Landau Damping in Particle Accelerators (DRAFT)

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Abstract

Coherent instabilities in longitudinal and transverse dynamics of any accelerator, linear or circular, put severe constrains to the accelerator's operations and performances. Fortunately the current limitations can be mitigated by a natural stabilizing mechanism, called Landau damping.

The concept of Landau damping is introduced here starting from the system for which it was firstly described: the plasma oscillations. The dispersion relation of electromagnetic waves in plasma that is derived is very similar to what it is found in particle accelerators. After a description of the physical mechanism behind the effect by using a simple mechanical model consisting of linear harmonic oscillators, we discuss the Landau damping in circular accelerators by considering first the longitudinal beam dynamics, either in the coasting and in the bunched beam case, and then the transverse one.

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Glossary

Electron density: $\rho(x, v_x, t) = ef(x, v_x, t)$
Fourier Transform (space): $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$
Fourier Inverse Transform : $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$
Fourier Transform (time): $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$
Fourier Inverse Transform : $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega$
Laplace Transform: $\mathcal{F}(p) = \int_0^{\infty} f(t)e^{-pt}dt$
Laplace Inverse Transform : $f(t) = \frac{1}{2\pi i}\int_C \mathcal{F}(p)e^{pt}dp$

1 Introduction

Landau damping is a physical effect named after his discoverer, the Soviet physicist Lev Davidovich Landau, who studied in 1946 the wave propagation in a warm plasma. According to Landau, an initial perturbation of longitudinal charge density waves in plasma is prevented from developing because of a natural stabilizing mechanism.

In the original paper[1], the theory of Landau damping was derived from a pure mathematical approach, without any physical explanation about the underlying mechanism.

A first experimental evidence of damped waves in plasma in absence of collisions was observed only eighteen years later, in 1964[2].

The application of Landau damping to the accelerator beam physics was first formulated for coasting beams for the longitudinal case[3, 4], and then applied to the transverse case[5]. Eventually, Landau damping theory was applied to bunched beams[6, 7]

Still today however, there are several aspects of the physics of Landau damping that need a clear physical interpretation: paper devoted to the explanation and teaching of Landau damping are still appearing regularly more than sixty years after its discovery[8].

2 Basics of plasma physics

2.1 Cold plasma stationary oscillations

A cold plasma[1] of ionized gas, first observed in 1929[9], consists of ions and free electrons distributed over a region in space. The positive ions are very much heavier than the electrons, so that we can neglect their motion in

[1]The term cold or warm depends on the average kinetic energy of electrons. Generally a cold plasma is a plasma with all the particles having the same velocity.

comparison to that of electrons.

The plasma at the equilibrium, being neutral, is characterized by the same local density n_0 [$1/\mathrm{m}^3$] for both electrons and ions. If, for some reason, electrons are displaced from their equilibrium position, the local density changes producing electrical forces that tend to restore the equilibrium. As in any classical harmonic oscillator, the electrons gain kinetic energy, and instead of coming to rest, they start oscillating back and forth, at a frequency called "plasma frequency".

To simplify the study of these oscillations, we assume that the motion occurs only in one dimension, say x. Referring to Fig. 1, electrons initially in x are displaced from their equilibrium position by a small amount $s(x,t)$, and consequently, their local density n is perturbed. Due to the conservation of the charge, the number of electrons, initially confined between the planes A and B, $n_0\Delta x$, is the same as the number of electrons in the region $A'B'$ of width $\Delta x + \Delta s$ (after the displacement). The local electron density is then then written as:

$$n = \frac{n_0 \Delta x}{\Delta x + \Delta s} = \frac{n_0}{1 + (\Delta s/\Delta x)} \tag{1}$$

In case of small density perturbation, we can write

$$n \simeq n_0 \left(1 - \frac{\Delta s}{\Delta x}\right) \tag{2}$$

Since positive ions do not move appreciably, their density remains n_0, and the total charge density in the perturbed point becomes

$$\rho = -(n - n_0)e \tag{3}$$

where $e = 1.6 \times 10^{-19}$ C is the electron charge.

In the limit of $\Delta \to 0$, the charge density

$$\rho = n_0 e \frac{ds}{dx} \tag{4}$$

is source of an the electric field through the Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{5}$$

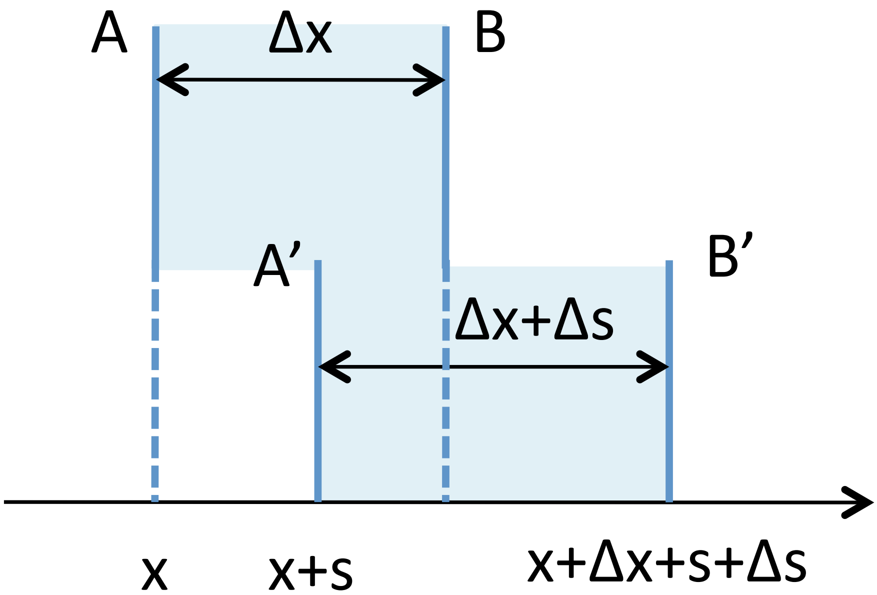

Figure 1: Electron motion in a plasma wave. The region AB shifts and the number of electrons in AB is the same of that in the region $A'B'$.

For a one dimensional motion, from eqs. (4) and (5), we get

$$\frac{\partial E_x}{\partial x} = \frac{n_0 e}{\varepsilon_0}\frac{\partial s}{\partial x} \tag{6}$$

which, integrated over x, gives the longitudinal electric field due to the perturbation:

$$E_x = \frac{n_0 e}{\varepsilon_0} s \tag{7}$$

being $E_x = 0$ at $s = 0$.

The force acting on an electron in the displaced position s is

$$F_x = -\frac{n_0 e^2}{\varepsilon_0} s \tag{8}$$

It is easy to recognize in (8) a restoring force proportional to the displacement s of the electron, which leads to an harmonic oscillation motion of the electrons. For a displaced electron, Newton's second law is then:

$$m_e \frac{d^2 s}{dt^2} = -\frac{n_0 e^2}{\varepsilon_0} s \tag{9}$$

which corresponds to a plasma oscillation with frequency

$$\omega_p^2 = \frac{n_0 e^2}{\varepsilon_0 m_e} \tag{10}$$

called "plasma frequency", a characteristic parameter of a cold plasma. The equivalent elastic constant is $k = n_o e^2/\varepsilon_0$.

Thus, a perturbation in the plasma will set up free oscillations with a frequency ω_p proportional to the square root of the density.

This natural resonance of a plasma has some interesting effects. For example, it is well known in satellite communication that the propagation of a radiowave through the ionosphere needs a frequency higher than its plasma frequency, while lower frequencies are reflected to the earth, allowing communication with a radio station beyond the horizon.

2.2 Perturbation of a plasma: Vlasov equation

2.2.1 Boltzmann equation for a collisionless system

Let us consider now the more general case of an ensemble of N particles having a distribution function $f(\mathbf{r}, \mathbf{v}, t)$ in the phase space, where $\mathbf{r}$ and $\mathbf{v}$ are position and velocity, respectively, such that

$$\int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{r} d\mathbf{v} = N \tag{11}$$

with $d\mathbf{r} = dxdydz$, and $d\mathbf{v} = dv_x dv_y dv_z$.

In order to study the time evolution of the above distribution function in the collisionless regime, we first make the following considerations:

- In a deterministic system, a particle trajectory in the phase space $(\mathbf{r}, \mathbf{v})$ is completely determined by its initial conditions, therefore two particles with the same initial conditions will follow exactly the same trajectory in the phase space. The consequence is that the trajectories of two particles in the phase space never intersect, unless they completely coincide. Furthermore, if we consider the particles inside a given area in the phase space, since their trajectories cannot intersect the trajectories of the particles at the boundary, we have that particles inside the area cannot jump outside; similarly, particles outside the area cannot jump inside.

- If the system of particles with mass m experiences a conservative force $\mathbf{F}(\mathbf{r}, t)$, the phase space area enclosing a number of particles at time t can be distorted at a next time $t + dt$ but it remains constant[10].

In other words, particles, which at time t are in a phase space area $d\mathbf{r}d\mathbf{v}$ at position $\mathbf{r}$ and velocity $\mathbf{v}$, at time $t + dt$, they will be in the area $d\mathbf{r}d\mathbf{v}$ at position $\mathbf{r} + \mathbf{v}dt$ and velocity $\mathbf{v} + \frac{\mathbf{F}}{m}dt$.

In terms of the distribution function f we can write:

$$dN = f(\mathbf{r} + \mathbf{v}dt, \mathbf{v} + \frac{\mathbf{F}}{m}dt, t + dt)d\mathbf{r}d\mathbf{v} = f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{r}d\mathbf{v} \tag{12}$$

We make the first order expansion of the term

$$f(\mathbf{r} + \mathbf{v}dt, \mathbf{v} + \frac{\mathbf{F}}{m}dt, t + dt) = f(\mathbf{r}, \mathbf{v}, t) + \nabla f \cdot \mathbf{v}dt + \nabla_{\mathbf{v}} f \cdot \frac{\mathbf{F}}{m}dt + \frac{\partial f}{\partial t}dt \tag{13}$$

where ∇ is the space coordinate gradient and $\nabla_{\mathbf{v}}$ the gradient with respect to the velocity.

Using the above expansion in (12), we obtain[2]

$$\frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{v} + \nabla_{\mathbf{v}} f \cdot \frac{\mathbf{F}}{m} = 0 \tag{14}$$

which is known as "collisionless Boltzmann equation". It is worth noting that equation (14) is often confused with the Liouville equation which actually holds for a discrete N-particle system.

2.2.2 Vlasov-Poisson equations and the dispersion relation

Due to the existence in plasma of long range collective forces, in 1938 Vlasov showed[11] that Boltzmann equation is suitable for a description of plasma dynamics only if the long range collective forces are taken into account through a self-consistent field. Indeed, he suggested, for the correct description of the system, a set of equations known today as Vlasov-Poisson equations.

Let us assume then that the collisionless system is formed by neutral warm plasma characterized by a non-relativistic motion[3], in a region with zero-magnetic field. Assuming the case of heavy ions ("frozen" motion), from equation (14), we derive here the kinetic equation for the evolution of

[2] It is worth noting that equation (14) is equivalent to writing $\frac{df}{dt} = 0$.

[3] Here we have already considered non-relativistic particles. For relativistic particles we had to use the momentum instead of the velocity $\mathbf{v}$.

a density perturbation in the plasma, the so-called Vlasov equation[12]. To simplify the study, we assume the motion only in one dimension, that we indicate with x, the corresponding velocity being v_x.

In this case, the density function $f\left(x, v_x, t\right)$ represents the electrons distribution function in the plasma. The electric forces generated by the charge distribution will act on the charges and modify the distribution. Then eq. (14) can be written as[4]

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{e}{m_e} E_x \frac{\partial f}{\partial v_x} = 0 \tag{15}$$

where E_x is the electric field satisfying

$$E_x = -\frac{\partial \phi}{\partial x} \tag{16}$$

ϕ being the electric field potential given by the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0}\left(n_0 - \int f dv_x\right) \tag{17}$$

Equations (15) - (17) are the Vlasov-Poisson equations.

We can get an approximate solution of the Vlasov-Poisson equations by using a perturbation technique, assuming that the electron distribution function $f(x, v_x, t)$ is given by the sum of the unperturbed density function $f_0(v_x)$ and a perturbation[5] $f_1\left(x, v_x, t\right)$

$$f(x, v_x, t) = f_0(v_x) + f_1\left(x, v_x, t\right) \tag{18}$$

Since the equilibrium state is neutral, we have that $\int f_0 dv_x = n_0$ and $\int f_1 dx dv_x = 0$. Moreover, f_0 does not depend on time and position, thus it results:

$$\frac{\partial f_0}{\partial t} = 0 \tag{19}$$

$$\frac{\partial f_0}{\partial x} = 0 \tag{20}$$

[4]For our convenience we invert here the scalar products.

[5]The perturbation is characterized by small amplitude and slope.

For what concern the electric field, we know that it vanishes in the unperturbed neutral state, and its value is related to the amplitude of the perturbation f_1 only. If we consider now the last term of the LHS of eq. (15), we can write

$$E_x \frac{\partial f}{\partial v_x} = E_x \frac{\partial}{\partial v_x}(f_0 + f_1) = E_x \frac{\partial f_0}{\partial v_x} \tag{21}$$

where we have neglected $E_x \partial f_1 / \partial v_x$, being of second order in the perturbation. By using eqs. (19) - (21) into eq. (15), we get

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} - \frac{e}{m_e} E_x \frac{\partial f_0}{\partial v_x} = 0 \tag{22}$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} \int f_1 dv_x \tag{23}$$

A solution of eqs. (22) and (23) was first derived by Vlasov who applied the double Fourier transforms from the domain (x, t) to the domain (k, ω), getting for the perturbation and the potential respectively

$$\tilde{f}_1(v_x, k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x, v_x, t) e^{i(\omega t - kx)} dx dt \tag{24}$$

and

$$\tilde{\phi}(k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, t) e^{i(\omega t - kx)} dx dt \tag{25}$$

and for the differential equation (22)

$$i(kv_x - \omega)\tilde{f}_1 + i\frac{e}{m_e} k\tilde{\phi} \frac{\partial f_0}{\partial v_x} = 0 \tag{26}$$

Accordingly, eq. (23) becomes:

$$-k^2 \tilde{\phi} = \frac{e}{\varepsilon_0} \int \tilde{f}_1 dv_x \tag{27}$$

If we take $\tilde{f}_1$ from (26) and substitute into (27), we obtain the following dispersion relation

$$1 + \frac{e^2}{\varepsilon_0 m_e k} \int \frac{\partial f_0 / \partial v_x}{\omega - kv_x} dv_x = 0 \tag{28}$$

Integration of (28) over v_x provides a relation between k and ω which depends only on the slope of the unperturbed distribution function $f_0(v_x)$. The dispersion relation contains a divergent integral, because of the singularity at $\omega = kv_x$. To overcome this difficulty, without giving a solid explanation, Vlasov calculated the principal value of the integral, getting, as result, only a frequency shift without any kind of damping.

