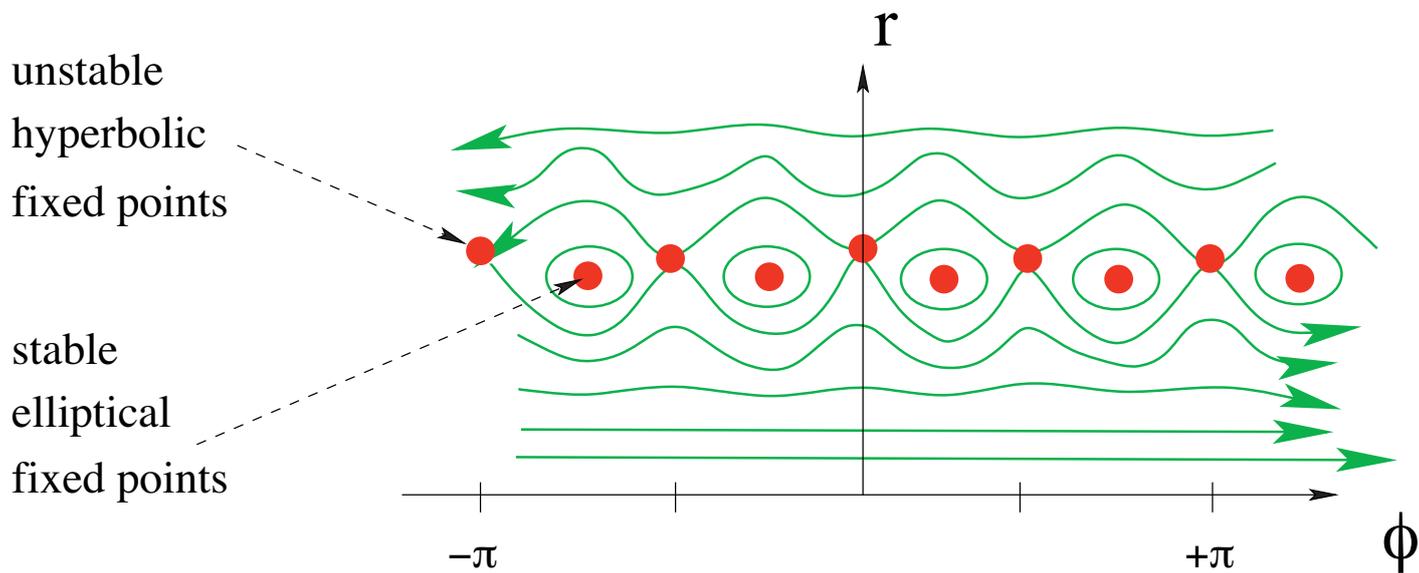
$$\Delta \phi_{i+1} = l \cdot k_3 \cdot \beta^2 (3 \pm 1) / 24 \cdot \Delta r_i$$

Stability for '−' sign and $k_3 > 0$?



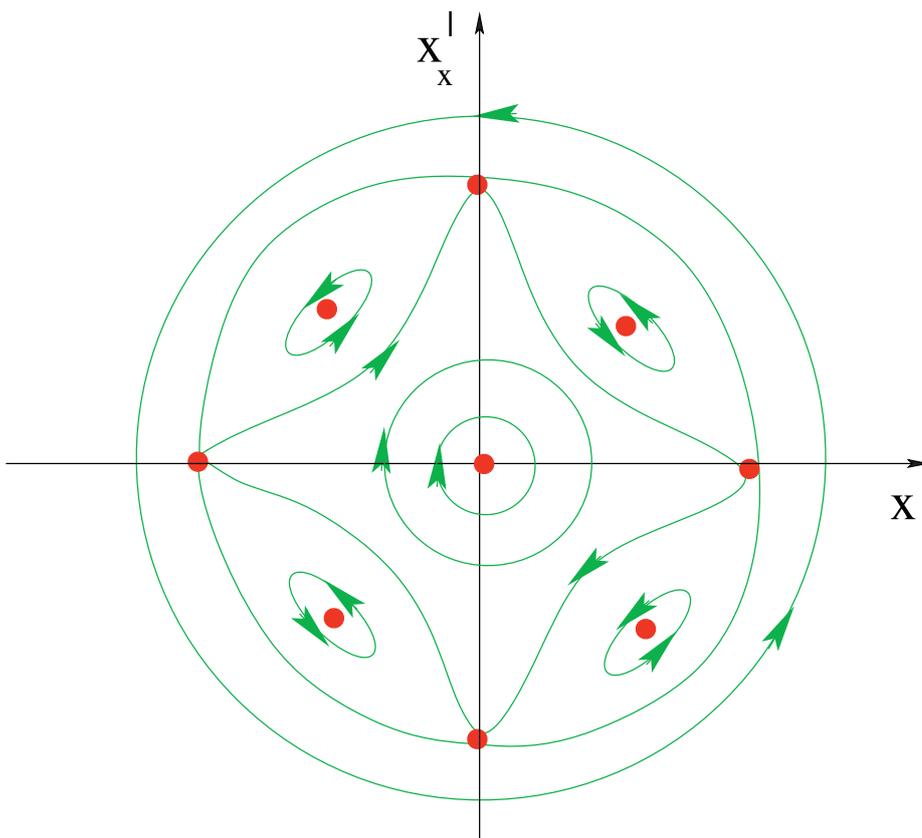
Perturbation XXI

■ Poincare Section for 'r' and ϕ :



island structure

■ Poincare section in normalized coordinates:



Perturbation XXII

Octupole x'

Poincare
Section

from

Simulations:

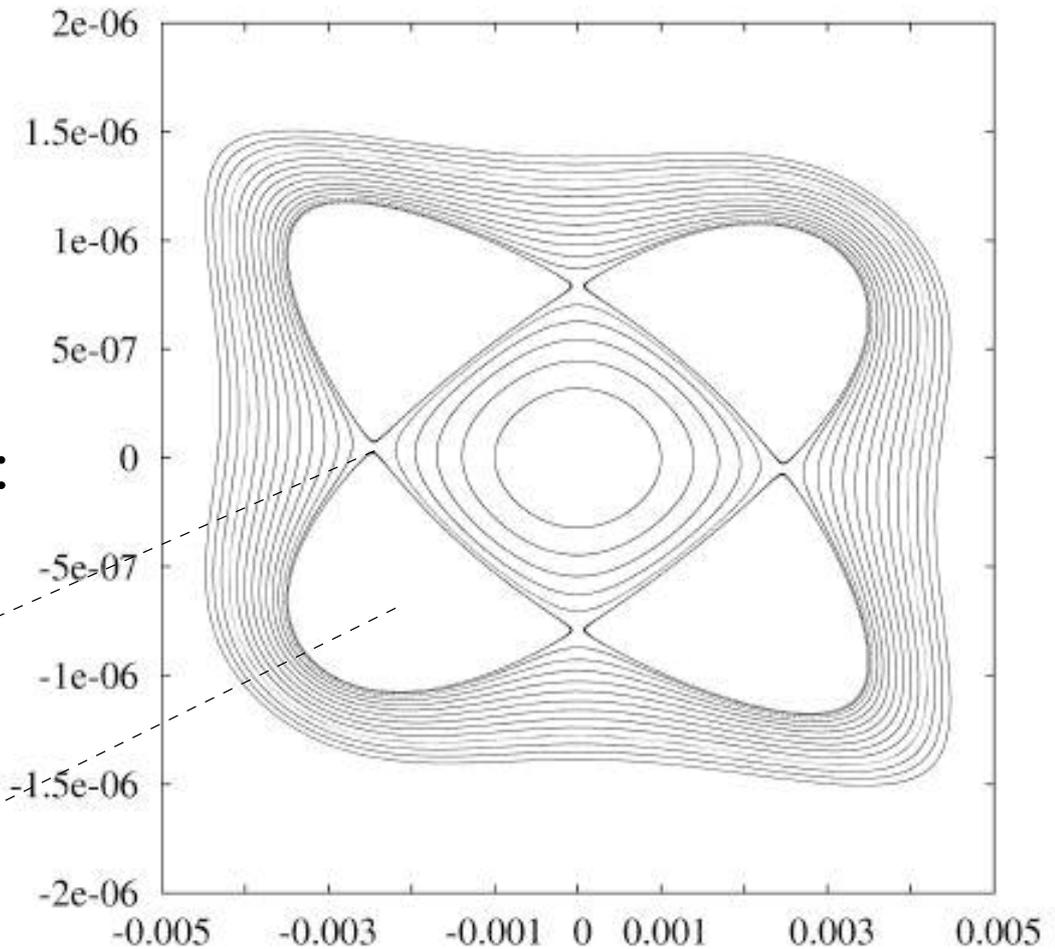
island structure

unstable

hyperbolic fixed point

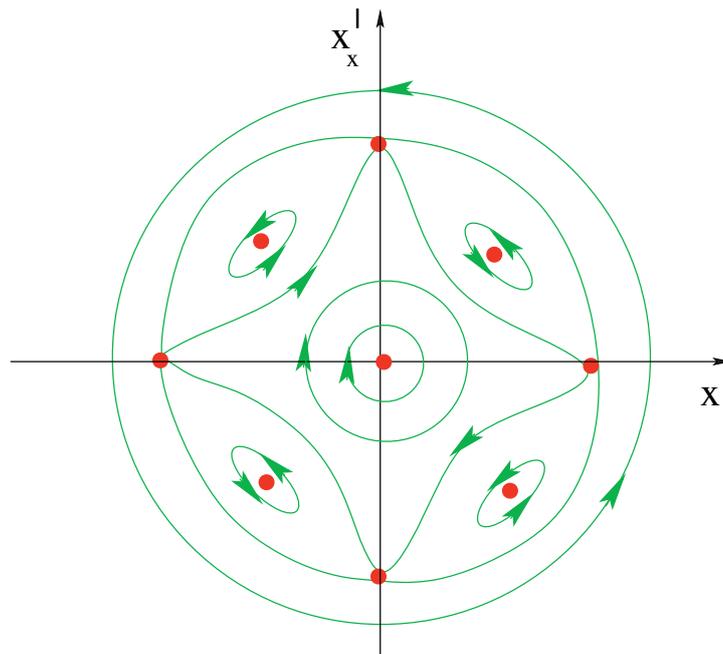
stable

elliptical fixed point



x

Poincare section in normalized coordinates:



generic signature of non-linear resonances:

