CERN Accelerator School
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Low Emittance Rings

Solutions to Tutorial Problems

Lecture 1 Problem

1. The DIAMOND storage ring has a circumference of 560 m, and contains 48 dipole magnets (without quadrupole gradient), each of length 1 meter. The beam energy is 3 GeV.

   a. Calculate:

      i.   the bending radius of each dipole;

      ii. the second and third synchrotron radiation integrals;

      iii. the energy loss of each particle per turn through the ring;

      iv. the horizontal, vertical and longitudinal damping times;

      v.   the equilibrium energy spread.

   b. Discuss the effects on the beam parameters, and the impact on machine performance, if the existing dipoles were replaced by shorter dipoles.

Solution:

a) i. The dipoles on their own would form a ring of circumference 48 m. Therefore, the bending radius of each dipole is:

$$\rho = \frac{48 \, \text{m}}{2\pi} \approx 7.64 \, \text{m}$$

a) ii. The second synchrotron radiation integral is:

$$I_2 = \oint \frac{1}{\rho^2} ds = \frac{2\pi}{\rho} \approx 0.822 \, \text{m}^{-1}$$

The third synchrotron radiation integral is:

$$I_3 = \oint \frac{1}{|\rho|^3} ds = \frac{2\pi}{\rho^2} \approx 0.108 \, \text{m}^{-2}$$

a) iii. The energy loss per turn is:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \approx 937 \, \text{keV}$$

a) iv. Assuming that the damping partition numbers are $j_x = j_y = 1$, and $j_z = 2$ (good approximations for j_x and j_z, given no quadrupole gradient in the dipoles), the transverse damping times are:

$$\tau_x = \tau_y = 2\frac{E_0}{U_0}T_0 \approx 12.0\,\text{ms}$$

and the longitudinal damping time is:

$$\tau_z = \frac{E_0}{U_0}T_0 \approx 5.98\,\text{ms}$$

a) v. The equilibrium energy spread is found from:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

which gives:

$$\sigma_\delta \approx 9.31\times10^{-4}$$

b.   Reducing the dipole length would have the following effects:

- The dipole field would need to increase in proportion, to maintain the same total bending angle.
- This would reduce the bending radius (in proportion to the dipole length).
- The dispersion would remain roughly constant: the increase in dipole field will tend to increase the dispersion, but this would be mostly cancelled by the reduction in the dipole length.
- There would be only a small change in the first synchrotron radiation integral, I_1 (given by the integral of the dispersion divided by the bending radius). This means that the momentum compaction factor would remain roughly constant.
- The second synchrotron radiation integral would be increased: hence, a larger radiation energy loss per turn, and a shorter damping time.
- The third synchrotron radiation integral would be increased more significantly than the second SR integral: hence, the natural energy spread would be increased.
- The increase in SR energy loss would mean (for the same RF voltage and frequency) a change in synchronous phase, such that the RF focusing is reduced. Hence, combined with the increased energy spread (and roughly constant momentum compaction factor), the bunch length would be increased.
- There would be some increase in I_4; but with no gradient in the dipoles, this would not in itself have any significant impact on the damping partition numbers.
- The impact of the dipole length on I_5 is not obvious: in fact (from Lecture 2), we expect only a small effect on the natural emittance.

Increasing the dipole field will change the SR spectrum from the dipole (to shorter wavelengths): this may well be a desirable impact. Reduction in damping time will generally be beneficial for machine operation (improved stability). The increase in energy spread could also improve stability (increased Landau damping); however, it may be undesirable from other considerations (increased beam size in dispersive regions). Also, if a constant bunch length is required, then additional RF power will be needed.