2.2.3 Landau solution of the Vlasov equation

In a very original paper of 1946 Landau proposed a new method of solution of Vlasov-Poisson equations putting the basis of the theory of plasma oscillations and instabilities[1]. He demonstrated that the problem had to be considered as an initial value or Cauchy problem, with a perturbation $f_1(x, v_x, t)$ known at $t = 0$. To this end he adopted the Laplace transform for the time domain and used the Fourier transform only for the space domain. Accordingly, the perturbation and the electric field are first transformed as

$$\tilde{f}_1\left(k, v_x, t\right) = \int_{-\infty}^{\infty} f_1\left(x, v_x, t\right) e^{-ikx} dx \tag{29}$$

$$\tilde{E}_x\left(k, t\right) = \int_{-\infty}^{\infty} E_x\left(x, t\right) e^{-ikx} dx \tag{30}$$

so that eqs. (22) and (23) become

$$\frac{\partial \tilde{f}_1}{\partial t} + ikv_x \tilde{f}_1 - \frac{e}{m_e} \tilde{E}_x \frac{\partial f_0}{\partial v_x} = 0 \tag{31}$$

$$ik\tilde{E}_x = -\frac{e}{\varepsilon_0} \int \tilde{f}_1 dv_x \tag{32}$$

Applying the Laplace transform to (29) and (30), we get

$$\mathcal{F}_1(k, v_x, p) = \int_0^{\infty} \tilde{f}_1(k, v_x, t) e^{-pt} dt \tag{33}$$

$$\mathcal{E}_x(k, p) = \int_0^{\infty} \tilde{E}_x(k, t) e^{-pt} dt \tag{34}$$

while for eqs. (31) and (32), reminding that the Laplace transform of the time derivative is $p\mathcal{F}_1 - \tilde{f}_1(t=0)$, we obtain

$$p\mathcal{F}_1 + ikv_x\mathcal{F}_1 = \frac{e}{m_e}\mathcal{E}_x\frac{\partial f_0}{\partial v_x} + \tilde{f}_1(t=0) \tag{35}$$

and

$$ik\mathcal{E}_x\left(k,p\right) = -\frac{e}{\varepsilon_0}\int \mathcal{F}_1 dv_x \tag{36}$$

Taking $\mathcal{F}_1$ from eq. (35) and substituting it in (36), we get

$$ik\mathcal{E}_x\left(k,p\right) = -\frac{e}{\varepsilon_0}\int\left[\frac{e}{m_e}\mathcal{E}_x\frac{\partial f_0/\partial v_x}{p+ikv_x} + \frac{\tilde{f}_1(t=0)}{p+ikv_x}\right]dv_x \tag{37}$$

Since $\mathcal{E}_x$ does not depend on v_x, we find the following expression of the electric field:

$$\mathcal{E}_x\left(k,p\right) = -\frac{e/\varepsilon_0}{ik\epsilon(k,p)}\int\frac{\tilde{f}_1(t=0)}{p+ikv_x}dv_x \tag{38}$$

where $\epsilon(k,p)$ is the plasma dielectric function defined as

$$\epsilon(k,p) = 1 + \frac{e^2}{\varepsilon_0 m_e k}\int\frac{\partial f_0/\partial v_x}{ip - kv_x}dv_x \tag{39}$$

It is worth noting that eq. (39), putting $\epsilon(k,p) = 0$ with $p = -i\omega$, gives the dispersion integral (28). From the Laplace transform of the electric field $\mathcal{E}_x\left(k,p\right)$ we can then obtain the perturbation function $\mathcal{F}_1$ as

$$\mathcal{F}_1(k,v_x,p) = \frac{e}{m_e}\mathcal{E}_x\left(k,p\right)\frac{\partial f_0/\partial v_x}{p+ikv_x} + \frac{\tilde{f}_1(t=0)}{p+ikv_x} \tag{40}$$

The electric field and the perturbation function in the time domain are obtainable by inverting the Laplace transforms:

$$\tilde{E}_x(k,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{E}_x\left(k,p\right)e^{pt}dp \tag{41}$$

$$\tilde{f}_1(k,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{F}_1\left(k,p\right)e^{pt}dp \tag{42}$$

where the path of integration has to be chosen in a region of the complex p-plane free of singularities.

Due to the complexity of the integrals (41) and (42), which can be treated in general only through numerical tools, Landau analyzed the time behavior of field and perturbation at very large times considering the singularities of $\mathcal{E}_x(k,p)$ in the p-complex plane, concluding that the electric field, generated by the perturbation at $t = 0$ decays exponentially with time, showing the existence of a new effect known as Landau damping.

His analysis was not trivial and relied on the properties of the integration in the complex p-plane. More details are given in Appendix A.

Landau showed that the integral in (39), as well as the others similar in the expressions of the electric field and of the perturbation, has to be performed along the real axis of v_x by deforming the path of integration such to surround the singularity at ip/k as shown in Fig. 2.

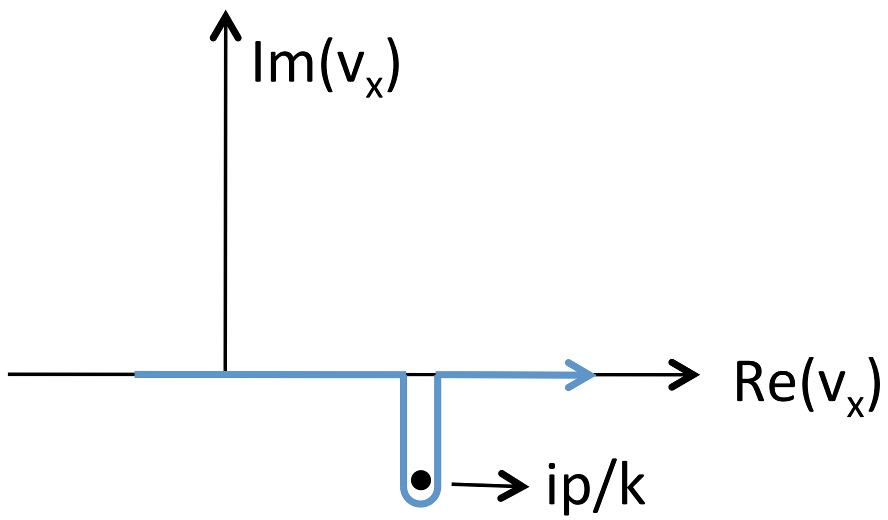

Figure 2: Path of integration for Landau damping.

Furthermore, the plasma dielectric function (39) dictates the region of regularity of the electric field $\mathcal{E}_x(k,p)$, which diverges for vanishing values of $\epsilon(k,p)$. The stability analysis can be traced back to the study of the condition $\epsilon(k,p) = 0$, which, for $p = -i\omega$, coincides with the Vlasov dispersion relation

(28). Therefore, as long as we are interested to the instability conditions through the analysis of the zeros of the plasma dielectric function (39), we may make use of the Vlasov dispersion integral (28), provided it is executed in the complex v_x plane deforming the integration contour as shown in Fig. 3. Accordingly, the dispersion function becomes:

$$1 + \frac{e^2}{\varepsilon_0 m_e k}\left[P.V. \int \frac{\partial f_0/\partial v_x}{\omega - k v_x} dv_x - \frac{i\pi}{k}\left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k}\right] = 0 \tag{43}$$

The imaginary term of the above equation produces the damping or anti-damping effect predicted by Landau, depending on the slope of the distribution function. With this procedure, we obtain straightforwardly the correct dispersion relations via Fourier transformation of the Vlasov equation.

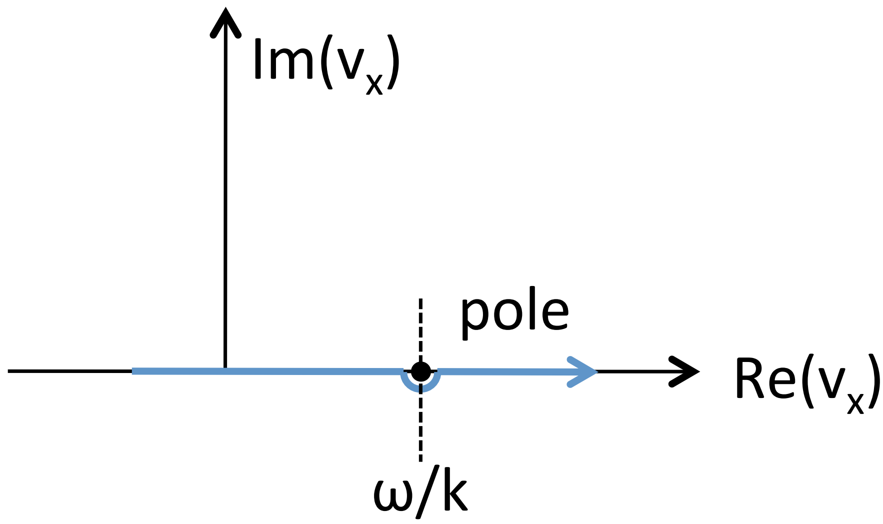

Figure 3: Path of integration of the dispersion relation in the velocity plane.

Example: Plasma with a Maxwellian velocity distribution

As an example to clarify the use of the dispersion relation for the analysis of the plasma stability, we consider a plasma with a velocity Maxwellian distribution function

$$f_0(v_x) = \frac{n_0}{(2\pi k_B T/m_e)^{1/2}} \exp\left(-\frac{m_e v_x^2}{2k_B T}\right) \tag{44}$$

where k_B is the Boltzmann constant. We can integrate by parts the principal value of eq. (43) obtaining

$$P.V. \int \frac{\partial f_0/\partial v_x}{\omega - k v_x} dv_x = \left. \frac{f_0(v_x)}{\omega - k v_x} \right|_{-\infty}^{\infty} - k \int \frac{f_0(v_x)}{(\omega - k v_x)^2} dv_x \tag{45}$$

The first term on the right hand side is zero since $f_0 \to 0$ as $v_x \to \pm\infty$ and the denominator helps. In order to perform the second integral of eq. (45), we remind that $\int f_0(v_x) dv_x = n_0$, and that $f_0(v_x)$ is an even function of v_x, while $\partial f_0/\partial v_x$ is an odd function of v_x. If the wave phase velocity is larger than the electron thermal speed, both f_0 and $\partial f_0/\partial v_x$ become smaller at larger v_x and the integral of eq. (45) is dominated by the range where $v_x \ll \omega/k$. We can then Taylor expand the denominator of the integral in the form

$$\frac{1}{(\omega - k v_x)^2} = \frac{1}{\omega^2} \left(1 + 2 \frac{k v_x}{\omega} + 3 \frac{k^2 v_x^2}{\omega^2} + ... \right) \tag{46}$$

so that we have

$$P.V. \int \frac{\partial f_0/\partial v_x}{\omega - k v_x} dv_x = -\frac{k}{\omega^2} \int f_0(v_x) \left(1 + 2 \frac{k v_x}{\omega} + 3 \frac{k^2 v_x^2}{\omega^2} + ... \right) dv_x \tag{47}$$

The first integral term is just n_0, the second one is zero since f_0 is an even function of v_x, and the third integral term is proportional to the variance of the velocity distribution, which from eq. (44) is $k_B T/m_e$, so that eq. (43) can be written as

$$\int \frac{\partial f_0/\partial v_x}{\omega - k v_x} dv_x = -\frac{k n_0}{\omega^2} - 3 \frac{k^3 n_0}{\omega^4} \frac{k_B T}{m_e} - \frac{i\pi}{k} \left(\frac{\partial f_0}{\partial v_x} \right)_{v_x = \omega/k} \tag{48}$$

By using the above results, the dispersion relation (28) becomes

$$1 - \frac{\omega_p^2}{\omega^2} - 3k^2 \frac{\omega_p^2}{\omega^4} \frac{k_B T}{m_e} - i \frac{\pi e^2}{\varepsilon_0 m_e k^2} \left(\frac{\partial f_0}{\partial v_x} \right)_{v_x = \omega/k} = 0 \tag{49}$$

namely

$$\omega^2 = \omega_p^2 \left(1 + 3k^2 \frac{k_B T}{\omega^2 m_e} \right) + i \omega^2 \frac{\pi e^2}{\varepsilon_0 m_e k^2} \left(\frac{\partial f_0}{\partial v_x} \right)_{v_x = \omega/k} \tag{50}$$

with ω_p the plasma frequency defined by eq. (10). In case of a cold plasma, with $T \to 0$, we recover the same results of section 2.1.

If we assume $T \neq 0$, such to produce a small perturbation of the plasma frequency ω_p, we can write $\omega = \omega_r + i\delta\omega_i$, with $\omega_r - \omega_p \ll \omega_p$ and $\delta\omega_i \ll \omega_p$, and approximate the above expression as

$$\omega^2 \simeq \omega_r^2 + 2i\omega_p\delta\omega_i \simeq \omega_p^2\left(1 + 3k^2\frac{k_BT}{\omega_p^2 m_e}\right) + i\omega_p^2\frac{\pi e^2}{\varepsilon_0 m_e k^2}\left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k} \tag{51}$$

The real part is

$$\omega_r^2 \simeq \omega_p^2\left(1 + 3k^2\frac{k_BT}{\omega_p^2 m_e}\right) = \omega_p^2\left(1 + 3k^2\lambda_D^2\right) \tag{52}$$

where we $\lambda_D = \sqrt{k_BT/m_e\omega_p^2}$ is the Debye length, having assumed $k\lambda_D \ll 1$. This result is the dispersion relation for waves [13] in warm plasma obtained by Vlasov in his paper[11].

For the imaginary part, that was not predicted by Vlasov, we have that

$$2\omega_p\delta\omega_i \simeq \omega_p^2\frac{\pi e^2}{\varepsilon_0 m_e k^2}\left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k} \tag{53}$$

so that

$$\delta\omega_i \simeq \frac{\pi}{2}\frac{\omega_p e^2}{\varepsilon_0 m_e k^2}\left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k} = -\frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_p}{(k\lambda_d)^3}\exp\left[-\frac{1}{(k\lambda_D)^2}\right] \tag{54}$$

Since this term is negative, it corresponds to an exponential decay which is the damping effect predicted by Landau. If the wavelength is much larger than the Debye length, the wave is lightly damped. However, if the wavelength becomes comparable to the Debye length, the damping becomes strong. Even if the above approximate solution is not very accurate, it clearly shows the existence of damped waves that have been observed experimentally[2]. A simple physical description of this effect can be found in Appendix B.

It is worth reminding that there are no dissipative effects included in the collisionless Vlasov equation.

3 A simple mechanical model

The Landau damping effect has been derived from a pure mathematical approach, and there are several aspects of its physics that are still surprising. We wonder, in fact, how it is possible that a collisionless, lossless system, perturbed from the equilibrium, can show such a behavior.

In order to get a physical insight in the mechanism which is responsible of the damping, despite the free loss nature of the system, we analyze now a simple model consisting of an infinite set of harmonic oscillators, with frequency distribution $G(\omega)$, such that

$$\int_{-\infty}^{\infty} G(\omega)d\omega = 1 \tag{55}$$

and with an average value ω_0.