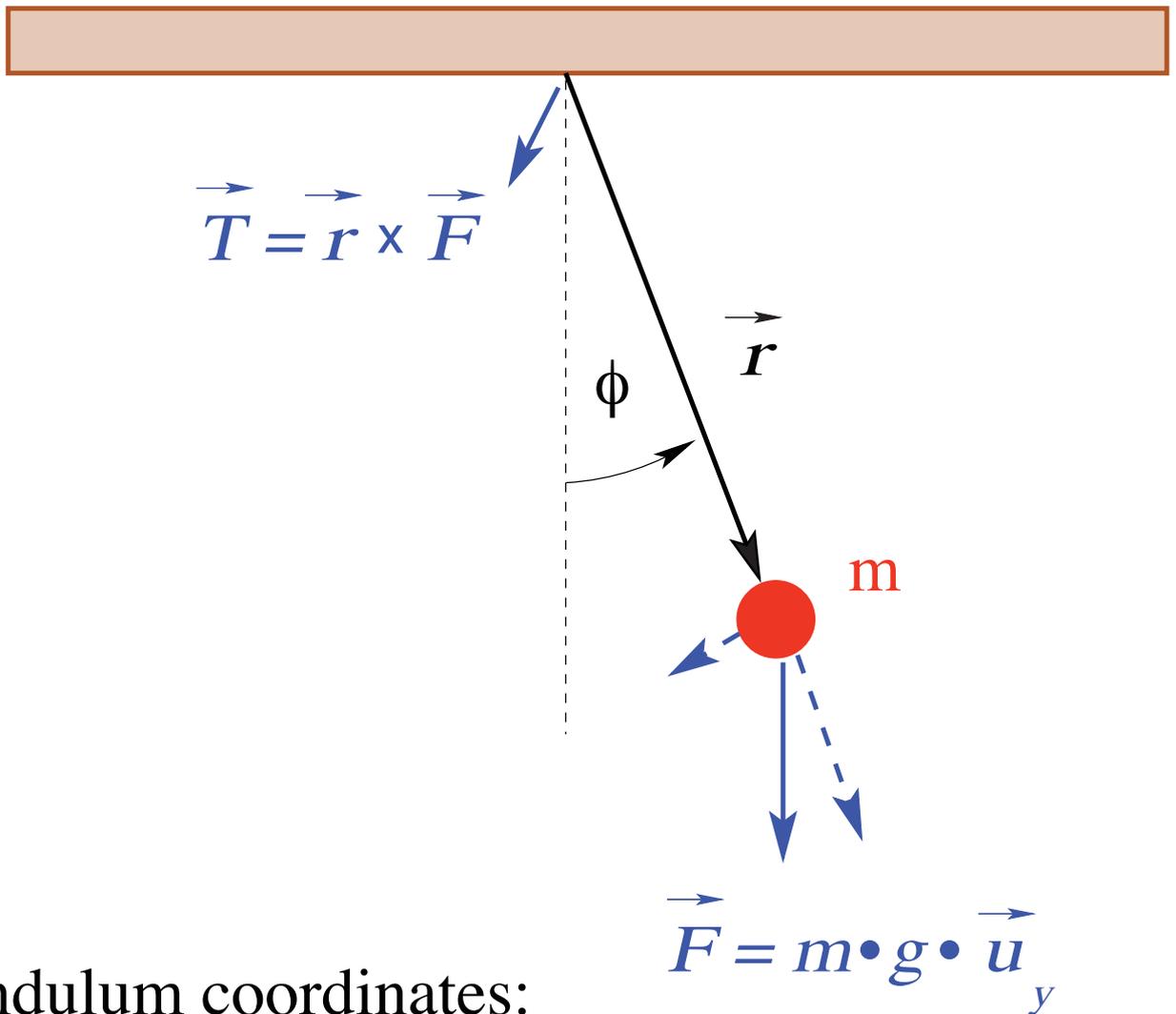
→ chain of resonance islands

Pendulum Dynamics I

generic signature of non-linear resonances:

→ chain of resonance islands

pendulum dynamics:



pendulum coordinates:

angle variable: ϕ

angular momentum: $L = m \cdot r \cdot v$

$$v = \frac{ds}{dt} = r \cdot \frac{d\phi}{dt} \longrightarrow L = m \cdot r^2 \cdot \frac{d\phi}{dt}$$

Pendulum Dynamics II

equations of motion:

$$\frac{d\phi}{dt} = \frac{1}{m \cdot r^2} \cdot L \qquad \frac{dL}{dt} = -r \cdot g \cdot m \cdot \sin(\phi)$$

generic form:

$$\frac{d\phi}{dt} = G \cdot p \qquad \frac{dp}{dt} = -F \cdot \sin(\phi)$$

constant of motion:

$$E_{\text{tot}} = E_{\text{kin}} + U_{\text{pot}}$$

$$\rightarrow E_{\text{kin}} = \frac{1}{2} G \cdot p^2 \qquad U_{\text{pot}} = -F \cdot \cos(\phi)$$

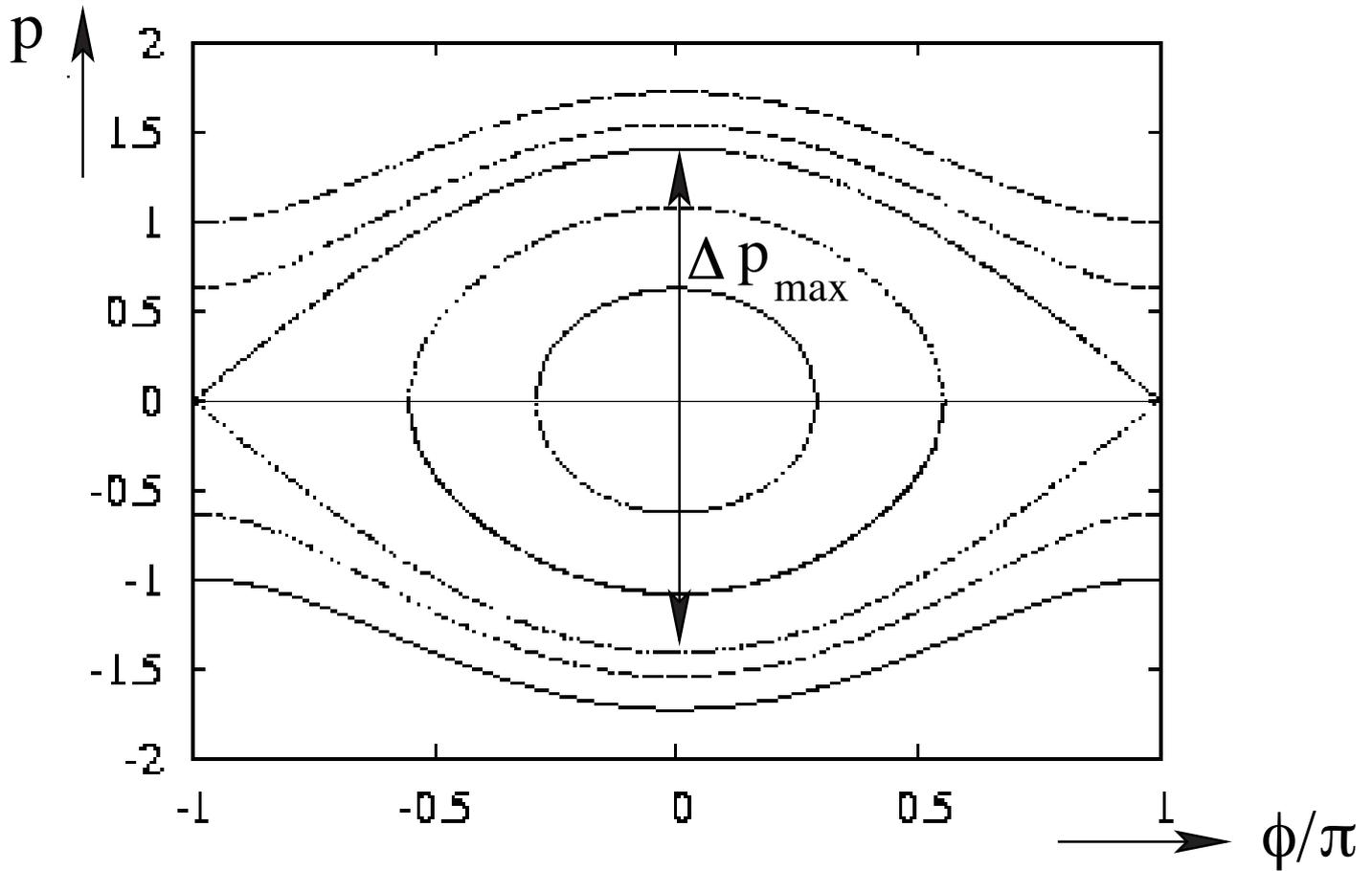
solution:

$$\frac{d\phi}{dt} = G \cdot p \qquad p = \sqrt{[E + F \cdot \cos(\phi)]} \cdot \sqrt{\frac{2}{G}}$$

$$\rightarrow t - t_0 = \sqrt{\frac{1}{2G}} \int \frac{d\phi}{\sqrt{[E + F \cdot \cos(\phi)]}}$$