Lecture 2 Problem

2. a. Show that, for small phase advance μ_x in a FODO cell, the natural emittance of a FODO lattice is given approximately by:

$$\varepsilon_0 \approx 8C_q\gamma^2\left(\frac{\theta}{\mu_x}\right)^3$$

where θ is the bending angle of one dipole in the cell, and γ is the relativistic factor.

b. Given data on a number of storage ring lattices:

Storage ring	Lattice type	Beam energy	Number of dipoles
SRS	72° FODO	2 GeV	16
SRS (HBL)	140° FODO	2 GeV	16
APS	DBA	7 GeV	80
DIAMOND	DBA	3 GeV	48
ALS	TBA	1.9 GeV	36

estimate the natural emittance for each ring, assuming that the lattice is tuned for the minimum emittance in each case (with zero dispersion in the straight sections of the achromats). *Note: use the small phase advance approximation for the FODO lattices – even though this may not be very accurate in these cases!*

Explain why the emittances for the achromat lattices are likely to be somewhat different (larger or smaller) in practice than the values you have calculated.

Solution:

a. The natural emittance in a FODO lattice is given approximately by:

$$\varepsilon_0 \approx C_q\gamma^2\left(\frac{2f}{L}\right)^3\theta^3 \tag{1}$$

where f is the focal length of a quadrupole, L is the length of a dipole (assumed to fill the space between quadrupoles) and θ is the bending angle of a dipole. We also have for the phase advance per cell:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2}$$

For $\mu_x \ll 1$, we can write:

$$\cos\mu_x \approx 1 - \tfrac{1}{2}\mu_x^2$$

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Hence:

$$\frac{f}{L} \approx \frac{1}{\mu_x}$$

Substituting into equation (1) gives:

$$\varepsilon_0 \approx 8C_q\gamma^2\left(\frac{\theta}{\mu_x}\right)^3$$

which is valid for $\mu_x \ll 1$.

b. The general expression for the minimum natural emittance in a lattice is:

$$\varepsilon_0 \approx FC_q\gamma^2\theta^3$$

where F is a numerical factor determined by the type of lattice, and θ is the bending angle of one dipole.

Storage ring	F	θ	ε_0
SRS	≈ 4.03	0.393	1430 nm
SRS (HBL)	≈ 0.548	0.393	195 nm
APS	$1/4\sqrt{15}$	0.0785	2.25 nm
DIAMOND	$1/4\sqrt{15}$	0.131	1.91 nm
ALS	$1/9\sqrt{15}$	0.175	0.81 nm

In practice, achromat lattices are often detuned from "ideal" emittance conditions, to improve dynamic aperture. This results in a larger emittance than might otherwise be achieved, given zero dispersion in the straights. On the other hand, it is possible to detune the dispersion, allowing non-zero dispersion in the straights, to approach the conditions for a TME lattice, thereby reducing the emittance.

The actual emittances in the above rings are:

Storage ring	ε_0
SRS	1500 nm
SRS (HBL)	110 nm
APS	7.5 nm
DIAMOND	2.7 nm
ALS	5.6 nm

Lecture 3 Problem

3. A design for the ILC damping rings, with arcs consisting of simple FODO cells, includes two families of sextupoles with the following parameters:

	SF sextupoles	SD sextupoles
Number of magnets	196	196
Integrated strength, k_2L	0.351 m^{-2}	-0.654 m^{-2}
Horizontal beta function, β_x	34.0 m	9.38 m
Vertical beta function, β_y	9.79 m	35.2 m
Horizontal dispersion, η_x	0.553 m	0.286 m

The horizontal and vertical tunes are 61.121 and 60.410, respectively; the natural emittance is
0.64 nm, and the natural energy spread is 0.13%. The dipoles have no quadrupole gradient.

a. Estimate the vertical emittance that would result from vertical sextupole alignment errors with 100 μm rms, in a lattice otherwise free of alignment and tuning errors.

b. What level of sextupole alignment would be required to achieve (under the same conditions as in part (a)) an expected vertical emittance of 2 pm?