We assume that the system is driven by an external sinusoidal force of frequency Ω, and that the oscillators do not interact each other.

For a single oscillator the differential equation of motion is

$$x'' + \omega^2 x = A\cos\Omega t \tag{56}$$

With the starting conditions $x(t=0)=0$ and $x'(t=0)=0$, its solution is

$$x(t>0) = -\frac{A}{\Omega^2-\omega^2}\left(\cos\Omega t - \cos\omega t\right) \tag{57}$$

We also assume that all the oscillator frequencies are sufficiently close each other and that Ω lies within the oscillator frequency spectrum. Moreover, we define the difference between the resonance frequency Ω and a resonator frequency ω as $\delta_\omega = \Omega - \omega \ll \omega_0$, such that $\Omega + \omega \simeq 2\omega_0$.

Under these assumptions, eq. (57) becomes

$$x(t>0) \simeq -\frac{A}{2\omega_0\delta_\omega}\left[\cos\left(\omega_0+\frac{\delta_\omega}{2}\right)t - \cos\left(\omega_0-\frac{\delta_\omega}{2}\right)t\right] = \frac{A}{\omega_0\delta_\omega}\sin\omega_0 t\sin\frac{\delta_\omega}{2}t \tag{58}$$

that can be seen as an oscillation at frequency ω_0 with an amplitude modulated at the lower frequency $\delta_\omega/2$. It is convenient to write eq. (58) as

$$x(t>0) = \frac{At}{2\omega_0} \frac{\sin\frac{\delta_\omega}{2}t}{\left(\frac{\delta_\omega}{2}t\right)} \sin\omega_0 t \tag{59}$$

Let's observe now the motion of two oscillators, the former with $\delta_\omega = 0$, and the latter with $\delta_\omega \neq 0$, as shown in Fig. 4 with blue and red curves respectively. Both are at rest at $t = 0$, and they start to oscillate due to the action of the same external force. While the amplitude of the "on resonance" oscillator growths linearly with time, the other reaches a maximum amplitude (beating of two close frequency) after a time $t = \pi/\delta_\omega$, (when $\sin(\delta_\omega t/2) = 1$), after which this oscillator is "out of resonance", and loses the phase synchronism with the external driving force. We can reverse this argument by saying that at a time t^*, only those oscillators with a frequency ω, such that $\delta_\omega < \pi/t^*$ maintain a phase relation with the external force. The longer we wait, the narrower the frequency bandwidth δ_ω of synchronous oscillators.

Therefore, at any instant t^* we can divide the oscillators into two groups: the oscillators having frequencies such that $\delta_\omega < \pi/t^*$ which keep their initial phase synchronism and their amplitude grow linearly with time; the oscillators with $\delta_\omega > \pi/t^*$, which are no longer in resonance with the external force. The "on resonance" oscillators are in phase with the external force and they contribute to absorb energy from the force, but their frequency bandwidth, and thus their number, decreases with time. The net effect is an absorption of energy by the system while the average amplitude of oscillation remains constant.

To demonstrate that, let us calculate the average of the particle displacements, given by

$$< x > (t) = -\int_{-\infty}^{\infty} G(\omega) \frac{A}{\Omega^2 - \omega^2} \left(\cos\Omega t - \cos\omega t\right) d\omega \tag{60}$$

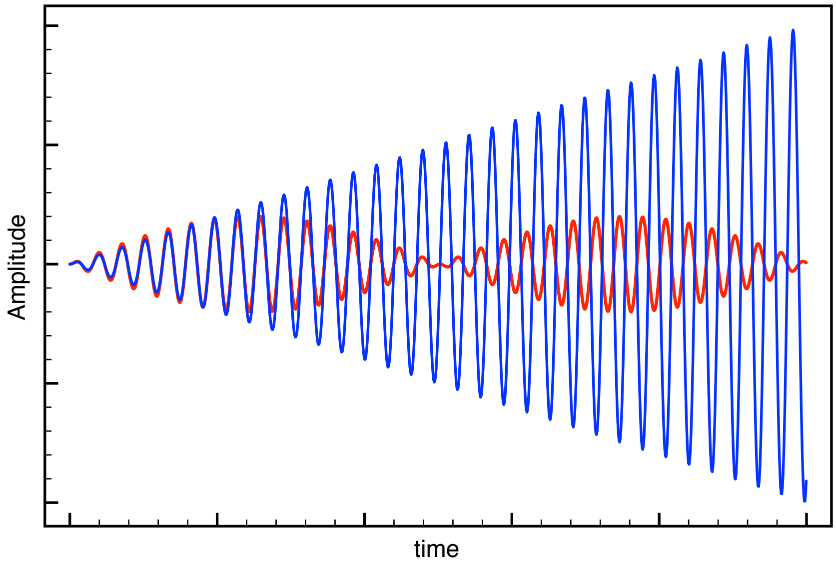

Figure 4: "On resonance" and "out of resonance" oscillations of two particles due to a sinusoidal external force in arbitrary units.

Since $\Omega - \omega \ll \omega_0$, we can justify the approximation

$$< x > (t) \simeq -\frac{A}{2\omega_0} \int_{-\infty}^{\infty} \frac{G(\omega)}{\Omega - \omega} \left(\cos \Omega t - \cos \omega t\right) d\omega \tag{61}$$

If we make a change of variable from ω to δ_ω, we get

$$\begin{aligned} < x > (t) \simeq \frac{A}{2\omega_0} \Bigg[& \sin \Omega t \int_{-\infty}^{\infty} G(\Omega - \delta_\omega) \frac{\sin \delta_\omega t}{\delta_\omega} d\delta_\omega \\ & - \cos \Omega t \int_{-\infty}^{\infty} G(\Omega - \delta_\omega) \frac{1 - \cos \delta_\omega t}{\delta_\omega} d\delta_\omega \Bigg] \end{aligned} \tag{62}$$

In order to evaluate the average displacement as the time growths, we first note that[14]

$$\lim_{t \to \infty} \frac{\sin xt}{x} = \pi \delta(x) \tag{63}$$

since, as t increases, the function $\sin(xt)/x$ is peaked around $x = 0$ and its integral is π, independent on t.

The second integral in eq. (62) can be simplified by observing that the function $(1 - \cos xt)/x$ approaches $1/x$ with increasing accuracy as $t \to \infty$, especially close to $x = 0$, and the cosine term removes the singularity of the integral at $x = 0$. So we can substitute the above term with $1/x$, with the condition that we take the principal value of the integral (to remove the singularity in $x = 0$), that is

$$\int_{-\infty}^{\infty} G(\Omega - \delta_\omega) \frac{1 - \cos \delta_\omega t}{\delta_\omega} d\delta_\omega = P.V. \int_{-\infty}^{\infty} \frac{G(\Omega - \delta_\omega)}{\delta_\omega} d\delta_\omega \tag{64}$$

Thus, the average particle displacement at large times (62), can be calculated by the following simplified expression

$$< x > (t) = \frac{A}{2\omega_0} \left[\pi G(\Omega) \sin \Omega t + \cos \Omega t P.V. \int_{-\infty}^{\infty} \frac{G(\omega)}{\omega - \Omega} d\omega \right] \tag{65}$$

In conclusion, the average oscillation amplitude of the system does not increase with time, but remains limited when time goes to infinity.

As example of frequency spectrum, let us consider a frequency distribution $G(\Omega)$ that is uniform between two frequencies ω_1 and ω_2, and zero elsewhere, that is

$$G(\Omega) = \begin{cases} \dfrac{1}{\omega_2 - \omega_1} & \omega_1 < \Omega < \omega_2 \\ 0 & \text{elsewhere} \end{cases} \tag{66}$$

The principal value in the integral of the average particle displacement can be easily done, and we obtain

$$< x > (t) = \frac{A}{2\omega_0(\omega_2 - \omega_1)} \left[\pi \sin \Omega t + \ln \left(\frac{\omega_2 - \Omega}{\Omega - \omega_1} \right) \cos \Omega t \right] \tag{67}$$

that represents an oscillation at frequency Ω with time independent amplitude. It is worth noting that the "sine" term is responsible of the power absorption, because, by doing the time derivative of equation (67), we get the velocity of the average particle displacement, whose "cosine" term is in phase with the external force producing an absorption of energy by the system.

This statement can be demonstrated by reminding that the energy U of an harmonic oscillator is proportional to the square of the oscillation amplitude. From equation (58) we get

$$U \propto \frac{A^2}{\omega_0^2 \delta_\omega^2} \sin^2 \frac{\delta_\omega}{2} t \tag{68}$$

which leads to the total energy of the system U_{tot}

$$U_{tot} \propto N \frac{A^2}{\omega_0^2} \int_{-\infty}^{\infty} G(\Omega - \delta_\omega) \frac{\sin^2 \frac{\delta_\omega}{2} t}{\delta_\omega^2} d\delta_\omega \tag{69}$$

where N is the total number of particles of the system. As time increases, the function $\sin^2(xt/2)/x^2$ becomes peaked around $x = 0$ and tends to a Dirac delta function

$$\lim_{t \to \infty} \frac{\sin^2 xt/2}{x^2} = \frac{\pi t}{2} \delta(x) \tag{70}$$

that, substituted in the integral of equation (69) gives

$$U_{tot} \propto N \frac{A^2}{\omega_0^2} \frac{\pi}{2} G(\Omega) t \tag{71}$$

Eq. (71) shows that the energy of the system increases linearly with time. This energy cannot be regarded as thermal energy of the system because is not distributed over al particles, but it is stored in a time narrowing range of frequencies around the driving frequency. Only the particles with frequency such that $|x - \Omega| < 1/t$ contribute to the "sine" response and are in resonance with the external force, but since their number decreases with time, the net contribution to the average displacement remains constant.

Eq. (65) can be compared with the results of the previous section making use of the complex notation and remembering that only the real parts are meaningful. Expressing the external sinusoidal force as $Ae^{-i\Omega t}$ we obtain

$$< x > (t) = -\frac{A}{2\omega_0} e^{-i\Omega t} \left[P.V. \int_{-\infty}^{\infty} \frac{G(\omega)}{\Omega - \omega} d\omega - i\pi G(\Omega) \right] \tag{72}$$

Observe that the term in the square bracket is similar to that of eq. (43). The difference is that, in this case, the driving force is external and independent of the system and, as a consequence, here we cannot find any dispersion relation[6].

By using the complex notations and considering only the particular solution of the equation of motion (56), the average displacement of the system can be written as

$$< x >= -\frac{A}{2\omega_0} e^{-i\Omega t} \int_{-\infty}^{\infty} \frac{G(\omega)}{\Omega - \omega} d\omega \tag{73}$$

that is a diverging integral analogous to equation (28). Here too we can avoid the singularity by doing the integration in the complex ω-plane similar to that of Fig. 3[7].

4 Longitudinal dynamics and Landau damping in coasting beams

Almost all accelerators suffer of collective instabilities that limit the current intensity. The instability mechanism is typical of a feedback system, as shown in Fig. 5: any perturbation of the beam equilibrium distribution, due to the interaction with the surrounding walls, produces electromagnetic fields that interact back on the beam. Under specific conditions a resonance in the beam motion can be excited with growing amplitude.

In section (4.1) we will first discuss the beam-wall interaction through the concept of wake field and of its Fourier transform, the coupling impedance,

[6] The dispersion relation can be obtained if we assume that the force acting on the system is proportional to the average value of the oscillations, that is $A \propto < x > (t)$.

[7] An equivalent way to obtain the same result is by considering the external force with a small increasing time dependent exponential. This driving force, which cannot be existed since $t \to -\infty$, has the same effect as introducing the initial conditions for what concerns the removing of the singularity.

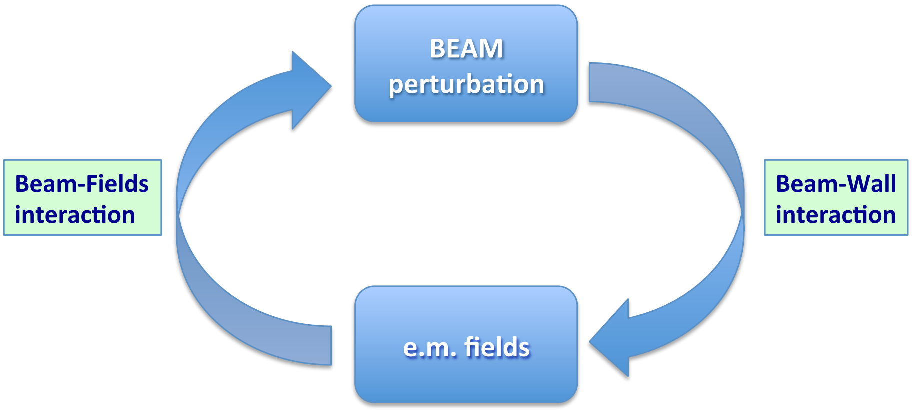

Figure 5: An accelerator as a feedback system.

in sections (4.2) and (4.3) we will analyze the dynamics of the beam under the effect of the electromagnetic forces to obtain the dispersion integral for the longitudinal motion of coasting beams that will be used in sections (4.4) and (4.5) to determine the stability conditions.

4.1 Longitudinal beam wall interaction

In the previous chapters we have introduced the Landau damping for two systems: a plasma and an ensemble of harmonic oscillators. We now focus our attention to a beam of particles stored in a circular accelerator, and, as a first example, we consider the longitudinal beam dynamics of a coasting beam subjected to the space charge and smooth wall interaction forces only. Additionally, we assume that the beam current is given by a stationary constant current I_0 plus a sinusoidal perturbation ΔI of the kind

$$I(s,t) = I_0 + \Delta I e^{i(ks-\omega t)} \tag{74}$$

As shown in Fig. 6, the perturbation behaves like a wave traveling along the ring moving with the same velocity of the charges. According to the notation adopted in particle accelerators, the longitudinal coordinate s represents the azimuthal position of the charge along its orbit of radius R_0. The

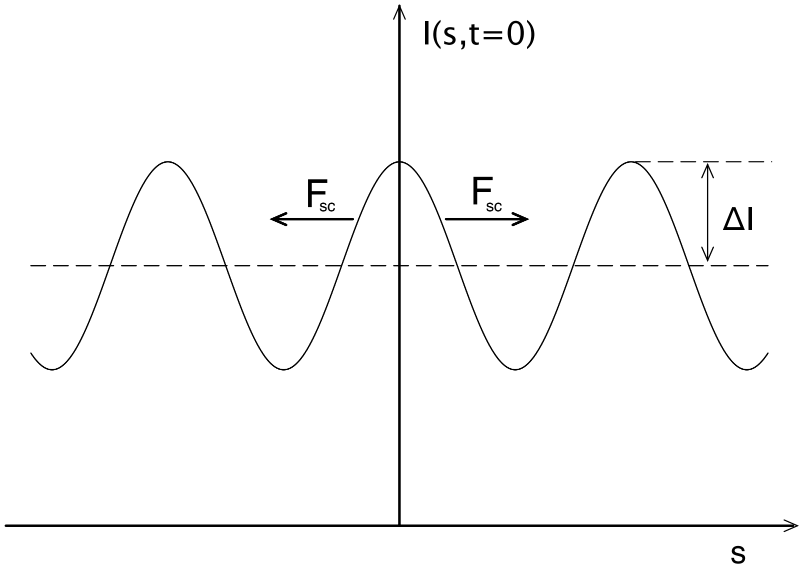

Figure 6: Longitudinal beam distribution for a coasting beam.

wavelength of the perturbation is a submultiple of the machine length L_0, such that:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{L_0} = \frac{n}{R_0} \tag{75}$$

Following from eq. (74), also the electromagnetic fields produced by the beam can be seen as a sum of those of the stationary distribution, that we do not examine here because they modify the stationary distribution but do not lead to instabilities, plus those of the perturbation on which we focus our attention. These fields, acting back to the beam, cause an energy variation which can be positive in case of energy gain or negative in case of energy loss. Due to the phase relationship between perturbation current and perturbation electromagnetic forces, we may have also a pure "reactive" interaction, in which the net exchange of energy is zero.