Pendulum Dynamics III

phase space:



island width: $\Delta p_{\max} = 4 \sqrt{F / G}$

$E_{\text{tot}} = F$ and $\phi = 0$

island oscillation frequency: $\omega_{\text{island}} = \sqrt{F \cdot G}$

pendulum motion:

libration: oscillation around stable fixed point

rotation: continuous increase of phase variable

separatrix: separation between the two types

Cylindrical Coordinates I

linear solution:

$$x = \sqrt{\beta} \cdot \sqrt{R} \cdot \cos(\phi) \quad x' = -\sqrt{R} \cdot \sin(\phi) / \sqrt{\beta}$$

with: $\frac{d\phi}{ds} = \omega = \frac{2\pi Q}{L} = \frac{1}{\beta}$

perturbed Hill's equation:

$$\frac{d^2 x}{ds^2} + \omega^2 \cdot x = \frac{F_x(x,y)}{v \cdot p}$$

→ $x'' = \frac{-1}{n!} \cdot k_n(s) \cdot x^n - \omega^2 \cdot x$

equation of motion in cylindrical coordinates:

$$\frac{d\phi}{ds} = \frac{d\phi}{dx} \cdot x' + \frac{d\phi}{dx'} \cdot x''$$

$$\frac{dR}{ds} = \frac{dR}{dx} \cdot x' + \frac{dR}{dx'} \cdot x''$$

Cylindrical Coordinates II

radial coordinate:

$$R = \frac{x^2}{\beta} + x'^2 \cdot \beta$$



~~$$\frac{dR}{ds} = \frac{2xx'}{\beta} - 2\beta\omega^2 xx' + 2x'\beta \cdot \frac{F_x(s,r,\phi)}{v \cdot p}$$~~

$$\frac{dR}{ds} = \frac{-2}{n!} \cdot k_n(s) \cdot (R \cdot \beta)^{(n+1)/2} \cdot \sin(\phi) \cdot \cos^n(\phi)$$

angular coordinate:

$$\phi = \text{atan}\left(\frac{-x' \cdot \beta}{x}\right)$$

with:

$$\frac{d}{ds} \text{atan}(f[s]) = \frac{1}{f^2(s) + 1} \cdot \frac{df}{ds}$$

$$\left(\frac{1}{\beta} = \omega\right) \rightarrow \frac{d\phi}{ds} = \omega - \frac{x}{R} \cdot \frac{F_x(s,r,\phi)}{v \cdot p}$$

$$\frac{d\phi}{ds} = \omega + \frac{1}{n!} \cdot k_n(s) \cdot R^{(n-1)/2} \cdot \beta^{(n+1)/2} \cdot \cos^{n+1}(\phi)$$

Examples for Equation of Motion I

■ quadrupole: $n = 1$

$$\frac{dR}{ds} = -k_1(s) \cdot R \cdot \beta \cdot \sin(2\phi)$$

$$\frac{d\phi}{ds} = \omega + k_1(s) \cdot \beta \cdot \left(1 + \cos(2\phi)\right) / 2$$

→ similar expressions as with the map approach
but we can now treat distributed perturbations!

■ sextupole: $n = 2$

$$\frac{dR}{ds} = \frac{-1}{4} \cdot k_2(s) \cdot \left(R \cdot \beta\right)^{3/2} \cdot \left(\sin(\phi) + \sin(3\phi)\right)$$

$$\frac{d\phi}{ds} = \omega + \frac{1}{8} \cdot k_2(s) \cdot R^{1/2} \cdot \beta^{3/2} \cdot \left(3\cos(\phi) + \cos(3\phi)\right)$$

→ similar expressions as with the map approach

Examples for Equation of Motion II

■ octupole: $n = 3$

$$\frac{dR}{ds} = \frac{-1}{24} \cdot k_3(s) \cdot R^2 \cdot \beta^2 \cdot \left(2 \sin(\phi) + \sin(4\phi) \right)$$

$$\frac{d\phi}{ds} = \omega + \frac{1}{48} \cdot k_3(s) \cdot R \cdot \beta^2 \cdot \left(3 + 4\cos(2\phi) + \cos(4\phi) \right)$$

■ one single kick at one location:

$$\rightarrow \frac{F(s)}{v \cdot p} = 1 k_n(s) \cdot \delta_L(s - s_0)$$

$$\text{with: } \delta = \begin{cases} 1 & \text{for } s = s_0 + n \cdot L \\ 0 & \text{else} \end{cases}$$

→ Fourier series of δ -function:

$$\frac{F(s)}{v \cdot p} = 1 k_n(s) \cdot \frac{1}{L} \cdot \sum_{n=-\infty}^{+\infty} \cos(n \cdot 2\pi \cdot s/L)$$

Examples for Equation of Motion III

single octupole magnet at s_0 : $n = 3$

$$\frac{dR}{ds} = \frac{-1}{24 \cdot L} \cdot lk_3(s) \cdot R^2 \cdot \beta^2 \cdot \sum_{n=0}^{+\infty} \left(2 \sin(\phi + n \cdot 2\pi \cdot s/L) + \sin(4\phi + n \cdot 2\pi \cdot s/L) \right)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \frac{1}{48 \cdot L} \cdot lk_3(s) \cdot R \cdot \beta^2 \cdot \sum_{n=0}^{+\infty} \left(3 + 2 \cos(\phi + n \cdot 2\pi \cdot s/L) + \cos(4\phi + n \cdot 2\pi \cdot s/L) \right)$$

resonance: $\phi = \frac{2\pi \bar{Q}}{L} \cdot s + \phi_0$

with $\bar{Q} = N + 1/n$

→ all but one term change rapidly with s !

→ method of averaging!

Examples for Equation of Motion IV

1/4 resonance :

$$p = 4$$

$$\frac{dR}{ds} = \frac{-1}{24 \cdot L} \cdot l k_3 \cdot R^2 \beta^2 \cdot \sin(4\phi_0)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \frac{1}{48 \cdot L} \cdot l k_3 \cdot R \beta^2 \cdot (3 + \cos(4\phi_0))$$

fixed point conditions: $Q_0 \lesssim p/4; k_3 > 0$

$$\Delta R / \text{turn} = 0 \quad \text{and} \quad \Delta\phi / \text{turn} = 2\pi p / 4$$