Solution:

Sextupole parameters	SF	SD
Number of magnets	196	196
Integrated strength, k_2L (m^{-2})	0.351	-0.654
Horizontal beta function, β_x (m)	34	9.38
Vertical beta function, β_y (m)	9.79	35.2
Horizontal dispersion, η_x (m)	0.553	0.286

General lattice parameters	
ν_x	0.121
ν_y	0.410
σ_δ	1.30E-03
ε_0 (m)	6.40E-10
sqrt($<\Delta Y_s^2>$) (m)	1.00E-04
j_y	1.0
j_z	2.0

$<\eta_y^2/\beta_y>$ (m)	4.25E-07	from equation (29)
dispersion contribution to ε_y (m)	2.87E-12	from equation (40)
$<\kappa^2/\Delta\omega^2>$	0.000108	from equation (22)
coupling contribution to ε_y (m)	1.73E-14	from equation (19)
total expected ε_y (m)	2.89E-12	sum of dispersion and coupling contributions

Note that the vertical dispersion is expected to make a dominant contribution to the vertical emittance in this case. This is not terribly unusual, but depends a lot on the lattice design.

The expected emittance should only be taken as an indicative value: the real value will depend very much on the particular set of machine errors present, and can easily range over an order of magnitude, for a given rms of alignment errors.

Including only sextupole alignment errors is a very artificial situation. Quadrupole alignment errors will generate significant orbit distortion, and this is likely to dominate the beam offset in the sextupoles.

Since the emittance varies as the mean square of the sextupole alignment, to reduce the emittance by a factor of 2.00/2.89 (to give an expected vertical emittance of 2 pm), the sextupole alignment rms must be reduced by a factor $\sqrt{2.00/2.89} \approx 0.83$, i.e. to 83 µm. This is not a realistic alignment accuracy using survey techniques alone (even aside from the contributions from alignment errors on dipoles and quadrupoles). Beam-based techniques are certainly required to achieve vertical emittances of order a few picometers.

Optional Extra Problem: ILC Damping Rings Case Study

4. The damping rings in the International Linear Collider (ILC) serve the purpose of accepting large-emittance electron or positron beams from the particle sources, and producing low-emittance, highly stable beams for acceleration in the linacs and collision at the interaction point. Each beam is stored for the time between machine pulses, which is 200 ms in the case of ILC.

   The damping rings are synchrotrons, similar to the storage rings in third generation light sources, but rather larger. Some of the parameter specifications for the damping rings are as follows:

Circumference	6.6 km
Energy	5 GeV
Injected emittance (x and y)	1 μm
Extracted horizontal emittance	0.8 nm
Extracted vertical emittance	2 pm
Equilibrium vertical emittance	1.4 pm
Maximum extracted energy spread	0.13%
Beam store time	200 ms
Lattice type	TME
Number of dipoles	120
Dipole length	6 m

   a. Calculate the transverse damping times required to achieve the extracted emittances starting with the specified injected emittances, in the given store time.

   b. Estimate (i) the damping times, and (ii) the natural emittance that would be achieved in the lattice without any damping wiggler (i.e. with the only synchrotron radiation energy loss provided by the dipoles). Assume that the lattice is properly tuned for the minimum possible natural emittance.

   c. Estimate the maximum wiggler peak field allowed by the specified extracted energy spread.

   d. Assuming the wiggler peak field is the maximum allowed by the energy spread, estimate the length of damping wiggler needed to achieve the required damping times.

   e. Assuming an average horizontal beta function in the wiggler of 20 m, estimate the maximum wiggler period in order to achieve the specified extracted horizontal emittance.

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Solution:

a. The emittance evolves as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-\frac{2t}{\tau}\right)\right]$$

$$= \varepsilon(\infty) + [\varepsilon(0) - \varepsilon(\infty)]\exp\left(-\frac{2t}{\tau}\right)$$

where τ is the damping time. Therefore:

$$\frac{t}{\tau} = \frac{1}{2}\ln\left(\frac{\varepsilon(0) - \varepsilon(\infty)}{\varepsilon(t) - \varepsilon(\infty)}\right)$$

Substituting in values for the injected, equilibrium and extracted vertical emittance:

$$\frac{t}{\tau} \approx 7.16$$

So, with the store time $t = 200$ ms, we get:

$$\tau \approx \frac{t}{7.16} \approx 27.9 \text{ ms}$$

The horizontal damping time will be approximately the same; but since a very much smaller extracted emittance is required in the vertical, the damping time requirements are set by the vertical emittance.