In order to calculate the rate of energy variation of a single particle in one turn due to the beam-wall electromagnetic interaction, we introduce the longitudinal wake function $W_{||}(\Delta z)$, defined as:

$$W_{||}(\Delta z) = -\frac{\left\langle F_{||}(\Delta z)\right\rangle L_0}{qq_1} \tag{76}$$

where $\left\langle F_{||}(\Delta z)\right\rangle$ is the electromagnetic force averaged along the accelerator

$$\left\langle F_{||}(\Delta z)\right\rangle = \frac{1}{L_0}\int_0^{L_0} F_{||}ds \tag{77}$$

acting on the charge q_1, q is the charge producing the electromagnetic fields, and Δz is the distance between q and q_1.

Eq. (76) represents the energy lost (in J/C) in one turn by q_1 due to the electromagnetic fields produced by q leading the motion and passing through a machine device at an earlier time t' such that, by supposing ultra-relativistic charges with $\beta = 1$, we have $\Delta z = c(t' - t)$.

If we indicate with E_0 the beam nominal energy, and with $\varepsilon = \Delta E/E_0$ its relative variation, and if we take into account the effects of the fields generated by all the particles belonging to the longitudinal perturbation, the rate of relative energy variation ε can be written in terms of wake function as

$$\frac{\partial\varepsilon}{\partial t} \simeq \frac{\Delta\varepsilon}{\Delta t} = -\frac{e}{E_0T_0}\int_{-\infty}^{t} W_{||}\left(ct' - ct\right)\Delta I e^{i(ks-\omega t')}dt' \tag{78}$$

that is, by changing the integration variable,

$$\frac{\partial\varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{cE_0T_0}\int_{-\infty}^{0} W_{||}\left(y\right)e^{-i\frac{\omega}{c}y}dy \tag{79}$$

Here T_0 represents the beam revolution time given by $T_0 = L_0/c$.

For ultra-relativistic charges, due to the causality principle, the wake function is zero ahead of the test particle so that the integral becomes c times the Fourier transform of the wake field which, by definition, is the longitudinal coupling impedance[15]; we then get

$$\frac{\partial\varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{E_0T_0}Z_{||}(\omega) \tag{80}$$

Before concluding this section we show some common longitudinal coupling impedances that are generally found in a particle accelerator[10, 15]. For a perfectly conducting smooth and circular vacuum chamber of radius b, also the space charge effect due to the non relativistic velocity of the charges can be written in terms of coupling impedance, and it gives

$$Z_{||}(\omega) = iZ_0 \frac{R_0\omega}{c(\beta\gamma)^2} \ln \frac{b}{r} \tag{81}$$

with Z_0 the impedance of the free space, and r the transverse position of the test charge q_1.

In case of the resistive wall of the circular pipe with beam at center and high conductivity σ_c, such that $c^2/(\omega^2 b)$ and b are much bigger that the skin depth, we have

$$Z_{||}(\omega) = \frac{R_0}{b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}\,[1 - i\,\mathrm{sign}(\omega)] \tag{82}$$

For a resonating mode of a cavity, the longitudinal coupling impedance can be written as

$$Z_{||}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)} \tag{83}$$

with R_s the shunt resistance, Q the quality factor, and ω_r the resonant frequency.

Note that the sign of the imaginary part of the coupling impedance depends on the notations that has been used (j or i to indicate the imaginary term).

4.2 Longitudinal beam dynamics of coasting beams

A particle with nominal energy E_0 moves in the circular machine with velocity c on a closed orbit, called the reference orbit, of length L_0. A particle with a small energy deviation ΔE, with $\Delta E = c\Delta p$, travels along a different path with a different speed. The change $\Delta\omega$ of its revolution frequency is due to a combination of two effects[16]: the speed and the dispersion, so that

$$\frac{\omega_0 - \bar{\omega}_0}{\bar{\omega}_0} = \frac{\Delta\omega}{\bar{\omega}_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0} = \\ = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E_0} = -\eta\frac{\Delta E}{E_0} = -\eta\varepsilon \quad (84)$$

with $\bar{\omega}_0$ the revolution frequency of a particle with nominal energy E_0, α_c the momentum compaction (a property of the guide fields) and γ the relativistic factor. When $\eta > 0$ the machine works above the transition energy and a positive deviation ε causes a longer trajectory which produces a reduction in the revolution frequency.

The change in the revolution frequency influences the longitudinal position of a particle. If we use the quantity z to define the longitudinal coordinate of a particle with respect to the reference one, which has a nominal energy E_0, we observe that a revolution frequency different from $\bar{\omega}_0$ produces a change in the longitudinal position z in one turn given by the relation

$$\frac{\Delta z}{L_0} = \frac{\Delta\omega}{\bar{\omega}_0} \quad (85)$$

from which

$$\frac{\Delta z}{T_0} = \Delta\omega R_0 \quad (86)$$

In the above relations we have assumed $z > 0$ ahead of the reference particle.

Even though we start with a monochromatic beam, all the particles having the same energy E_0, space charge and beam-wall interaction will produce electromagnetic forces that, interacting back on the beam, modify the particle energy and may lead to instability.

For example, the longitudinal effect of the space charge in a perfectly conducting pipe is a force proportional to $-\partial I/\partial s$[17]. As a consequence, the particles that are on the front slope of the sinusoidal perturbation will experience a positive force, and, in one turn, their energy will increase. The contrary will happen to the rear slope of the perturbation. If we are above transition, from equation (84), an increase of energy implies a decrease of the revolution frequency. Therefore the particles in the front slope will delay and those in the back crest will anticipate, giving, as a net result, an increase of the height of the crest. An initial sinusoidal perturbation is thus increased leading to instability, known as negative mass instability. On the contrary, below transition, the longitudinal space charge forces stabilize the beam.

4.3 Dispersion relation of longitudinal coasting beam

This dynamics of the coasting beam can be formalized by treating the motion of the particles by means of the Vlasov equation. The formalism is very similar to that we have used for the waves in a perturbed plasma. Here we use $f(z, \varepsilon; t)$ for the beam distribution function such that its integration over longitudinal space and energy gives the total number of particles N in the beam

$$\int \int f(z, \varepsilon; t) dz d\varepsilon = N \tag{87}$$

The beam current I, defined by eq. (74), can be obtained from the beam distribution function as

$$I(z; t) = ec \int f(z, \varepsilon; t) d\varepsilon \tag{88}$$

Here we have performed a change of variable from s to $z = s - ct$. Therefore, by using eq. (75) the beam current I of eq. (74) becomes

$$I(z, t) = I_0 + \Delta I e^{i[kz - (\omega - n\bar{\omega}_0)t]} \tag{89}$$

and also the beam distribution function can then be written as

$$f(z, \varepsilon; t) = f_0(\varepsilon) + f_1(\varepsilon) e^{i[kz - (\omega - n\bar{\omega}_0)t]} \tag{90}$$

In writing the above equations we have considered that the stationary distribution does not depend either on time or on z being the circular machine azimuthally symmetric.

The beam distribution function satisfies the Vlasov equation[3] that we write here in the form

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} = 0 \tag{91}$$

The terms $\partial z/\partial t$ and $\partial \varepsilon/\partial t$ represent the rate of change of the longitudinal position and energy of the particle. The characteristic time in which all the involved variables have significant changes, also in presence of instability, is generally much longer than the revolution time T_0; therefore we may assume T_0 to be the minimum time step. Under this assumption we can write

$$\frac{\partial z}{\partial t} \simeq \frac{\Delta z}{T_0} = \Delta\omega R_0 \tag{92}$$

where the last identity has been obtained by using equation (86).

The relative energy variation in one turn arises from the longitudinal forces produced by the interaction of the beam with the surroundings and by the space charge. These forces vanish if the longitudinal distribution is uniform along the accelerator and, as we have shown, they can be expressed by means of the longitudinal wake function[15].

As a matter of facts, by applying equation (88), the instantaneous current can also be written as

$$I(z;t) = \frac{ecN}{L_0} + ece^{i[kz-(\omega-n\bar{\omega}_0)t]} \int f_1(\varepsilon)d\varepsilon \tag{93}$$

As we have shown in section (4.1), the rate of energy variation is then

$$\frac{\partial \varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{E_0 T_0} Z_{||}(\omega) = -\frac{ce^2 e^{i[kz-(\omega-n\bar{\omega}_0)t]}}{E_0 T_0} Z_{||}(\omega) \int f_1(\varepsilon)d\varepsilon \tag{94}$$

We now linearize the Vlasov equation by substituting eq. (90) into (91) and ignore second order terms in the perturbation, that is the term containing $(\partial f_1/\partial\varepsilon)\int f_1(\varepsilon)d\varepsilon$. With the use of the eqs. (75), (92), and (94), we then obtain

$$\begin{aligned} -i\left(\omega - n\bar{\omega}_0 - n\Delta\omega\right) f_1 e^{i[kz-(\omega-n\bar{\omega}_0)t]} &= \\ &= \frac{\partial f_0}{\partial \varepsilon}\frac{e^2 c Z_{||}(n\bar{\omega}_0)}{E_0 T_0} e^{i[kz-(\omega-n\bar{\omega}_0)t]} \int f_1 d\varepsilon \end{aligned} \tag{95}$$

To first order of perturbation the coupling impedance has been evaluated at the unperturbed frequency $n\bar{\omega}_0$.

By using now eq. (84) the above equation can be written as

$$f_1 = i\frac{\partial f_0/\partial\varepsilon}{\omega - n\bar{\omega}_0 + n\bar{\omega}_0\eta\varepsilon}\frac{e^2c^2Z_{||}(n\bar{\omega}_0)}{E_0L_0}\int f_1 d\varepsilon \tag{96}$$

If we integrate both the members by ε, and use the definition of the average current of eq. (93) then we obtain the dispersion integral

$$1 = i\frac{(Z_{||}/n)I_0L_0}{2\pi N(E_0/e)\eta}\int \frac{\partial f_0/\partial\varepsilon}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta} + \varepsilon}d\varepsilon \tag{97}$$

Observe that the above dispersion integral, derived from the Vlasov equation, has a very close similarity to eq. (28) obtained for the plasma oscillations. As in that case, we now know that we must execute the above integral in the complex ε-plane by deforming the contour of the integration, in order to avoid the singularity[8]. If we do that, by using the same rule of eq. (43), we get

$$1 = i\frac{(Z_{||}/n)I_0L_0}{2\pi N(E_0/e)\eta}\left[P.V.\int \frac{\partial f_0/\partial\varepsilon}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta} + \varepsilon}d\varepsilon - i\pi\left(\frac{\partial f_0}{\partial\varepsilon}\right)_{\varepsilon=\frac{(n\bar{\omega}_0-\omega)}{n\bar{\omega}_0\eta}}\right] \tag{98}$$

4.4 Monochromatic beam

Let us use the dispersion integral to discuss the stability of a monochromatic beam, namely a beam without energy spread. In this case the stationary distribution can be written as

$$f_0(\varepsilon) = N\frac{\delta(\varepsilon)}{L_0} \tag{99}$$

with δ the Dirac delta function. Since in this case there is no energy spread, the dispersion integral does not diverge, and we can use directly eq. (97).

[8]Observe that the singularity exists only if the frequency shift $\omega - n\bar{\omega}_0$ due to the coupling impedance lies within the frequency spread due to the energy distribution. Outside this frequency range there is no Landau damping.

With the above relation, we get

$$1 = i\frac{(Z_{||}/n)I_0}{2\pi(E_0/e)\eta}\int \frac{\delta'(\varepsilon)}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}d\varepsilon =$$

$$= -i\frac{(Z_{||}/n)I_0}{2\pi(E_0/e)\eta}\frac{\partial}{\partial\varepsilon}\left(\frac{1}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}\right)\Bigg|_{\varepsilon=0} \tag{100}$$

that is

$$1 = i\frac{\eta(Z_{||}/n)I_0}{2\pi(E_0/e)}\left(\frac{n\bar{\omega}_0}{\omega-n\bar{\omega}_0}\right)^2 \tag{101}$$

so that the frequency is

$$\omega = n\bar{\omega}_0 \pm n\bar{\omega}_0\sqrt{i\frac{\eta(Z_{||}/n)I_0}{2\pi(E_0/e)}} \tag{102}$$

When ω has an imaginary part ω_i, we obtain a perturbation with a time exponential growing amplitude that leads to instability (actually in the above equation there is a second solution that produces an exponential decay) . The real part of ω, ω_r, gives the frequency of the perturbed current term. If we ignore the machine coupling impedance $Z_{||}$ this frequency is $n\bar{\omega}_0$.

If the machine coupling impedance $Z_{||}$ has a real part, that is a resistive component, ω will always have an imaginary part and therefore the beam will be unstable. For a pure imaginary impedance $Z_{||} = iZ_{||,i}$, stability or instability will depend on the sign of η and $Z_{||,i}$. Above transition energy ($\eta > 0$), if $Z_{||,i} > 0$ (a capacitive impedance due to the space charge as in the previous example) we find the negative mass instability. The overall behavior can be summarized by saying that when $\eta Z_{||,i} < 0$, the beam is stable.

Eq. (102) can also be used to determine the instability growth rate once we know the machine coupling impedance. For example, if we plot $-\,\mathrm{sign}(\eta)Z_{||,i}$ as a function of $\mathrm{sign}(\eta)Z_{||,r}$ at constant values of ω_i, we obtain the Fig. 7 that represents the stability diagram for zero energy spread. Observe that ω_i is related to the instability rise time by the relation:

$$\omega_i = \frac{1}{\tau} \tag{103}$$

Positive value of τ produce instability and the curves allow to evaluate the rise time once the coupling impedance is known.