$$\rightarrow \phi_{\text{fixed point}} = \pi/2; \pi; 3\pi/2; 2\pi$$

$$R_{\text{fixed point}} = \frac{96 \pi (p/4 - Q_0)}{l k_3 \beta^2 (3+1)}$$

$$\rightarrow \phi_{\text{fixed point}} = \pi/4; 3\pi/4; 5\pi/4; 7\pi/4$$

$$R_{\text{fixed point}} = \frac{96 \pi (p/4 - Q_0)}{l k_3 \beta^2 (3-1)}$$

Example Octupole



$$lk_3 = 4 \text{ m}^{-3}$$

$$Q = 0.2495$$

$$L = 4.7 \text{ km:}$$



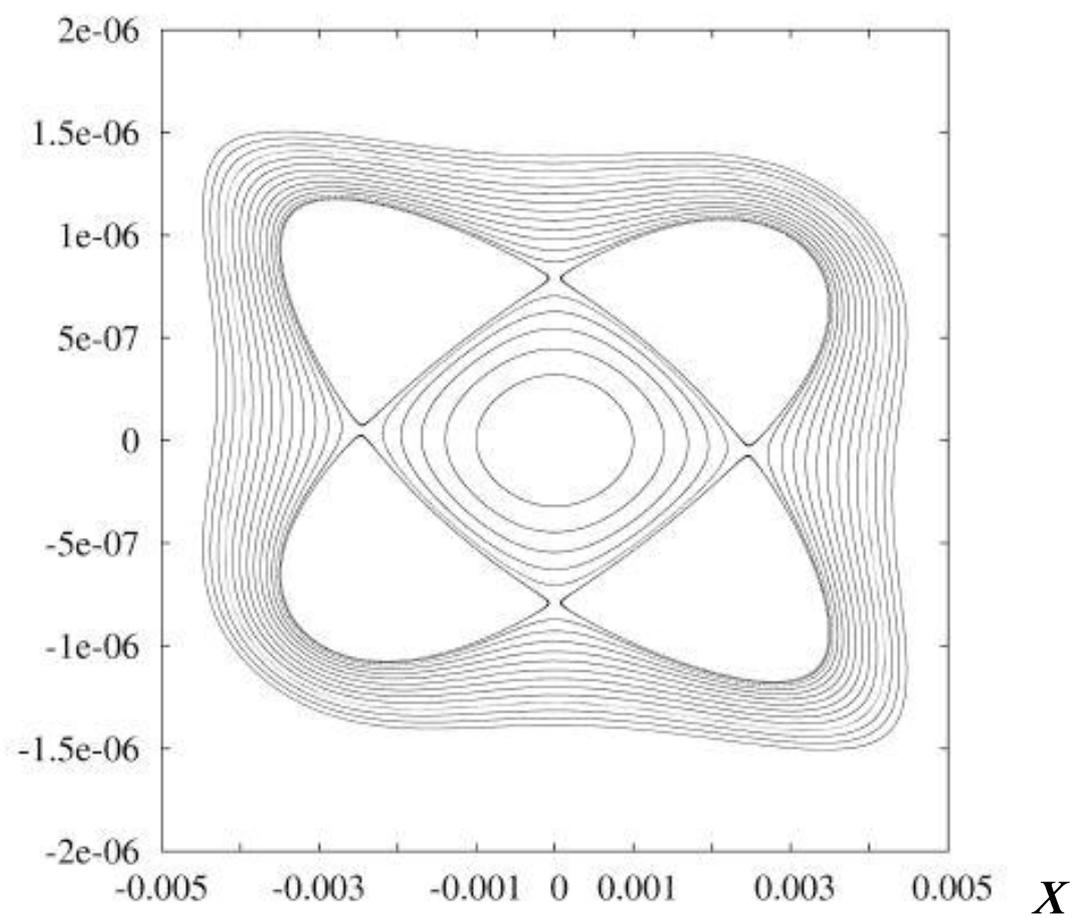
$$\beta = 3000$$



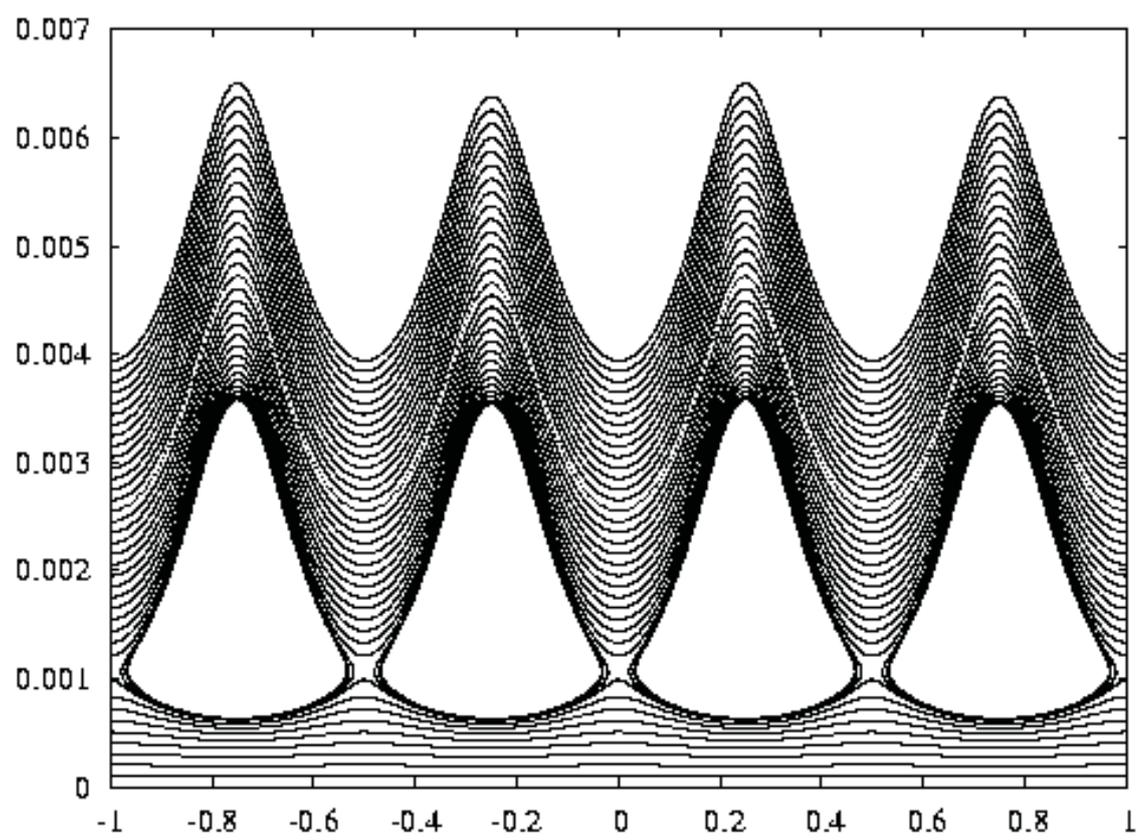
$$R = 2 \cdot 10^{-9}$$

$$R = 1 \cdot 10^{-9}$$

x'



$$R \cdot 10^6$$



ϕ / π

Examples for Equation of Motion V

expand motion around stable fixed point:

$$\phi = \frac{2\pi Q}{L} s + \phi_{\text{fix}} + \Delta\phi$$

$$R = R_{\text{fix}} + \Delta R \quad \text{and keep only first order in } \Delta R$$

$$\frac{d\Delta R}{ds} = \frac{-1}{24 \cdot L} \cdot I k_3 \cdot R_{\text{fix}}^2 \cdot \beta^2 \cdot \sin(4\Delta\phi)$$

$$\begin{aligned} \frac{d\phi}{ds} &= \frac{2\pi Q_0}{L} + \frac{1}{48 \cdot L} \cdot I k_3 \cdot R_{\text{fix}} \cdot \beta^2 \cdot \left(3 - \cancel{\cos(4\Delta\phi)} \right) \\ &\quad + \frac{1}{48 \cdot L} \cdot I k_3 \cdot \Delta R \cdot \beta^2 \cdot \left(3 - \cancel{\cos(4\Delta\phi)} \right) \end{aligned}$$

change to new angular variable:

$$\varphi = 4\phi - 8\pi Q \cdot s / L \quad r = 4 \cdot \Delta R$$

with $Q = Q_0 + \frac{1}{48 \cdot \pi} \cdot I k_3 \cdot R_{\text{fix}} \cdot \beta^2$

Examples for Equation of Motion VI

pendulum approximation:

$$\frac{d r}{d s} = -F \cdot \sin(\varphi)$$

with

$$F = \frac{4}{24 \cdot L} \cdot l k_3 \cdot \beta^2 \cdot R_{\text{fix}}^2$$

$$\frac{d \varphi}{d s} = G \cdot r$$

and

$$G = \frac{1}{24 \cdot L} \cdot l k_3 \cdot \beta^2$$

resonance width:

$$\Delta r_{\text{max}} = 4 \sqrt{F / G} = 8 \cdot \Delta R_{\text{fix}}$$

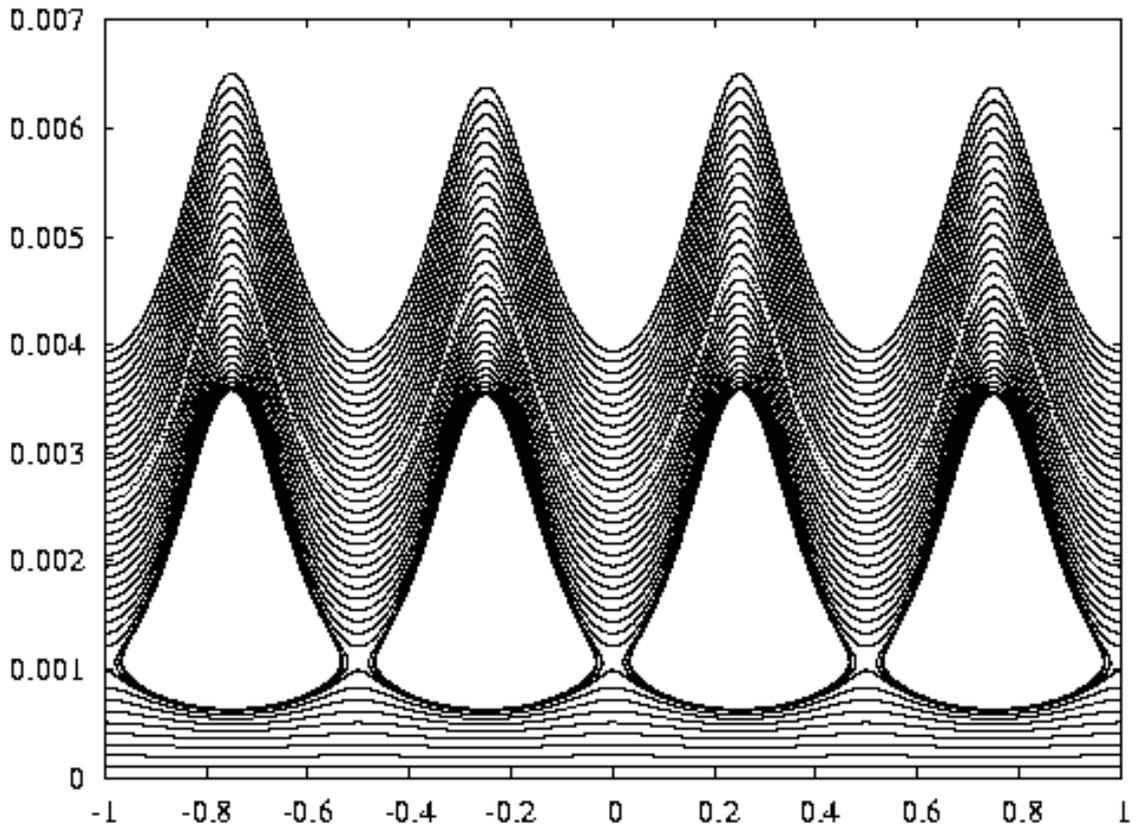
$$\longrightarrow \Delta R_{\text{max}} = 2 \cdot \Delta R_{\text{fix}}$$

resonance width equals twice the stable fixed point

resonance width increases with decreasing k_3 !