b. (i) The dipoles would form a ring of circumference 720 m, so the bending radius must be 720 m / 2π = 114.6 m. Therefore, the dipoles make a contribution to the second synchrotron radiation integral:

$$I_{2,dip} = \frac{2\pi}{\rho} \approx 0.0548 \text{ m}^{-1}$$

If the dipoles were the only source of synchrotron radiation energy loss, the energy loss per turn would be:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_{2,dip} \approx 482 \text{ keV}$$

The transverse damping times (assuming the damping partition numbers are equal to 1) would then be:

$$\tau_x = \tau_y = 2\frac{E_0}{U_0}T_0 \approx 457 \text{ ms}$$

(ii) For a TME lattice, the natural emittance is given by:

$$\varepsilon_0 = \frac{1}{12\sqrt{15}}C_q\gamma^2\theta^3$$

So with a beam energy of 5 GeV, and 120 dipoles, we find:

$$\varepsilon_0 \approx 0.11 \text{ nm}$$

c. If the synchrotron radiation energy loss is dominated by the wiggler (which will need to be the case to achieve the specified damping times), the equilibrium energy spread is related to the wiggler peak field by:

$$\sigma_\delta^2 \approx \frac{4}{3\pi} C_q \frac{\gamma^2}{\rho_w} = \frac{4}{3\pi} \frac{e}{mc} C_q \gamma B_w$$

With the maximum equilibrium energy spread 0.15%, the maximum wiggler peak field is:

$$B_w \approx 1.81\,\text{T}$$

d. Let us start by calculating the required energy loss per turn. This is related to the transverse damping time by:

$$\tau_x = \tau_y = 2\frac{E_0}{U_0}T_0 \approx 27.9\,\text{ms}$$

With the given beam energy and circumference, we find:

$$U_0 \approx 7.89\,\text{MeV}$$

Now we calculate the required value for the second synchrotron radiation integral, I_2. This is related to the energy loss per turn by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \approx 7.89\,\text{MeV}$$

Hence, we find that:

$$I_2 \approx 0.897\,\text{m}^{-1}$$

From part (b), we know that the dipoles make a contribution to the second synchrotron radiation integral:

$$I_{2,dip} = \frac{2\pi}{\rho} \approx 0.0548\,\text{m}^{-1}$$

which is much smaller than the total value of I_2 required. The rest must be contributed by the wiggler:

$$I_{2,wig} \approx 0.842\,\text{m}^{-1}$$

The wiggler contribution is related to the peak field, wiggler length, and beam rigidity by:

$$I_{2,wig} = \frac{1}{(B\rho)^2} \frac{B_w^2 L_w}{2} \approx 0.842\,\text{m}^{-1}$$

With a beam energy of 5 GeV, the rigidity is 16.68 Tm; so with a peak field of 1.81 T, the total length of wiggler required is:

$$L_w \approx 143\,\text{m}$$

e. The synchrotron radiation energy loss is dominated by the wiggler (94%), so we assume that we can neglect the dipole contribution to the natural emittance (this isn't completely true, but a good approximation).

In this case, the natural emittance is given by:

$$\varepsilon_0 \approx \frac{8}{15\pi} C_q \gamma^2 \frac{\langle \beta_x \rangle}{\rho_w^3 k_w^2}$$

where ρ_w is the bending radius corresponding to the peak field of the wiggler, and:

$$k_w = \frac{2\pi}{\lambda_w}$$

where λ_w is the wiggler period.

Assuming an average beta function of 20 m, and a natural emittance close to the specified extracted emittance, we find:

$$\lambda_w \approx 0.445 \text{ m}$$

In practice, the wiggler period will probably need to be somewhat smaller than this, to allow some margin between the natural emittance and the specified extracted emittance, and (in the likely case that the lattice isn't perfectly tuned for the lowest possible emittance) to allow for the quantum excitation from the main dipole magnets in the arcs.