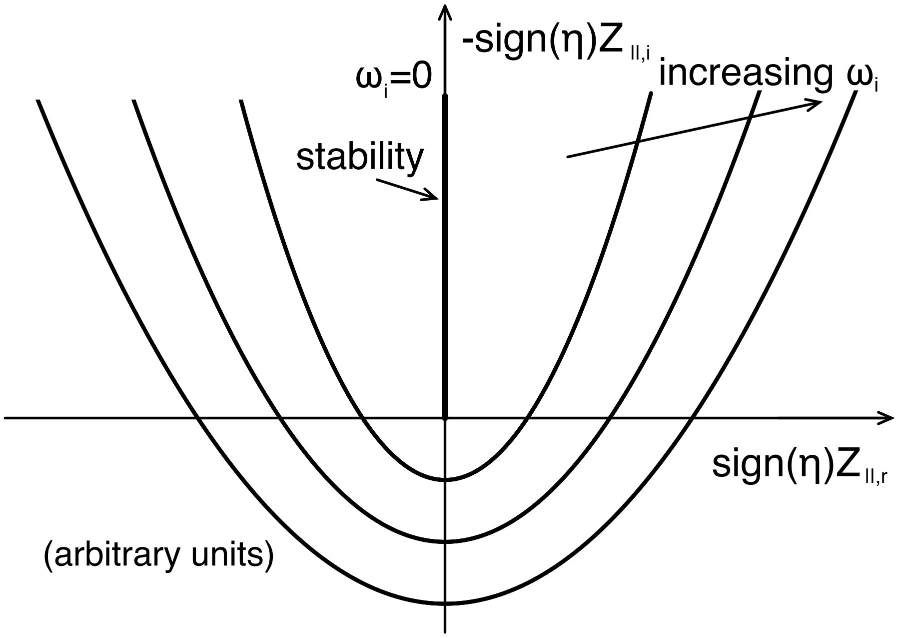

Figure 7: Stability diagram relating growth rate and impedance for zero energy spread (in arbitrary units).

4.5 Beam with energy spread

Till now we have seen that a monochromatic beam is stable only when the machine impedance is purely imaginary with a proper sign. This, however, never occurs, a resistive impedance due to the energy loss of the beam wall interaction can cause instability of the beam. Fortunately, when we take into account also the beam energy spread, the general relation (98), which may produce a Landau damping, holds. Let us consider, for example, the case of a stationary parabolic energy distribution of the kind[18]

$$f_0(\varepsilon) = \frac{3N}{4L_0\varepsilon_m}\left[1 - \left(\frac{\varepsilon}{\varepsilon_m}\right)^2\right] \tag{104}$$

such that the relative energy deviation ranges between $-\varepsilon_m$ and ε_m. Due to the fact that the frequency ω must lie within the frequency spread produced by the energy distribution, if the real part of frequency ω_r lies within the frequency spread given by the above energy distribution, by using eq. (98), we can write

$$1 = -i\frac{3(Z_{||}/n)I_0}{4\pi(E_0/e)\eta\varepsilon_m^2}\left[P.V.\int_{-1}^{1}\frac{x}{y+x}dx + i\pi y\right] \tag{105}$$

with

$$y = \frac{\omega - n\bar{\omega}_0}{n\bar{\omega}_0\eta\varepsilon_m} \tag{106}$$

We remind that the imaginary term in eq. (105) exists only if

$$-1 < Re(y) < 1 \tag{107}$$

The $P.V.$ of the integral can be easily done, and we get

$$1 = -i\frac{3(Z_{||}/n)I_0}{4\pi(E_0/e)\eta\varepsilon_m^2}\left[2 - 2y\,\text{arctanh}\,y + i\pi y\right] \tag{108}$$

This equation relates the frequency ω of the perturbation to the machine coupling impedance. The stability limit condition requires that $\omega_i \leq 0$. In Fig. 8 this is represented by the shaded area. The other curves are obtained from eq. (108) with the condition $\omega_i > 0$ (instability), and they allow to obtain the growth rate of the instability once the machine coupling impedance is known.

If the coupling impedance $Z_{||}$ is inside the stable area, then the coherent oscillation energy of the beam is transferred to the incoherent kinetic energy of a smaller and smaller number of particles inside the beam, thus stabilizing the perturbation. This is the Landau damping effect for the longitudinal instability of coasting beams[19]. Observe that, with respect to the monochromatic beam, here the stability area is expanded in regions where $Z_{||,r} \neq 0$.

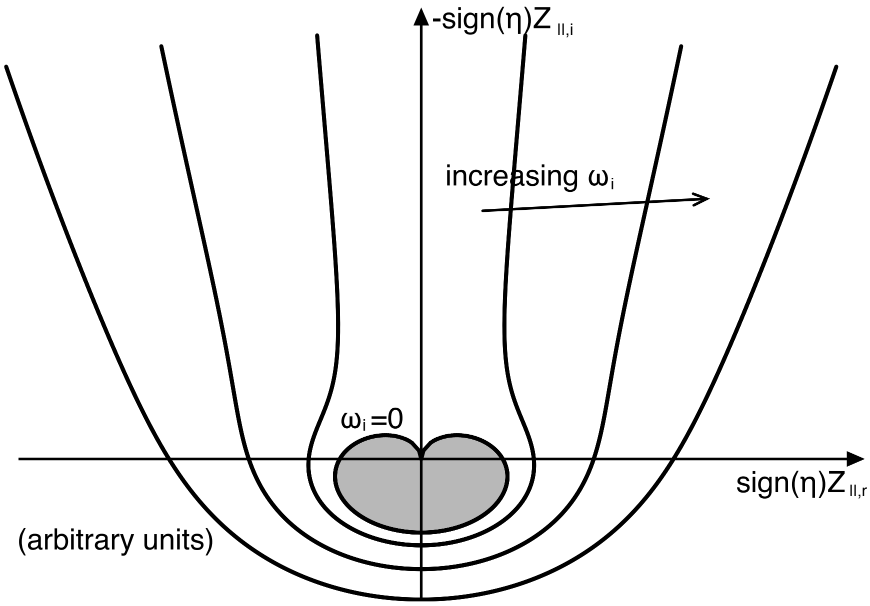

Figure 8: Stability diagram relating growth rate and impedance for a parabolic energy distribution (in arbitrary units).

The curves of Fig. 8 and the shape of the stability limit depend on the energy distribution and in particularly on its edges. Sharp edge distributions, as the parabolic one, are less stable than the ones with long tails, such as the Gaussian distribution[18].

If we consider, as another example, a tri-elliptical energy distribution of the kind[20]

$$f_0(\varepsilon) = \frac{8N}{3\pi L_0 \varepsilon_m} \left[1 - \left(\frac{\varepsilon}{\varepsilon_m} \right)^2 \right]^{3/2} \tag{109}$$

we obtain the dispersion relation

$$1 = -i \frac{4(Z_{||}/n) I_0}{\pi^2 (E_0/e) \eta \varepsilon_m^2} \left[P.V. \int_{-1}^{1} \frac{x\,(1-x^2)^{1/2}}{y+x} dx + i\pi y \left(1-y^2\right)^{1/2} \right] \tag{110}$$

with y given by equation (106). The $P.V.$ of the integral can be easily done, and we get

$$1 = \frac{4(Z_{||}/n)I_0}{\pi(E_0/e)\eta\varepsilon_m^2}\left[y\left(1-y^2\right)^{1/2} - i\left(\frac{1}{2}-y^2\right)\right] \tag{111}$$

The real and imaginary part of the impedance are represented in Fig. 9 as a function of ω with constant $\omega_i > 0$. The curves are similar to those of Fig. 8 except that in this case the stable area is a circle the radius of which can be found by the condition that y be real, from which we get

$$Z_{||,r}/n = -\frac{\pi(E_0/e)\eta\varepsilon_m^2}{4I_0}4y\left(1-y^2\right)^{1/2} \tag{112}$$

and

$$Z_{||,i}/n = -\frac{\pi(E_0/e)\eta\varepsilon_m^2}{4I_0}\left(4y^2-2\right) \tag{113}$$

that is

$$\left|\frac{Z_{||}}{n}\right| = \frac{\pi(E_0/e)|\eta|\varepsilon_m^2}{2I_0} \tag{114}$$

If we substitute ε_m with the half width at half maximum $\varepsilon_{1/2}$, that for the tri-elliptical distribution is

$$\varepsilon_m = \varepsilon_{1/2}\left(1-2^{-2/3}\right)^{-1/2} = 1.64\varepsilon_{1/2} \tag{115}$$

we obtain

$$\left|\frac{Z_{||}}{n}\right| = 0.68\frac{2\pi(E_0/e)|\eta|\varepsilon_{1/2}^2}{I_0} \tag{116}$$

that is known as Kheil - Schnell stability criterion[21, 22].

We can generalize the above equation for other energy distributions by writing a simplified stability criterion

$$\left|\frac{Z_{||}}{n}\right| \leq F\frac{(E_0/e)|\eta|\varepsilon_{1/2}^2}{I_0} \tag{117}$$

with the form factor F, of the order of unity, that determines the radius of the approximating circle.

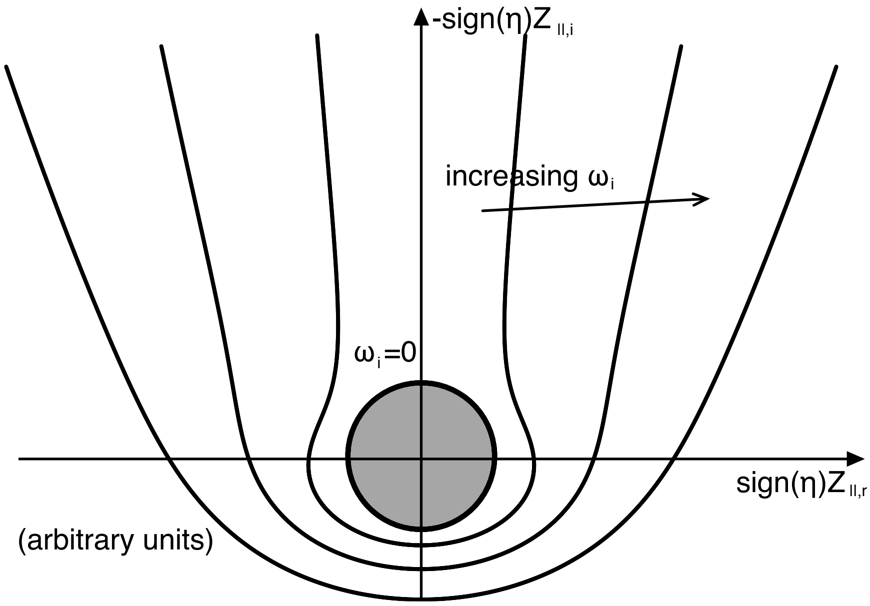

Figure 9: Stability diagram relating growth rate and impedance for a tri-elliptical energy distribution (in arbitrary units).

5 Longitudinal dynamics and Landau damping in bunched beams

The longitudinal beam dynamics of bunched beams, as for the coasting beam case, is described by the Vlasov equation (91), but with different equations of motion. Actually the rate of change of the longitudinal coordinate z is still given by eq. (92), but, due to the presence of the longitudinal focusing force of RF cavities, which is responsible of the synchrotron oscillations, the rate of change of energy is given by the contribution of two terms, one due to the RF and the other due to the wake field, such that, for small synchrotron oscillation amplitudes, instead of eq. (78), we now have

$$\frac{\partial \varepsilon}{\partial t} = \frac{\omega_{s0}^2 z}{\eta c} - \frac{eN}{(E_0/e)T_0} \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} W_{||}(z' + qL_0 - z)dz' \int_{-\infty}^{\infty} f_1(z', \varepsilon')e^{-i\omega(t-qT_0)}d\varepsilon' \quad (118)$$

with ω_{s0} the unperturbed synchrotron frequency.

In writing the dependence on the wake field, we have taken into account either the long term (the summation over q) and the short term (the integral over z') wake fields. It is important to observe that we are considering here the distribution function as the sum of an unperturbed distribution plus a small perturbation of the kind $f_1(z, \varepsilon)e^{-i\omega t}$.

The study of the solution of the Vlasov equation for longitudinal bunched beams with the inclusion of the wake fields goes beyond the scope of this paper. In general the perturbation is written in terms of oscillation modes and the study is essentially focused in finding the mode frequencies and the mode patterns that may lead to beam instability. The mode analysis was primarily developed by F. Sacherer[23] and the interested reader can find several papers in the works of A. W. Chao[24], J. L. Laclare[25], B. Zotter[26], C. Pellegrini[27] and others[28].

The effect of Landau damping on bunched beam dynamics is a complex problem. However, a simplified and approximated expression, similar to the Keil-Schnell stability criterion for the coasting beam, has been proposed by D. Boussard[29] in case of short range wake field and broad band impedance[15]. The idea is that at the high frequency of the signals emitted by the bunch in the instability regime, a bunched beam can be considered as a coasting beam with a current equal to the bunched beam peak current. As a consequence, we can use eq. (116) by substituting I_0 with $\hat{I}$, $\varepsilon_{1/2}^2$ with $2 \ln 2\sigma_\varepsilon^2$ (we consider a Gaussian energy distribution), and, since $0.68 \times 2 \ln 2 = 0.94 \simeq 1$, then we end up with the Boussard criterion

$$\left|\frac{Z_{||}}{n}\right| = \frac{2\pi(E_0/e)|\eta|\sigma_\varepsilon^2}{\hat{I}} \tag{119}$$

The Boussard criterion can be used to give a first evaluation of the threshold single bunch current in a storage ring before the microwave instability occurs[30].

Let us now evaluate the effects of the Landau damping for a more simple case, by considering an instability of N_b equally spaced bunches in a storage ring, produced by a single high order resonant mode at the frequency $p\bar{\omega}_0$ (long range wake field), and by supposing that this resonant mode drives the instability of a single azimuthal beam oscillation mode m. We omit here the details of the calculations, but it is possible to obtain a dispersion integral similar to eq. (97), which assumes now the form[31]

$$1 = -i\frac{mcI_{tot}}{(E_0/e)}\frac{Z_{||}(p\bar{\omega}_0)}{p}\int_0^\infty \frac{\partial f_0}{\partial \hat{z}} J_m^2\left(\frac{p\bar{\omega}_0\hat{z}}{c}\right)\frac{1}{\omega - m\omega_s(\hat{z})}d\hat{z} \tag{120}$$

with $I_{tot} = ceNN_b/L_0$ the total beam current, J_m the Bessel function of the first kind and m-th order, $\omega_s(\hat{z})$ the amplitude dependent synchrotron frequency, and $f_0(\hat{z})$ the unperturbed distribution function expressed in terms of the synchrotron oscillation amplitude $\hat{z}$. If the beam particles have all the same synchrotron frequency, that means $\omega_s(\hat{z})$ is constant, there is no Landau damping: it is actually the spread of the synchrotron frequency the responsible of this effect. The spread is caused, for example, by the non linearities of the RF fields.

The above dispersion relation permits to draw the stability diagrams relating growth rate and impedance for a given distribution function as we did for the coasting beam cases.