Example Octupole

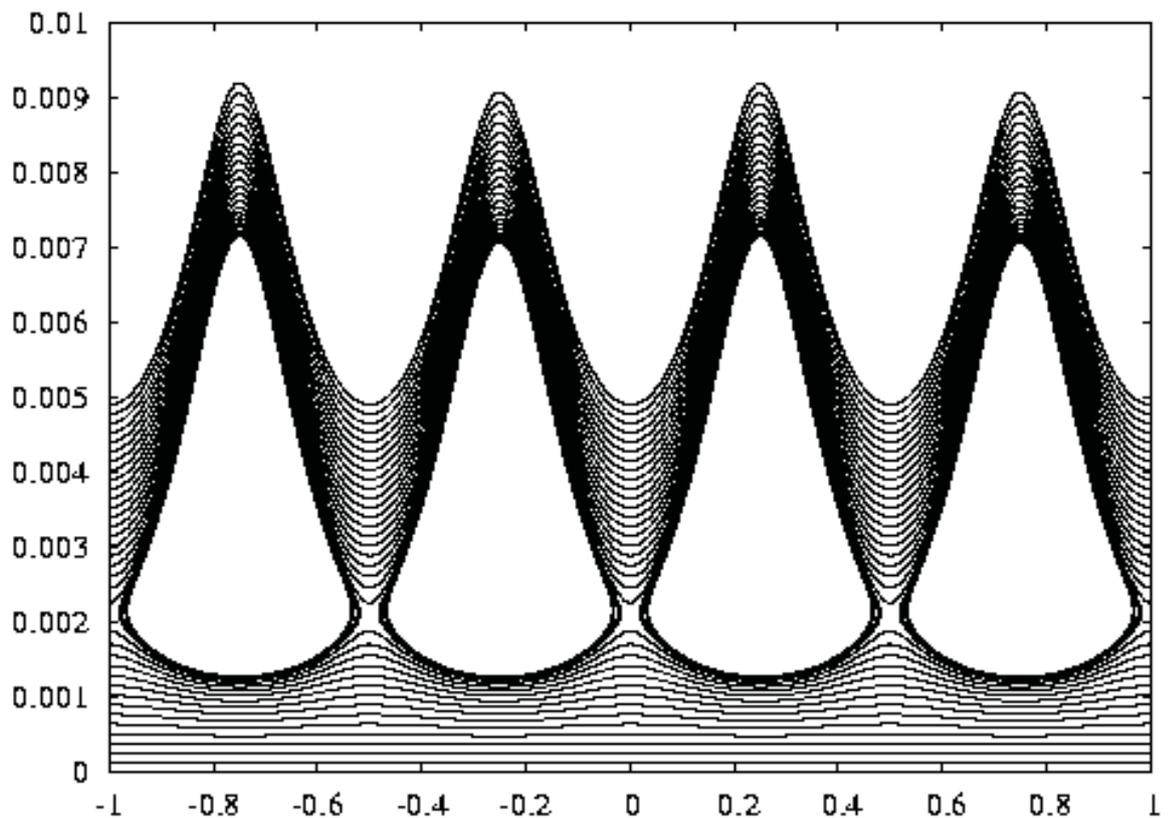
$R \cdot 10^6$



$lk_3 = 4 \text{ m}^{-3}$

ϕ / π

$R \cdot 10^6$



$lk_3 = 2 \text{ m}^{-3}$

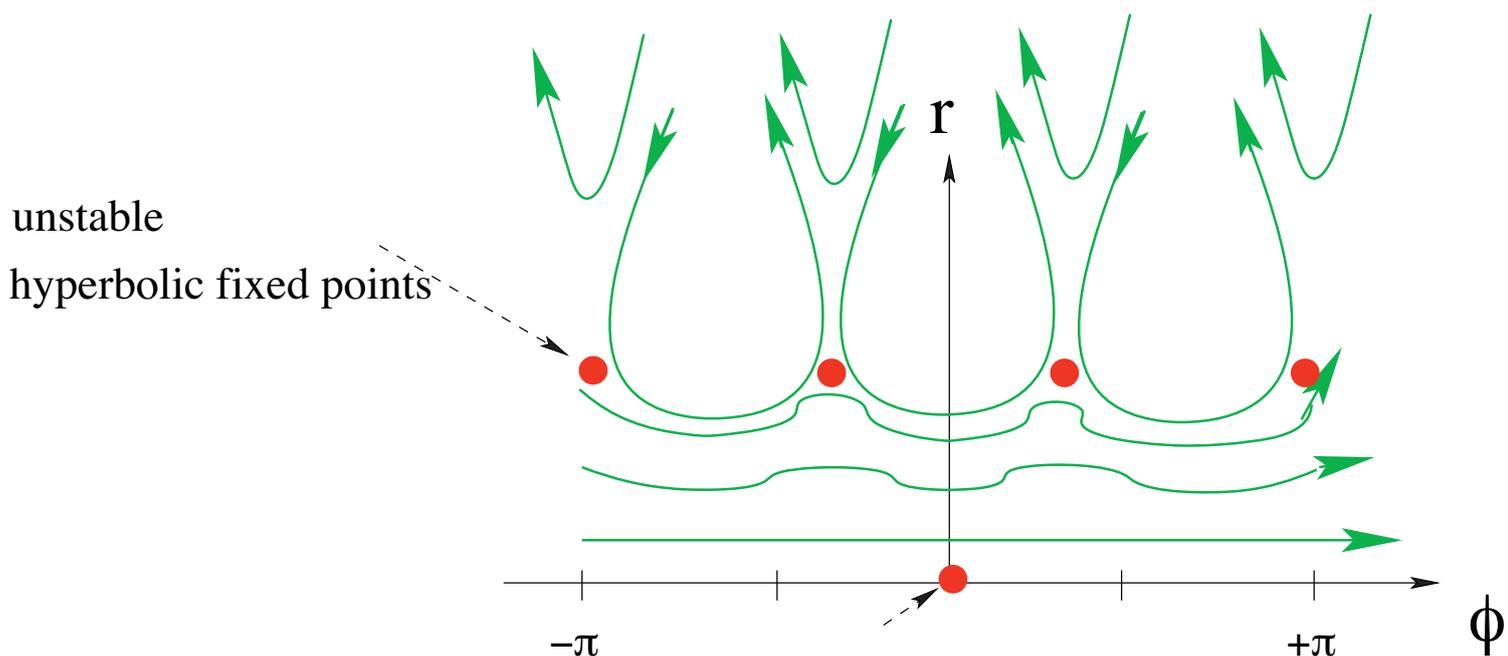
ϕ / π

Example Sextupole

■ why did we not find islands for a sextupole?

➔ the pendulum approximation requires an amplitude dependent tune!

➔
$$\frac{d\phi}{ds} = G \cdot r$$

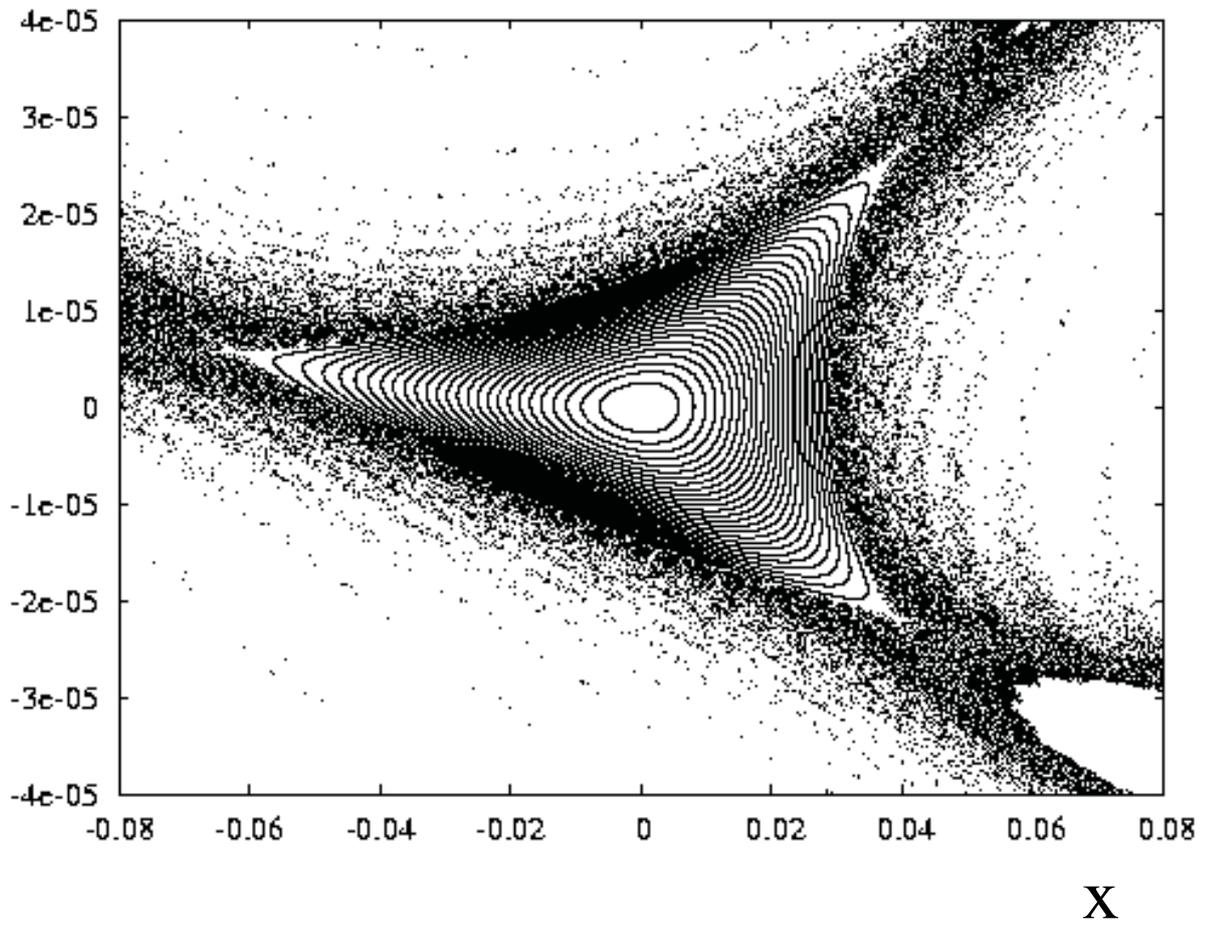


■ the sextupole perturbation has no amplitude dependent tune (to first order)

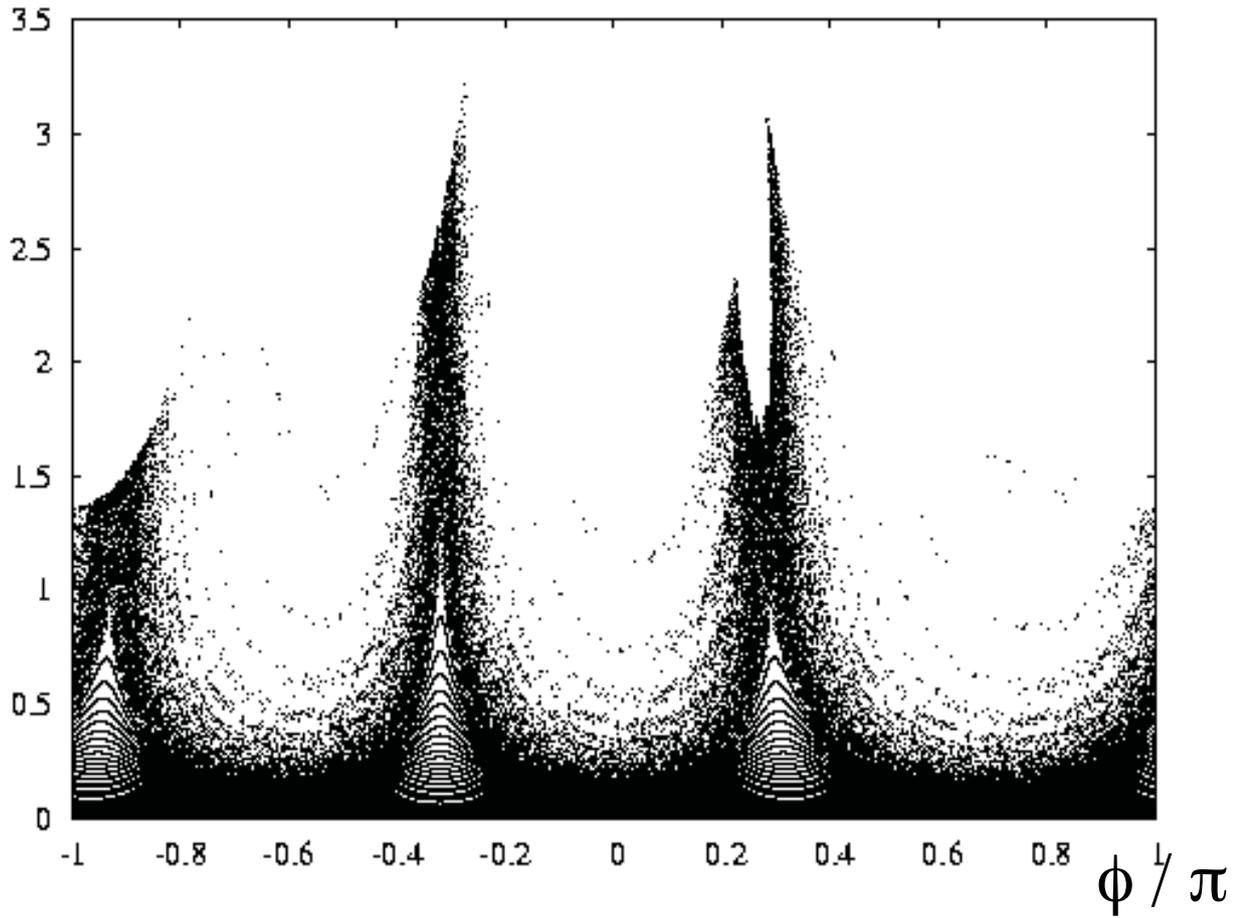
➔ stabilization by an octupole term?