For example, in case of a Gaussian distribution, we have

$$f_0(\hat{z}) = \frac{\eta}{2\pi c\omega_{s0}\sigma_{\tau 0}^2}e^{-\frac{\hat{z}^2}{2(c\sigma_{\tau 0})^2}} \tag{121}$$

with $\sigma_{\tau 0}$ the unperturbed temporal bunch length.

In order to solve the integral of the dispersion relation (120) we need to know the dependence of the synchrotron frequency on the oscillation amplitude. If we consider only the non linearities of the RF fields, substituting eq. (121) into (120), and after some manipulations, we obtain[31]

$$1 = i\frac{4\eta I_{tot}}{\pi\omega_{s0}^2(E_0/e)\sigma_{\tau 0}^4 h^2\bar{\omega}_0^2}\frac{Z_{||}(p\bar{\omega}_0)}{p}\int_0^\infty \frac{e^{-x}J_m^2\left(p\bar{\omega}_0\sigma_{\tau 0}\sqrt{2x}\right)}{x-y}dx \qquad (122)$$

with

$$y = -\frac{8(\omega - m\omega_{s0})}{m\omega_{s0}h^2\bar{\omega}_0^2\sigma_{\tau 0}^2} \qquad Re(y) > 0 \qquad (123)$$

and h the RF harmonic number.

As already illustrated for the coasting beam case, the integral (122) has to be executed in the complex x-plane by deforming the contour of the integration, in order to avoid the singularity. In Fig. 10 we show the stability diagram of the impedance for the longitudinal dipole mode $m = 1$ found by imposing $\omega_i = 0$ for the threshold curve and $\omega_i > 0$ for the instability curves. The intersection of the threshold curve with the real axis gives the maximum shunt impedance the resonant mode can have such that the instability growth rate can be contrasted by the Landau damping.

When, instead of the Gaussian longitudinal distribution, we consider a parabolic one, it is possible to see that the threshold curve intersect the real axis in the origin, and in order to remain in the stable region, it is necessary an imaginary part of the impedance. The physical reason is due to the fact that the parabolic distribution has a limited maximum oscillation amplitude, and, as a consequence, also the synchrotron frequency spread is limited. For the Gaussian distribution, in principle the amplitude of the oscillations $\hat{z}$ can be unlimited, the synchrotron frequency spread is much wider, and this has the consequence that also a pure real impedance can be Landau damped.

The synchrotron frequency spread can be increased by adding non linearities to the RF fields. This can be achieved, for example, by inserting

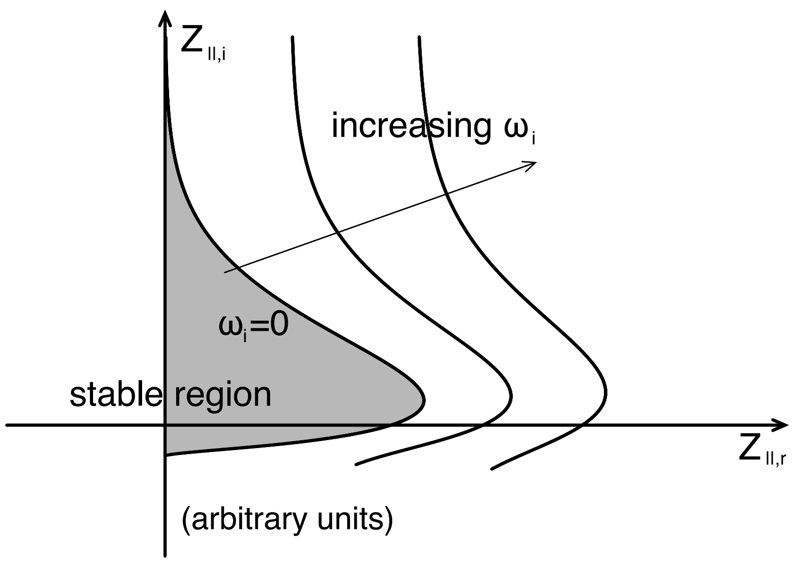

Figure 10: Stability diagram for a Gaussian longitudinal beam distribution (in arbitrary units) in case of a high order resonant mode for a dipole oscillation mode.

in a storage ring a higher harmonic cavity that is generally used to control the bunch length[32]. It has been shown[33] that in presence of a higher harmonic cavity, the Landau damping is amplified, on the real part of the impedance, that is on the real axis of Fig. 10, approximatively by a factor

$$\left|1 \pm \frac{kn^3}{\sin \phi_{s0}}\right| \tag{124}$$

where the sign $\pm$ depends if the higher harmonic cavity is operated in the shortening (+ sign) or lengthening ($-$ sign) regime, k is the ratio between the peak voltages of the main and the higher harmonic cavities, n is the harmonic cavity number, and $\sin \phi_{s0}$ is the main voltage synchronous phase.

As example, in Fig. 11 we show a qualitative comparison of the stability curves with and without the presence of the higher harmonic cavity by considering a third harmonic cavity with a peak voltage about one fifth of the main RF voltage working in the shortening regime.

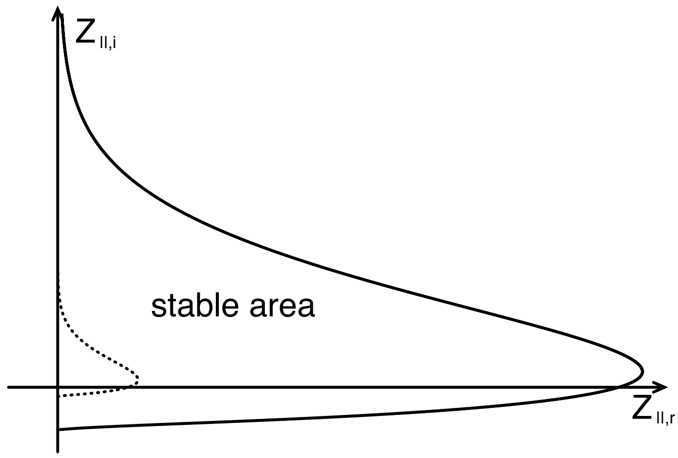

Figure 11: Stability diagram for a Gaussian longitudinal beam distribution (in arbitrary units) in case of a high order resonant mode for a dipole oscillation mode in presence of a third harmonic cavity compared to the case without the harmonic cavity (dotted curve).

6 Transverse dynamics and Landau damping in coasting beams

The study of transverse beam dynamics follows very closely the longitudinal one: we will define in section 6.1 the transverse wake field, and in section 6.2 we will introduce the transverse equations of motion that will be used in section 6.3 in the Vlasov equation to obtain the dispersion relation for the transverse case. In the following sections 6.4 and 6.5 we will apply the dispersion relation to some distribution functions in analogy with what we have done for the longitudinal coasting beam dynamics.

6.1 Transverse beam wall interaction

To study the transverse beam wall interaction we introduce here a transverse electromagnetic force produced, as for the longitudinal wake field, by the perturbed distribution function. Before doing that, we have to define a new couple of variables that describe the transverse particle motion. Here we will adopt a simplified model where particles execute simple transverse harmonic oscillations around the reference orbit. This is the case where the accelerator focusing term is constant. Although this condition is never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities. Let's call x the transverse position of a particle with respect to the reference orbit. In order to study its motion in the transverse phase space, it is convenient to adopt the following complex notations:

$$x = \hat{x} e^{i\theta} \tag{125}$$

$$v_x = \frac{d\hat{x}}{dt} e^{i\theta} - i\omega_\beta \hat{x} e^{i\theta} \tag{126}$$

where

$$\frac{\partial \theta}{\partial t} = -\omega_\beta \tag{127}$$

is the betatron frequency.

The distribution function in this case depends either on the longitudinal and transverse coordinates $\psi(z, \varepsilon, \theta, \hat{x})$, and we define it such that its integration gives the total number of particles in the beam

$$\int\int\int\int \psi(z, \varepsilon, \theta, \hat{x})\, dz d\varepsilon d\theta d\hat{x} = N \tag{128}$$

Here again we use a perturbation formalism by writing $\psi = \psi_0 + \psi_1$. The unperturbed distribution function can be factorized due to the uncoupled longitudinal and transverse motion, and due to the machine symmetry, it does not depend on the longitudinal coordinate z and on the angular coordinate θ, so that

$$\psi_0(\varepsilon, \hat{x}) = f_0(\varepsilon) g_0(\hat{x}) \tag{129}$$

Also the perturbation is considered uncoupled, and, as for the longitudinal case of equation (90), we use a sinusoidal wave traveling along the ring, that allows us to write

$$\psi_1(z, \varepsilon, \theta, \hat{x}, t) = f_1(\varepsilon) g_1(\theta, \hat{x}) e^{i(ks - \omega t)} \tag{130}$$

with $s = z + ct$.

In order to define the transverse wake function, as for the longitudinal case, we consider the transverse electromagnetic force produced by a charge q and acting on a charge q_1, averaged along the accelerator, as a transverse deflecting kick divided by the machine length

$$\langle F_\perp(x, \Delta z) \rangle = \frac{1}{L_0} \int_0^{L_0} F_\perp ds \tag{131}$$

Observe that the transverse force depends on the longitudinal distance Δz between the two charges and on the transverse position of the test charge q_1. From the average transverse force we define the transverse wake field as

$$W_\perp(x, \Delta z) = -\frac{\langle F_\perp(x, \Delta z) \rangle L_0}{q q_1} \tag{132}$$

Similarly to the longitudinal case, the transverse force acting on a charge and produced by the whole perturbed distribution function can be written in terms of transverse wake function of the dipole moment of the beam as

$$F_\perp = -\frac{e^2}{L_0} \frac{\int c dt' \int d\varepsilon \int d\hat{x} \int d\theta f_1(\varepsilon) g_1(\theta, \hat{x}) e^{i(ks - \omega t')} W_\perp(x, ct' - ct)}{\int \int g_0(\hat{x}) d\hat{x} d\theta} \tag{133}$$

The transverse wake function is linearized for small transverse displacements as

$$W_\perp(x, \Delta z) = W'_\perp(\Delta z) x \tag{134}$$

so that we get

$$F_\perp = -\frac{e^2}{L_0} e^{iks} \frac{\int W'_\perp(ct' - ct) e^{-i\omega t'} c dt' \int f_1(\varepsilon) d\varepsilon \int \int x g_1(\theta, \hat{x}) d\hat{x} d\theta}{\int \int g_0(\hat{x}) d\hat{x} d\theta} \tag{135}$$

The term

$$\frac{\int\int xg_1(\theta,\hat{x})\,d\hat{x}d\theta}{\int\int g_0(\hat{x})\,d\hat{x}d\theta} = D \tag{136}$$

is, by definition, the dipole displacement D of the distribution, so that

$$F_\perp = -\frac{e^2 D}{L_0} e^{iks} \int W'_\perp(ct' - ct) e^{-i\omega t'} c dt' \int f_1(\varepsilon) d\varepsilon \tag{137}$$

By changing the variable $(ct' - ct)$ in y and using the variable z instead of s we end up with a transverse force produced by the perturbation of the kind

$$F_\perp = -\frac{e^2 D}{L_0} e^{i[kz-(\omega-n\bar{\omega}_0)t]} \int W'_\perp(y) e^{-i\omega y/c} dy \int f_1(\varepsilon) d\varepsilon \tag{138}$$

If we write the force as a function of the transverse coupling impedance per unit of displacement, defined, for ultra-relativistic charges, as

$$Z'_\perp(\omega) = i \int W'_\perp(\tau) e^{-i\omega\tau} d\tau \tag{139}$$

we obtain

$$F_\perp = i\frac{e^2 Dc}{L_0} e^{i[kz-(\omega-n\bar{\omega}_0)t]} Z'_\perp(\omega) \int f_1(\varepsilon) d\varepsilon \tag{140}$$

As we did for the longitudinal case, we show here some common transverse coupling impedances per unit of displacement that are generally found in a particle accelerator[10, 15]. For a perfectly conducting smooth and circular vacuum chamber of radius b, the space charge effect due to the non relativistic velocity of the charges gives

$$Z'_\perp(\omega) = iZ_0 \frac{R_0}{(\beta\gamma)^2} \left(\frac{1}{r^2} - \frac{1}{b^2}\right) \tag{141}$$

with r the transverse position of the test charge q_1.

For the resistive wall of the circular pipe with beam at center and high conductivity σ_c, such that $c^2/(\omega^2 b)$ and b are much bigger that the skin depth, we have

$$Z'_\perp(\omega) = \frac{R_0}{b^3} \sqrt{\frac{2cZ_0}{\sigma_c|\omega|}} \left(\mathrm{sign}(\omega) - i\right) \tag{142}$$

For a resonating mode of a cavity, the transverse coupling impedance can be written as

$$Z'_{\perp}(\omega) = \frac{c}{\omega}\frac{R_s/b^2}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)} \tag{143}$$

with R_s the shunt resistance, Q the quality factor, and ω_r the resonant frequency.

6.2 Transverse dynamics in coasting beams

The transverse equation of motion of a single particle in a circular accelerator in the linear regime is derived from Newton's second law. The betatron equation of motion is influenced by the transverse deflecting force $F_{\perp}$ produced by the wake field and can be written as

$$\frac{\partial^2 x}{\partial t^2} + \omega_\beta^2 x = \frac{c^2}{E_0} F_{\perp} \tag{144}$$

Actually the transverse wake field is also responsible of a longitudinal force that changes the particle energy. Here we ignore this effect. Therefore, while the betatron motion is affected by the transverse wake, the longitudinal beam dynamics is unperturbed[9]. This has the consequence that the particle energy is considered constant during the transverse motion. However, due to the energy spread characterizing the ensemble of particles, the betatron frequency of a charge with energy deviation ε is given by the relation

$$\omega_\beta = \omega_{\beta 0}\left(1 + \xi\varepsilon\right) \tag{145}$$

with $\omega_{\beta 0}$ the betatron frequency at the nominal energy E_0 and ξ the chromaticity defined by

$$\xi = \frac{\Delta\omega_\beta}{\omega_{\beta 0}}\frac{E_0}{\Delta E} \tag{146}$$

[9]The consequence of ignoring the perturbation of the dipole moment of the beam on the longitudinal motion is that the system is not rigorously Hamiltonian. Anyway this is a good approximation if the transverse and the longitudinal motions are uncoupled.

In case of free oscillations ($F_\perp = 0$), the particle executes an harmonic motion which is described by a circumference in the phase space $(x, v_x/\omega_\beta)$. The action of the force will modify the particle dynamics, affecting in general both amplitude and betatron frequency. Since we are dealing with dipole forces, we assume that only the motion amplitude is affected.

By inserting eq. (125) into the equation of motion (144), we get

$$\frac{\partial^2 \hat{x}}{\partial t^2} - 2i\omega_\beta \frac{\partial \hat{x}}{\partial t} = \frac{c^2}{E_0} e^{-i\theta} F_\perp \tag{147}$$

Since we are dealing with a perturbation of the transverse force on the particle motion, this perturbation changes the oscillation amplitude $\hat{x}$ with time, but we ignore its second order derivative and write[10]

$$\frac{\partial \hat{x}}{\partial t} \simeq i \frac{c^2}{2\omega_{\beta 0} E_0} e^{-i\theta} F_\perp \tag{148}$$

In writing the above perturbed equation we have also ignored the chromaticity effect on the betatron frequency (this is a good approximation only for this perturbed equation).

As concerning the longitudinal motion of the single particle, we use eq. (92) that can be written as

$$\frac{\partial z}{\partial t} = -\eta c \epsilon \tag{149}$$

for the longitudinal coordinate, and

$$\frac{\partial \varepsilon}{\partial t} = 0 \tag{150}$$

for the energy, since, as we have already said, we ignore the longitudinal force produced by the transverse wake field.

6.3 Dispersion relation of transverse coasting beam

By using the distribution function (128), the transverse Vlasov equation reads

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \psi}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t} = 0 \tag{151}$$

[10] This is equivalent to ignore the betatron frequency shift produced by the wake field.

If we substitute in eq. (151) the single particle equations of motion (127) and (148) - (150), we get

$$\frac{\partial \psi}{\partial t} - \eta c \epsilon \frac{\partial \psi}{\partial z} - \omega_\beta \frac{\partial \psi}{\partial \theta} + i \frac{c^2}{2\omega_{\beta 0} E_0} e^{-i\theta} F_\perp \frac{\partial \psi}{\partial \hat{x}} = 0 \tag{152}$$

By introducing the transverse force (140) in the Vlasov equation (151) with the distribution function given by eqs. (129) and (130), ignoring second order terms in the perturbation, and by simplifying the exponential term, we end up with

$$- i(\omega - n\bar{\omega}_0) f_1(\varepsilon) g_1(\hat{x}, \theta) - ik\eta c \varepsilon f_1(\varepsilon) g_1(\hat{x}, \theta) - \omega_\beta f_1(\varepsilon) \frac{\partial g_1(\hat{x}, \theta)}{\partial \theta} = \frac{e^2 D c^3}{2\omega_{\beta 0} E_0 L_0} Z'_\perp(\omega) e^{-i\theta} f_0(\varepsilon) \frac{\partial g_0(\hat{x})}{\partial \hat{x}} \int f_1(\varepsilon) d\varepsilon \tag{153}$$

In order to find a solution of this equation for the transverse perturbed distribution, let us suppose that the transverse variables $\hat{x}$ and θ are uncoupled. Therefore $g_1(\hat{x}, \theta)$ is the product of two functions

$$g_1(\hat{x}, \theta) = X(\hat{x}) \Theta(\theta) \tag{154}$$

that, substituted in eq. (153), allows to find easily that

$$X(\hat{x}) = C \frac{\partial g_0(\hat{x})}{\partial \hat{x}} \tag{155}$$

and

$$\Theta(\theta) = e^{-i\theta} \tag{156}$$

with C a constant to be related to the dipole displacement D of the distribution. In fact, by using eq. (136) with $g_1(\hat{x}, \theta) = C \frac{\partial g_0(\hat{x})}{\partial \hat{x}} e^{-i\theta}$ we obtain

$$D = \frac{\int\int x g_1(\theta, \hat{x})\, d\hat{x} d\theta}{\int\int g_0(\hat{x})\, d\hat{x} d\theta} = C \frac{\int\int \hat{x} e^{-i\theta} \partial g_0(\hat{x}) / \partial \hat{x} e^{i\theta} d\hat{x} d\theta}{\int\int g_0(\hat{x})\, d\hat{x} d\theta} = C \frac{2\pi \int \hat{x} \partial g_0(\hat{x}) / \partial \hat{x} d\hat{x}}{2\pi \int g_0(\hat{x})\, d\hat{x}} = C \frac{[g_o(\hat{x})\, \hat{x}]_0^\infty - \int g_0(\hat{x})\, d\hat{x}}{\int g_0(\hat{x})\, d\hat{x}} = -C \tag{157}$$

so that we end up with

$$g_1(\hat{x}, \theta) = -D \frac{\partial g_0(\hat{x})}{\partial \hat{x}} e^{i\theta} \tag{158}$$

that, inserted in the eq. (153), allows us to obtain

$$(\omega - n\bar{\omega}_0 + k\eta c\varepsilon - \omega_\beta) f_1(\varepsilon) = -i \frac{e^2 c^3}{2\omega_{\beta 0} E_0 L_0} Z'_\perp(\omega) f_0(\varepsilon) \int f_1(\varepsilon) d\varepsilon \tag{159}$$

or, equivalently,

$$f_1(\varepsilon) = -i \frac{c^2 I_0}{2N\omega_{\beta 0}(E_0/e)} Z'_\perp(\omega) \frac{f_0(\varepsilon)}{\omega - n\bar{\omega}_0(1 - \eta\varepsilon) - \omega_{\beta 0}(1 + \xi\varepsilon)} \int f_1(\varepsilon') d\varepsilon' \tag{160}$$

As for the longitudinal case, we evaluate the transverse coupling impedance at the unperturbed frequency $n\bar{\omega}_0 + \omega_{\beta 0}$, and, by integration over ε and simplifying $\int f_1(\varepsilon) d\varepsilon$, we finally obtain the dispersion relation

$$1 = -i \frac{c^2 I_0 Z'_\perp(n\bar{\omega}_0 + \omega_{\beta 0})}{2N\omega_{\beta 0}(E_0/e)} \int \frac{f_0(\varepsilon) d\varepsilon}{\omega - n\bar{\omega}_0(1 - \eta\varepsilon) - \omega_{\beta 0}(1 + \xi\varepsilon)} \tag{161}$$

The difference of this dispersion integral with respect to the longitudinal one given by eq. (97) is mainly due to the presence here of the unperturbed distribution function and not to its energy derivative. This will have as a major consequence, as we will see, that for the monochromatic beam, even without the presence of the Landau damping, a real positive transverse coupling impedance does not produce any instability. We remember that in the longitudinal case any resistive component of the longitudinal impedance produced instability. This characteristic can be traced back to the fact that in the longitudinal motion of a coasting beam there is no external focusing as in the transverse case[34].

6.4 Monochromatic beam

We proceed similarly to the longitudinal case by analyzing firstly the case of a monochromatic beam with the stationary energy distribution function given by eq. (99). By substituting this function into the dispersion integral we easily obtain

$$1 = -i \frac{c^2 I_0 Z'_\perp}{2\omega_{\beta 0}(E_0/e) L_0} \frac{1}{\omega - n\bar{\omega}_0 - \omega_{\beta 0}} \tag{162}$$

from which we get the complex frequency of the perturbation

$$\omega = n\bar{\omega}_0 + \omega_{\beta 0} - i \frac{c^2 I_0 Z'_\perp}{2\omega_{\beta 0}(E_0/e) L_0} \tag{163}$$

The instability rise time is given by the inverse of the imaginary part of this frequency, that is

$$\frac{1}{\tau} = \omega_i = -\frac{c^2 I_0}{2\omega_{\beta 0}(E_0/e) L_0} Re\left(Z'_\perp\right) \tag{164}$$

Positive values of the rise time gives instability. In this case it is easy to evaluate also the frequency shift $Re(\omega) - n\bar{\omega} - \omega_{\beta 0}$:

$$\Delta\omega = \frac{c^2 I_0}{2\omega_{\beta 0}(E_0/e) L_0} Im\left(Z'_\perp\right) \tag{165}$$

Differently from the longitudinal case, the real part of the transverse impedance can produce either stability and instability, whilst the imaginary part produces a frequency shift. As a consequence, if we plot $Z'_{\perp,i}$ as a function of $Z'_{\perp,r}$ at constant values of ω_i we obtain vertical lines, each one corresponding to a given instability rise time, in the part of the complex plane $Z'_{\perp,r}, Z'_{\perp,i}$ with negative real axis. The stable region, differently from the longitudinal counterpart, is not only the imaginary positive axis of Fig. 7, but all the half plane with $Z'_{\perp,r} > 0$, as shown in Fig. 12. For the transverse plane we do not find the equivalent of the negative mass instability of the longitudinal plane.

As an application, let us consider the transverse resistive wall impedance of a circular pipe of radius b given by eq. (142). By substituting this impedance

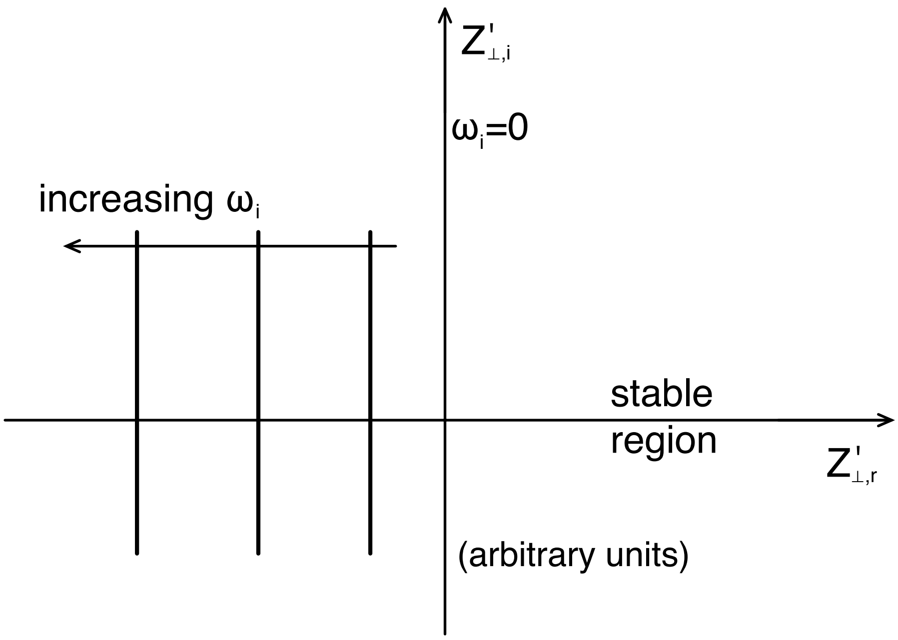

Figure 12: Stability diagram relating growth rate and impedance for zero energy spread (in arbitrary units) for transverse coasting beam.

in (163), we get

$$\omega = n\bar{\omega}_0 + \omega_{\beta 0} - \frac{c^2 I_0}{2\pi\omega_{\beta 0}(E_0/e)b^3} \sqrt{\frac{cZ_0}{2\sigma_c\,|n\bar{\omega}_0 + \omega_{\beta 0}|}}\,(1 + i\,\mathrm{sign}\,(n\bar{\omega}_0 + \omega_{\beta 0})) \quad (166)$$

from which we obtain the instability growth rate

$$\frac{1}{\tau} = -\frac{c^2 I_0}{2\pi\omega_{\beta 0}(E_0/e)b^3}\sqrt{\frac{cZ_0}{2\sigma_c\,|n\bar{\omega}_0 + \omega_{\beta 0}|}}\,\mathrm{sign}\,(n\bar{\omega}_0 + \omega_{\beta 0}) \quad (167)$$

The most dominant oscillation mode is given by n such that $|n\bar{\omega}_0 + \omega_{\beta 0}|$ is closer to zero. If we write $\omega_{\beta 0} = (N_\beta + \nu_\beta)\bar{\omega}_0$ (note that the quantity $N_\beta + \nu_\beta$ represents the betatron tune number), then the oscillation mode that has the highest growth rate is that with $n = -N_\beta$. For this mode the growth rate is

$$\frac{1}{\tau} = -\frac{c^2 I_0}{2\pi\omega_{\beta 0}(E_0/e)b^3}\sqrt{\frac{cZ_0}{2\sigma_c\,|\nu_\beta\bar{\omega}_0|}}\,\mathrm{sign}\,(\nu_\beta\bar{\omega}_0) \quad (168)$$

The above equation shows that if we are below the integer ($\nu_\beta < 0$) the growth rate is positive and the beam unstable.

6.5 Beam with energy spread

The effect of Landau damping produced by the energy spread is to increase the stability area by expanding it partially in the negative real axis of the impedance. As for the longitudinal case, let us consider the case of a stationary parabolic energy distribution given by eq. (104). By substituting this distribution in eq. (161), executing the integral in the complex ε-plane by deforming the contour of the integration in order to avoid the singularity as we have already seen, we obtain

$$1 = -i\frac{3}{8}\frac{c^2 I_0 Z'_\perp(n\bar{\omega}_0 + \omega_{\beta 0})}{\omega_{\beta 0}(E_0/e)L_0\varepsilon_m}\frac{1}{n\bar{\omega}_0\eta - \omega_{\beta 0}\xi}$$
$$\left[P.V.\int_{-1}^{1}\frac{1-x^2}{y+x}dx - i\pi(1-y^2)\right] \quad (169)$$

with

$$y = \frac{\omega - n\bar{\omega}_0 - \omega_{\beta 0}}{(n\bar{\omega}_0\eta - \omega_{\beta 0}\xi)\varepsilon_m} \quad (170)$$

The *P.V.* of the integral can be easily done, and we get

$$1 = -i\frac{3}{8}\frac{c^2 I_0 Z'_\perp(n\bar{\omega}_0 + \omega_{\beta 0})}{\omega_{\beta 0}(E_0/e)L_0\varepsilon_m}\frac{1}{n\bar{\omega}_0\eta - \omega_{\beta 0}\xi}$$
$$\left[2(y + \operatorname{arctanh} y - y^2 \operatorname{arctanh} y) - i\pi(1-y^2)\right] \quad (171)$$

As for the longitudinal case, this equation relates the frequency ω of the perturbation to the machine coupling impedance. The stability condition requires that $\omega_i = 0$. In Fig 13 this is represented by the shaded area. The other curves are obtained from eq. (171) with constant $\omega_i > 0$ (instability), and they allow to obtain the growth rate of the instability once the machine coupling impedance is known. We can see that all the half plane with $Z'_{\perp,r} > 0$ is still stable, and the stable region is expanded a bit also in the negative plane with $Z'_{\perp,r} < 0$.