Example Sextupole

■ X'
sextupole
only

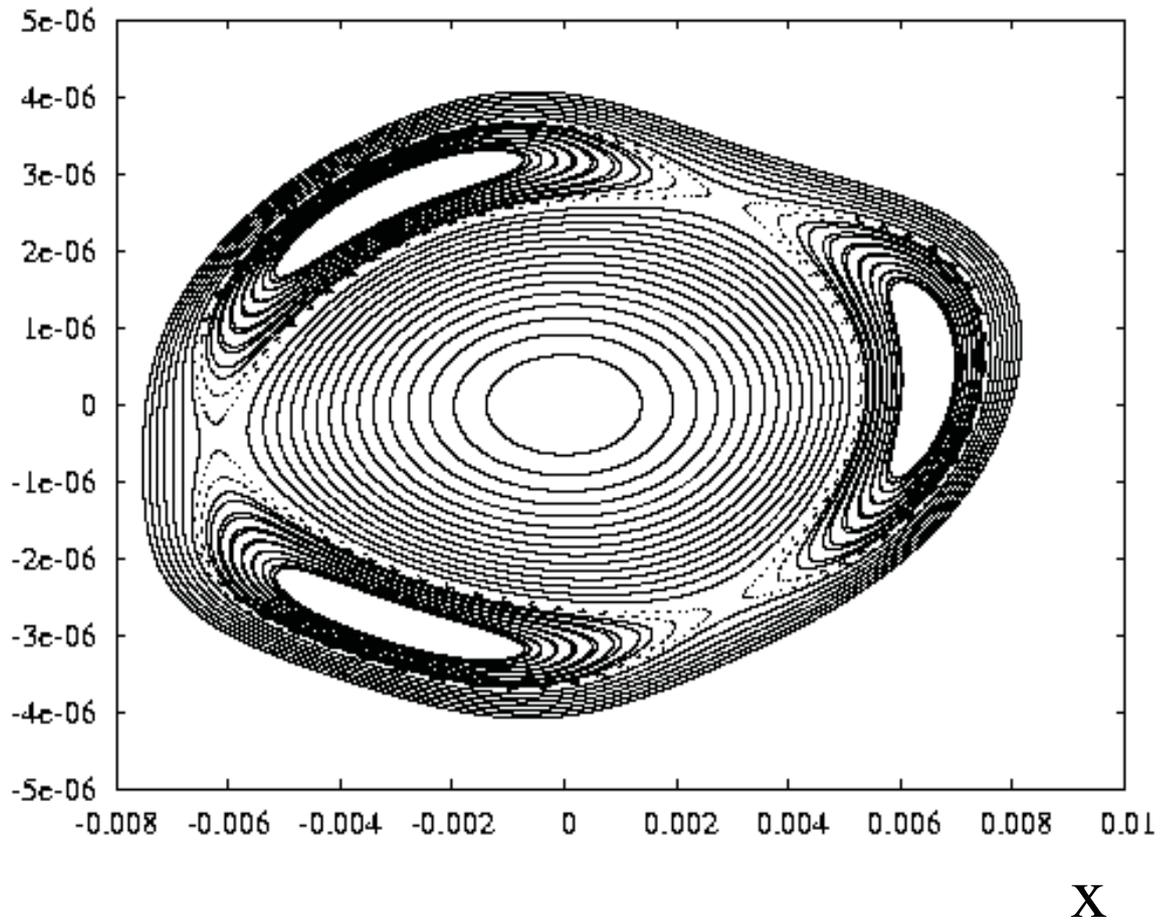


■ $R \cdot 10^6$

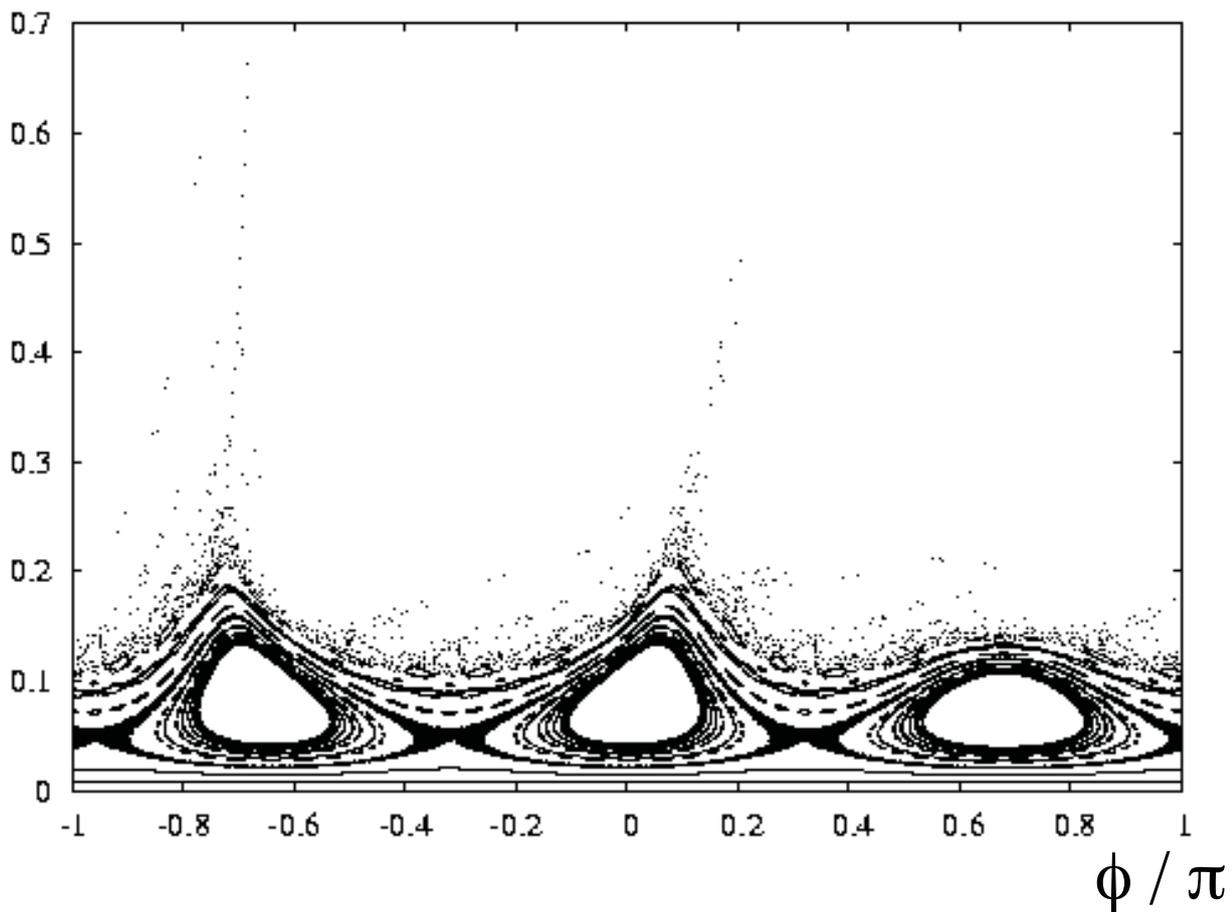


Example Sextupole + Octupole

 X'
sextupole
plus
octupole



 $R \cdot 10^6$



Higher Order

so far we assumed on the right-hand side:

$$\phi = 2\pi Q_0 \cdot s/L + \phi_{\text{fix}} + \Delta\phi$$

$$R = R_{\text{fix}} + \Delta R$$

and kept only first order terms in ΔR

higher order perturbation treatment:

$$R(s) = R_0(s) + \varepsilon R_1(s) + \varepsilon^2 R_2(s) + O(\varepsilon^3)$$

$$\phi(s) = \phi_0(s) + \varepsilon \phi_1(s) + \varepsilon^2 \phi_2(s) + O(\varepsilon^3)$$

$$\text{with: } \varepsilon = \left(\beta \cdot R_{\text{fix}} \right)^{(n+1)/2} \cdot l k_n / L$$

match powers of ε :

match powers of ε

solve lowest order without perturbation

substitute solution in next higher order equations

solve next order etc

Higher Order II

expand equation of motion into a Taylor series around zero order solution

$$\frac{dr}{ds} = F(r, \phi)$$

$$\frac{d\phi}{ds} = G(r, \phi)$$

→ single sextupole kick:

$$F = f(R) \cdot [\sin(3\phi) + 3\sin(\phi)]$$

$$G = g(R) \cdot [\cos(3\phi) + 3\cos(\phi)] + \frac{2\pi Q}{L}$$

$$\rightarrow \frac{dR}{ds} = \varepsilon \cdot f + \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial F}{\partial \phi} \cdot \phi_1 \right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \varepsilon \cdot g + \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial G}{\partial \phi} \cdot \phi_1 \right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

Higher Order III

match powers of ε and solve equation of motion in ascending order of ε^n :

zero order:
$$\phi_0(s) = \frac{2\pi Q}{L} \cdot s + \phi_0$$

$$R_0(s) = R_0 \quad (Q = p + v)$$

→ substitute into equation of motion and solve for $\phi_1(s)$ and $r_1(s)$

first order:

$$\phi_1(s) \propto \left[\sin\left(\frac{6\pi Q}{L} \cdot s + 3\phi_0\right)/3 + 3 \cdot \sin\left(\frac{2\pi Q}{L} \cdot s + \phi_0\right) \right]$$

$$R_1(s) \propto \left[\cos\left(\frac{6\pi Q}{L} \cdot s + 3\phi_0\right)/3 + 3 \cdot \cos\left(\frac{3\pi Q}{L} \cdot s + \phi_0\right) \right]$$

Perturbation IV

second order:

→ substitute $\phi_1(s)$ and $r_1(s)$ into equation of motion and order powers of ε^2

you get terms of the form: $\frac{dr_2}{ds} = \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial f}{\partial \phi} \cdot \phi_1 \right]$

→ $\frac{d\phi}{ds} = \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial g}{\partial \phi} \cdot \phi_1 \right]$

$\sin(3\phi) \cdot \cos(3\phi); \sin(3\phi) \cdot \cos(\phi); \sin(\phi) \cdot \cos(\phi)$

$\cos(3\phi) \cdot \cos(3\phi); \cos(3\phi) \cdot \cos(\phi); \cos(\phi) \cdot \cos(\phi)$

→ $\frac{d\phi}{ds} \propto \cos(6\phi); \cos(4\phi); \cos(2\phi); 1$

→ $\frac{dr}{ds} \propto \sin(6\phi); \sin(4\phi); \sin(2\phi)$

higher order resonances: ε^n

a single perturbation generates ALL resonances

driving term strength and resonance width

decrease with increasing order!