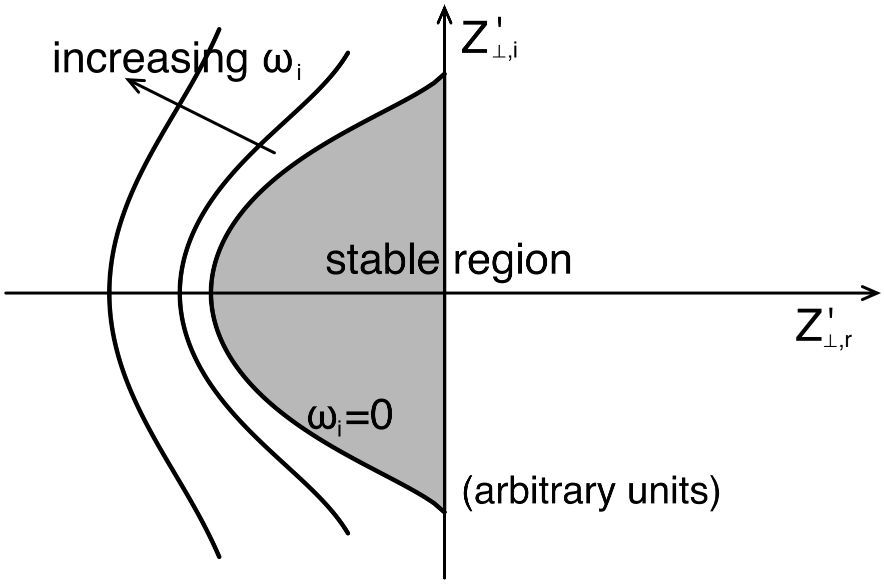

Figure 13: Stability diagram relating growth rate and impedance for a parabolic energy distribution (in arbitrary units) for transverse coasting beam.

In eq. (171) it appears the ration between the transverse coupling impedance per unit of displacement and the quantity $n\bar{\omega}_0\eta - \omega_{\beta 0}\xi$. If the coupling impedance is fixed (for example once the machine is given), the smaller value of $n\bar{\omega}_0\eta - \omega_{\beta 0}\xi$ produces the higher ω_i and the stronger instability. Therefore the most dangerous oscillation mode is that for which

$$n\bar{\omega}_0\eta - \omega_{\beta 0}\xi \simeq 0 \tag{172}$$

that is

$$n = \frac{\omega_{\beta 0}\xi}{\bar{\omega}_0\eta} \tag{173}$$

If the real part of the transverse impedance for this mode is positive, $Z'_{\perp,r}(n\bar{\omega}_0 + \omega_{\beta 0}) > 0$, that is if $(n\bar{\omega}_0 + \omega_{\beta 0}) > 0$, then this dangerous mode remains stable.

In the transverse case too it is possible to find a simplified criterion for the stability similar to the Kheil - Schnell criterion of the longitudinal coasting beam. Let us consider an elliptical energy distribution

$$f_0(\varepsilon) = \frac{2N}{\pi L_0 \varepsilon_m} \left[1 - \left(\frac{\varepsilon}{\varepsilon_m} \right)^2 \right]^{1/2} \tag{174}$$

that gives the dispersion relation

$$1 = -i \frac{c^2 I_0 Z'_\perp (n\bar{\omega}_0 + \omega_{\beta 0})}{\pi \omega_{\beta 0} (E_0/e) L_0 \varepsilon_m} \frac{1}{n\bar{\omega}_0 \eta - \omega_{\beta 0} \xi} \left[P.V. \int_{-1}^{1} \frac{(1-x^2)^{1/2}}{y+x} dx - i\pi (1-y^2)^{1/2} \right] \tag{175}$$

with y given by eq. (170)

By doing the $P.V.$ of the integral, we obtain

$$1 = -\frac{c^2 I_0 Z'_\perp (n\bar{\omega}_0 + \omega_{\beta 0})}{\omega_{\beta 0} (E_0/e) L_0 \varepsilon_m} \frac{1}{n\bar{\omega}_0 \eta - \omega_{\beta 0} \xi} \left[(1-y^2)^{1/2} + iy \right] \tag{176}$$

Is is easy to see that the stable area in this case is a circle the radius of which can be found by the condition that y be real, from which we get the absolute value of the transverse impedance

$$|Z'_\perp (n\bar{\omega}_0 + \omega_{\beta 0})| = \frac{\omega_{\beta 0} (E_0/e) L_0 \varepsilon_m}{c^2 I_0} (n\bar{\omega}_0 \eta - \omega_{\beta 0} \xi) \tag{177}$$

This equation is the transverse counterpart of (114).

7 Transverse dynamics and Landau damping in bunched beams

The transverse beam dynamics of bunched beams follows very closely that of coasting beams, with the difference that there is an energy variation with time due to the synchrotron oscillations, even by ignoring the longitudinal force produced by the transverse wake field, so that eq. (150) becomes here

$$\frac{\partial \varepsilon}{\partial t} = \frac{\omega_{s0}^2 z}{\eta c} \tag{178}$$

As a consequence the Vlasov equation (152) has one more term and it reads now

$$\frac{\partial\psi}{\partial t} - \eta c\epsilon\frac{\partial\psi}{\partial z} + \frac{\omega_{s0}^2 z}{\eta c}\frac{\partial\psi}{\partial\varepsilon} - \omega_\beta\frac{\partial\psi}{\partial\theta} + i\frac{c^2}{2\omega_{\beta 0}E_0}e^{-i\theta}F_\perp\frac{\partial\psi}{\partial\hat{x}} = 0 \tag{179}$$

with the distribution function given by

$$\psi(z,\varepsilon,\theta,\hat{x},t) = f_0(z,\varepsilon)g_0(\hat{x}) + f_1(z,\varepsilon)g_1(\theta,\hat{x})e^{-i\omega t} \tag{180}$$

Observe that the unperturbed longitudinal distribution function f_0 depends also on the longitudinal coordinate z, differently from the coasting beam case of eq. (129). Moreover, the longitudinal perturbation is not a sinusoidal wave traveling along the ring as in eq. (130), and thus we have generically indicated it as $f_1(z,\varepsilon)$.

Also the transverse force produced by the perturbation differs slightly from that of the coasting beam case. Instead of eq. (137) we have in this case

$$F_\perp = -\frac{e^2 D}{L_0}\sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} W'_\perp(z'+qL_0-z)dz'\int_{-\infty}^{\infty} f_1(z',\varepsilon')e^{-i\omega(t-qT_0)}d\varepsilon' \tag{181}$$

The procedure to study the transverse bunched beam instabilities is very similar to that of the longitudinal bunched beam case (see, e.g. [23]). The longitudinal perturbation is expanded in terms of azimuthal oscillation modes that, inserted into the Vlasov equation (179), result in an eigenvalue problem that goes beyond the scope of this paper.

In order to simplify the study and to put in evidence the effect of Landau damping, let us consider the situation in which we have a single high order resonant transverse mode that excites an instability of a single azimuthal oscillation mode m, as we did for the longitudinal bunched beam case. It is possible to demonstrate that the dispersion relation becomes

$$1 = -i\frac{\pi c I_{tot}\omega_{s0}}{L_0\omega_{\beta 0}\eta(E_0/e)}Z'_\perp(\omega')\int_0^\infty \frac{f_0(\hat{z})J_m^2\left(\frac{\omega'-\omega_\xi}{c}\hat{z}\right)}{\omega-\omega_\beta-m\omega_s}\hat{z}d\hat{z} \tag{182}$$

with

$$\omega' = p\bar{\omega}_0 + \omega_\beta + m\omega_s \tag{183}$$

and

$$\omega_\xi = \frac{\xi\omega_{\beta 0}}{\eta} \tag{184}$$

The above dispersion relation has to be compared with that given by eq. (120). Here we can observe that, with the exception of the $m = 0$ mode, there is a contribution to the transverse Landau damping also from a possible spread on the synchrotron frequencies ω_s. Moreover the betatron frequency energy dependent spread produced, for example, by sextupoles, and given by eq. (145), helps in damping transverse instabilities, but, in order to obtain an even more efficient betatron frequency spread, the so called Landau damping octupoles are sometimes used: they introduce an amplitude frequency dependence which is thought to be more effective than the energy dependent betatron frequency spread produced by sextupoles[35].

Once we know the dependence of the betatron and the synchrotron frequencies on the longitudinal oscillation amplitude, given a longitudinal unperturbed distribution function, the dispersion relation (182) allows to obtain the threshold values of the high order resonator mode impedance that can be Landau damped.

A Some considerations about the singularities

The Laplace transform of the electric field (38) is defined for $Re(p) > 0$, it is however possible to define $\mathcal{E}_x(k, p)$ also for $Re(p) < 0$ as analytical continuation of the equation.

We observe that in the Laplace transform of the electric field and of the perturbation we have integrals of the type

$$\int \frac{y(v_k)}{ip - kv_k} dv_k \tag{185}$$

that are functions of p and can be extended to the complex plane $Re(p) < 0$. Furthermore, if $y(v_k)$ does not contain singularities for finite v_k, then the above integral defines entire functions (with no singularities). If we now regard v_k as a complex variable of the kind $v_k = Re(v_k) + iIm(v_k)$, a singular point of the above integral is $p = -ikv_k$. Since for doing the Laplace inverse transform we must consider a vertical line γ such that all the poles are on the left of the line, that is $\gamma > Re(-ikv_k)$, then $Im(v_k) < \gamma/k$, that is in the complex v_k-plane we must consider a horizontal line such that all the poles are above it.

The integral

$$\int \frac{y(v_k)}{ip - kv_k} dv_k \tag{186}$$

can be done on the real axis of v_k even if the pole has the imaginary negative part, by deforming the path of integration passing below the pole as shown in Fig. 2. In this way the integral has no singularities and is defined in all the complex $p-$plane, and $\mathcal{E}_x(k,p)$ is a function equal to the ratio of two entire functions, and, as a consequence, the only poles are the roots of the denominator, that is of the dispersion relation.

Moreover we have that all the poles must be in the half plane of the p-plane such that $Re(p) < 0$. This is due to physical reasons: the poles p_k of $\mathcal{E}_x(k,p)$ determine the behavior of the inverse Laplace transform of the kind $e^{p_k t}$ that cannot increase indefinitely with time.

Rather than trying to obtain a general expression for the electric field, all these considerations allow us to determine the behavior of the electric field at very large times.

Since $\mathcal{E}_x(k,p)$ is now defined in all the p-plane, we can deform the path of integration to the left plane by turning around all the poles (the zeroes of the dispersion relation) as in Fig. 14. Let us consider the pole closer to the imaginary axis, that is less influenced by the time. The inverse Laplace transform (41) then contains a principal value that represents the line integral along the vertical axis $Re(p) = -\sigma$ and is proportional to $e^{-\sigma t}$ that converges rapidly, plus the residuals, the most important of which is p_k with the real part negative.

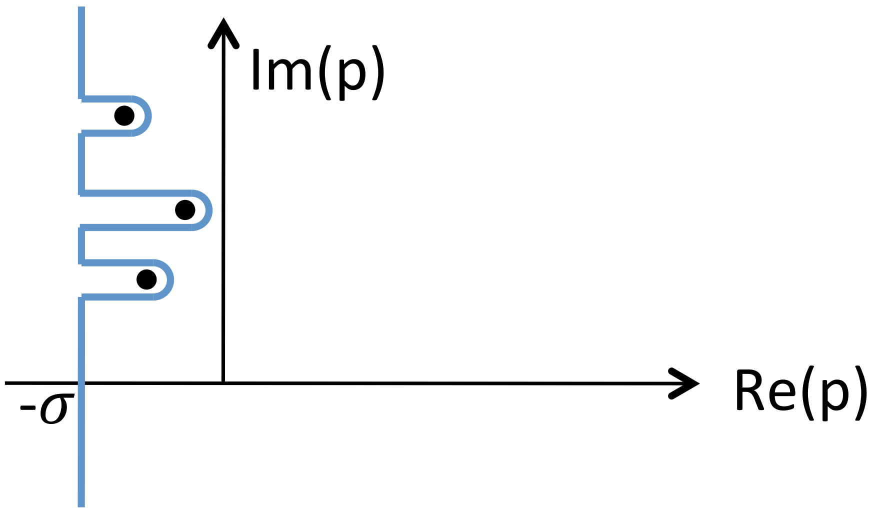

Figure 14: Distorted path of integration in the p-plane.

The result is that the electric field decays exponentially fast with time at large times. This exponential decays is the Landau damping[11].

By doing the same calculations for the perturbation, we obtain that this function is composed by two terms, one that has a similar behavior of the electric field, and the other that oscillates as $e^{-ikv_k t}$ and is not damped.

[11]It is important to underline that only the electric field decays exponentially due to the Landau damping, while the perturbed distribution function contains an undamped oscillating term.

B Simple physical description

In order to get a physical interpretation of the Landau damping, we consider a neutral plasma with non-relativistic electrons characterized by a Maxwellian velocity distribution as shown Fig. 15, and analyze the system from the energetic point of view. Let us consider a perturbation in the electron distribution such that a plasma wave propagates with a phase velocity v_{ph}. The electrons having velocities approximately equal to v_{ph} will interact more strongly with the wave, yielding energy to the wave and being decelerated by the wave electric field if they are faster than v_{ph}, gaining energy and being accelerated by the wave electric field if they are slightly slower than v_{ph}. Since v_{ph} is in the negative slope of the velocity distribution function, the number of "slower" electrons is greater than the number "faster" electrons. Hence, there are more particles gaining energy from the wave than losing to the wave, which leads to wave damping.

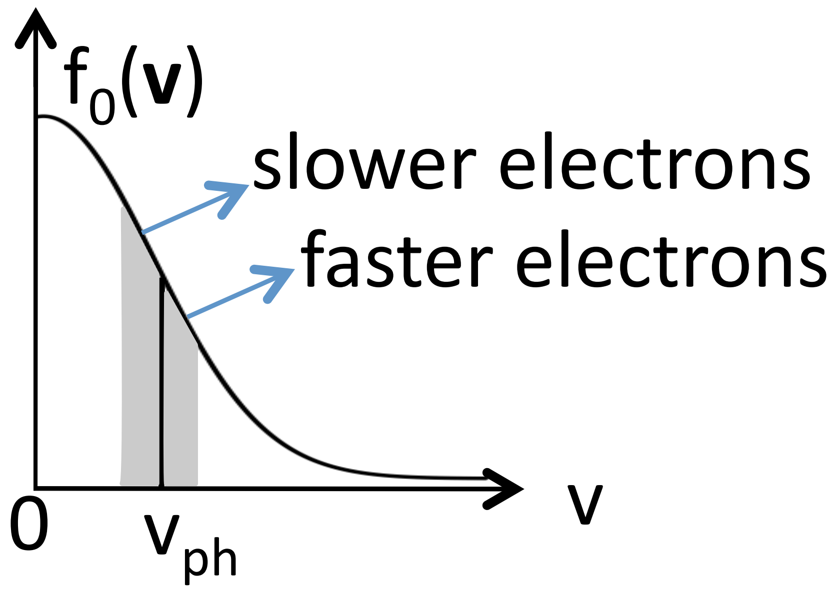

Figure 15: Maxwellian velocity distribution function.

A similarity that helps to visualize the phenomenon of Landau damping, although not strictly correct[36], is the following: let us imagine plasma waves as waves in the sea, and the particles as surfers trying to catch the wave, all moving in the same direction. If the surfer is moving on the water surface at a velocity slightly less than the waves, he will eventually be caught and pushed along the wave (gaining energy from the wave), while a surfer moving slightly faster than a wave will be pushing on the wave as he moves uphill (losing energy to the wave). Being higher the number of surfers gaining energy from the wave, the net effect is a flow of energy from the wave to the surfers, thus producing a wave damping.

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