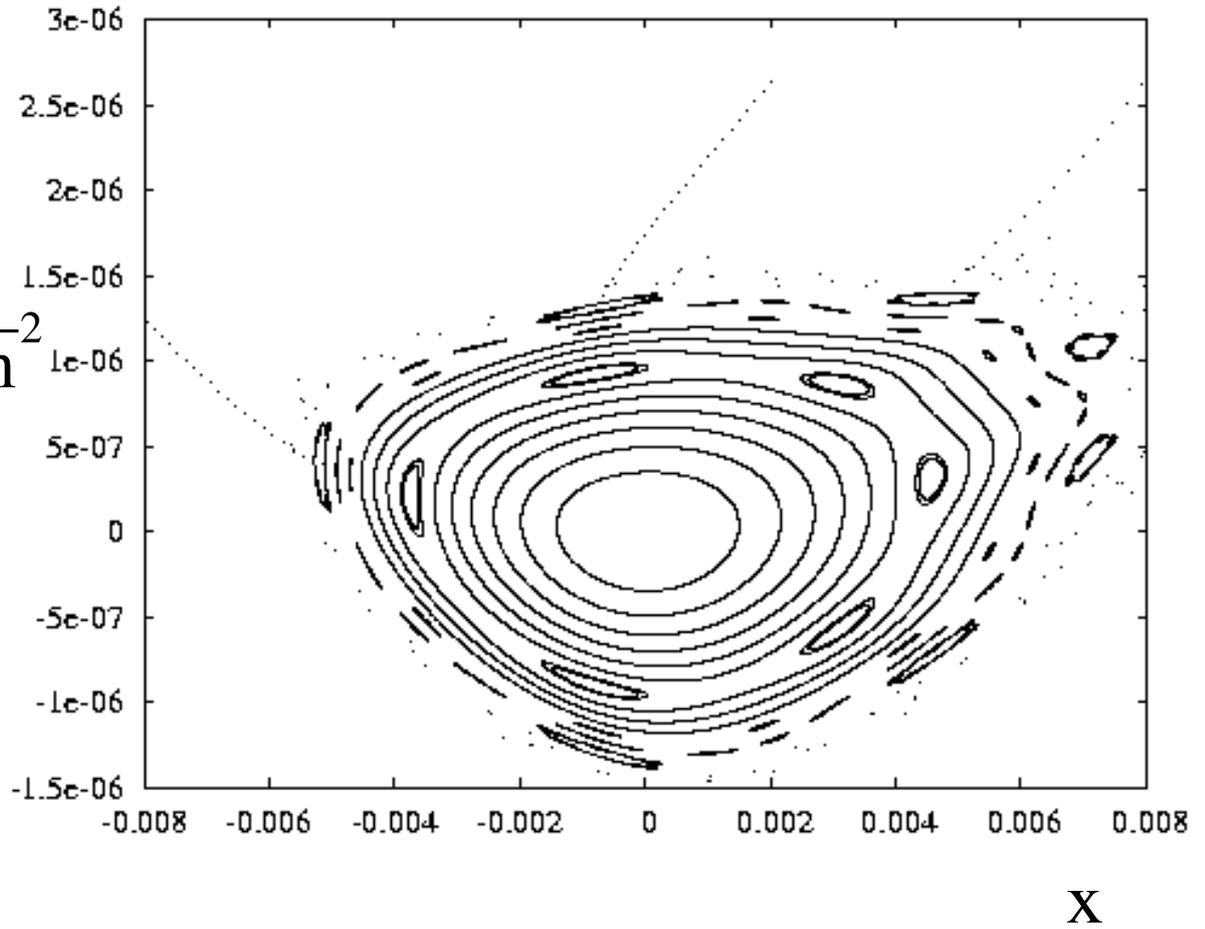
Perturbation V

x'

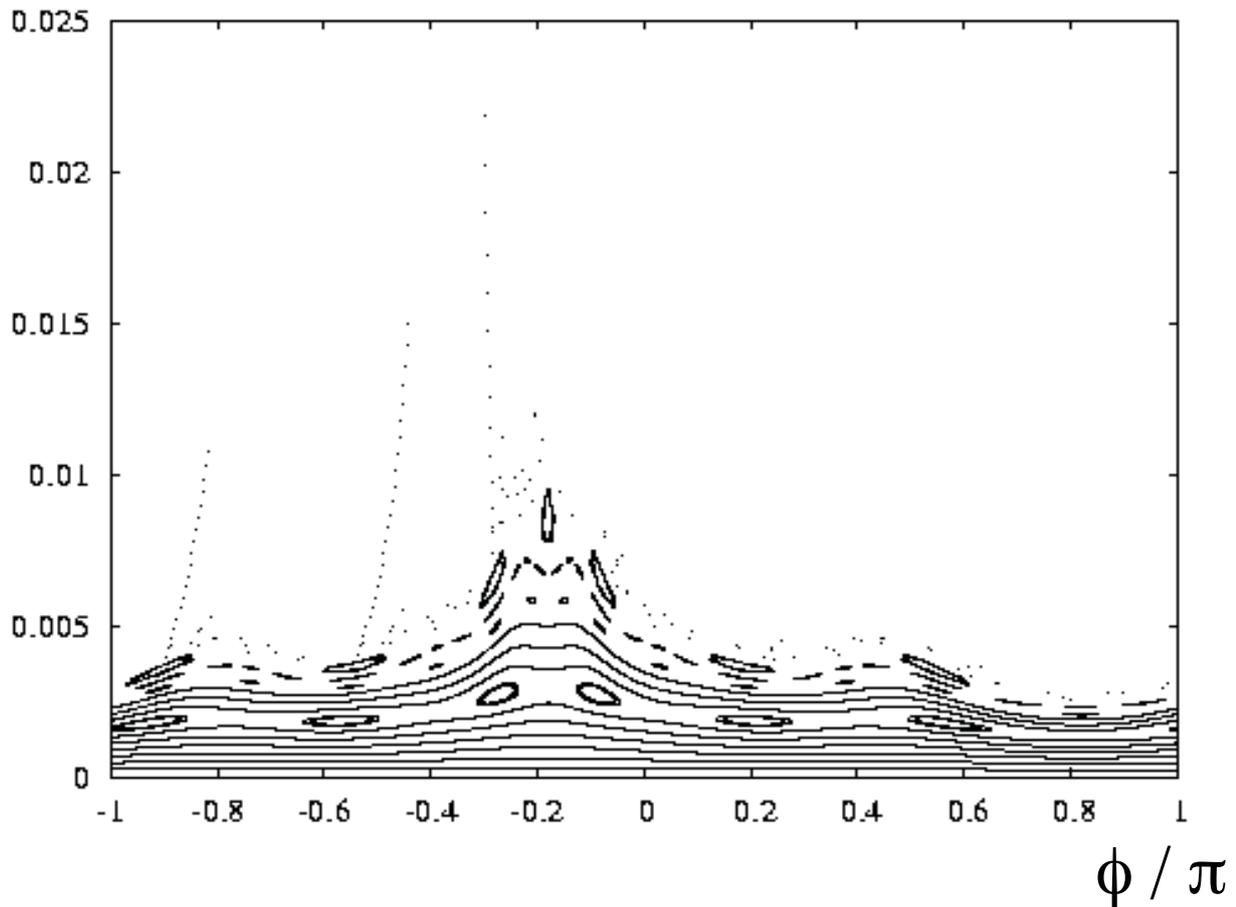
■ sextupole
only

$$k_2 = -0.06 \text{m}^{-2}$$

$$Q = 0.18$$



■ $R \cdot 10^6$



Integrable Systems

trajectories in phase space do not intersect

deterministic system

integrable systems:

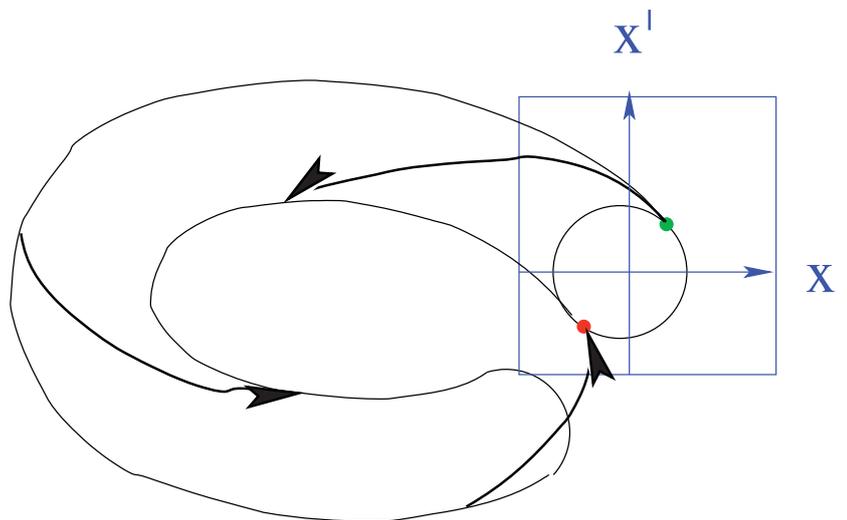
all trajectories lie on invariant surfaces

n degrees of freedom

→ n dimensional surfaces

two degrees of freedom:

x, s → motion lies on a torus



Poincare section for two degrees of freedom:

→ motion lies on closed curves

→ indication of integrability

Non-Integrable Systems

■ 'chaos' and non-integrability:

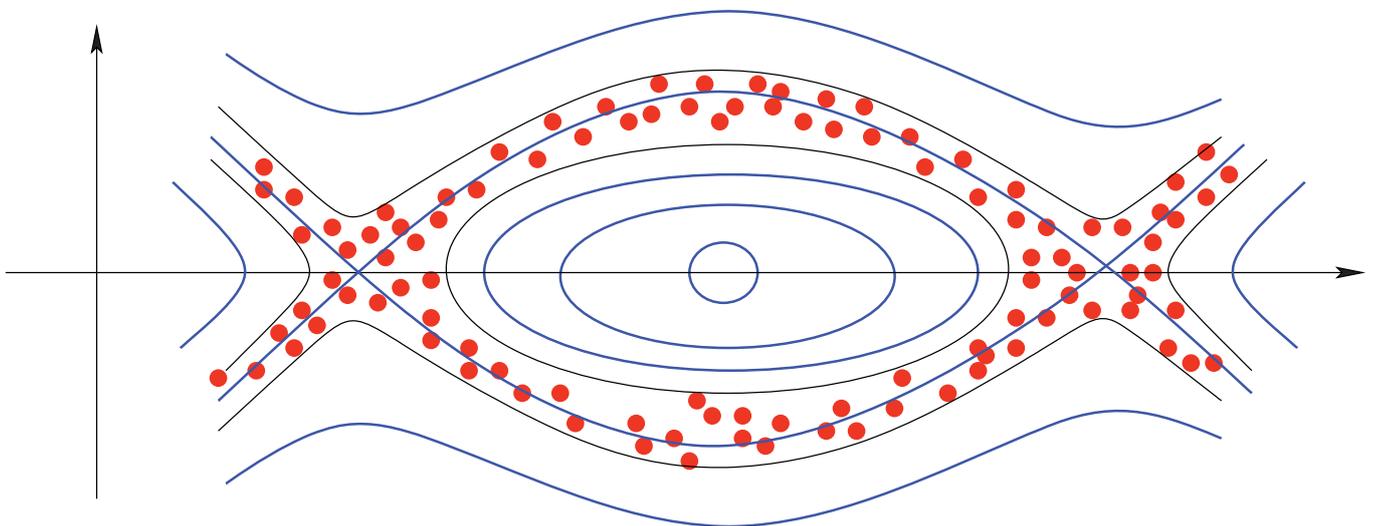
so far we removed all but one resonance
(method of averaging)

→ dynamics is integrable and therefore
predictable

re-introduction of the other resonances 'perturbs'
the separatrix motion

→ motion can 'change' from libration to rotation

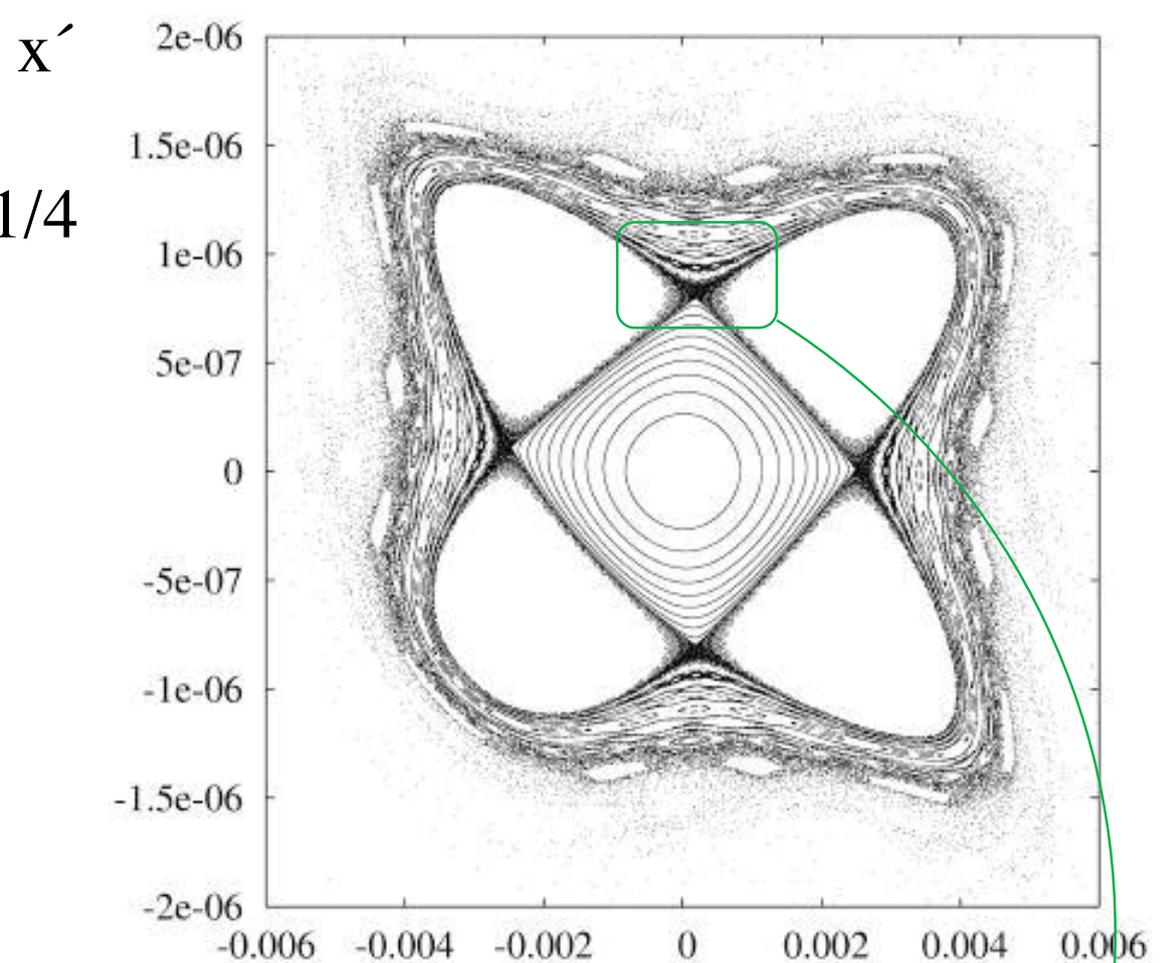
→ generation of a layer of 'chaotic motion'



no hope for exact deterministic solution in this area!

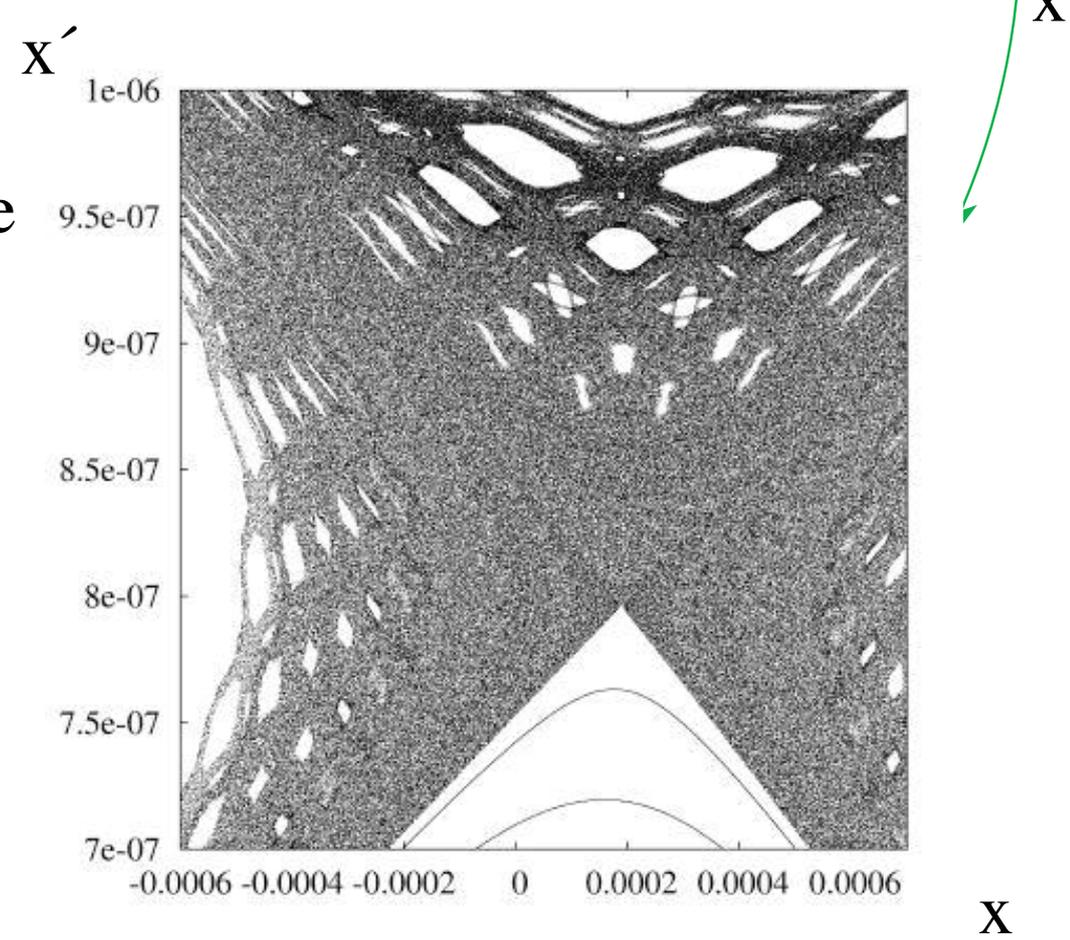
Sextupole + Octupole

■ motion near 1/4 resonance:



■ pendulum island structure appears on all scales!

→ renormalization theory



Non-Integrable Systems

■ slow particle loss:

particles can stream along the 'stochastic layer'
for 1 degree of freedom (plus 's' dependence)
the particle amplitude is bound by neighboring
integrable lines

not true for more than one degree of freedom

■ global 'chaos' and fast particle losses:

if more than one resonance are present their
resonance islands can overlap

→ the particle motion can jump from one
resonance to the other

→ 'global chaos'

→ fast particle losses and dynamic aperture

Summary

Non-linear Perturbation:

-  *amplitude growth*
-  *detuning with amplitude*
-  *coupling*



Complex dynamics:

3 degrees of freedom

+ 1 invariant of the motion

+ non-linear dynamics



no global analytical solution!



*analytical analysis relies on
perturbation theory*