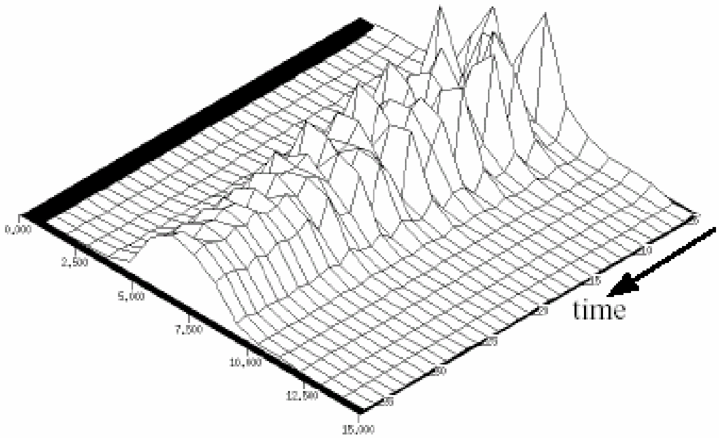
Injection Mismatch



Injection mismatch:

As a rule, proton/ion accelerators need their full aperture at injection, thus avoiding mismatch allows a beam of larger normalized emittance ε* and containing more Protons. In proton/ion ring accelerators any type of injection mismatch will lead to an emittance blowup. Off axis injection will lead to orbit oscillations. These oscillations can be detected easily by turn-by-turn BPMs in the ring (before Landau damping occurs). The orbit mismatch can be corrected by a proper setup of the steering magnets, kickers and septas. However, any mismatch of the optical parameters α, β (and therefore γ) will also lead to an emittance blowup (and beam losses) and is not detectable by BPMs.

Fig. 1a shows the phase ellipse at a certain location in a circular accelerator. The ellipse is defined by the optics of the accelerator with the emittance ε and the optical parameters β = beta function, γ = (1 + α)/β and the slope of the beta function α = -β'/2. Fig. 1b-d shows the process of filamentation after some turns.

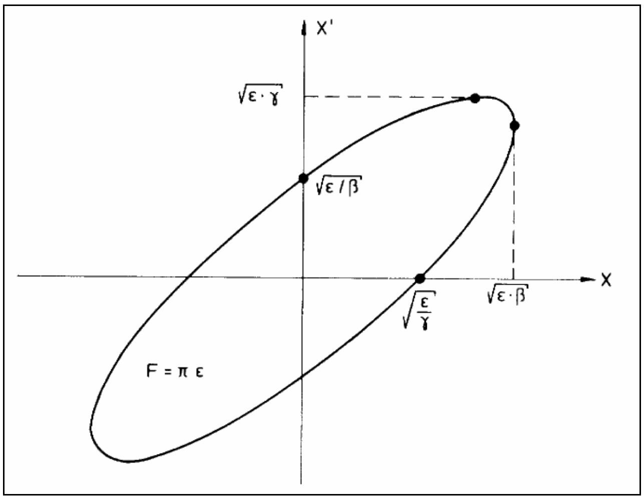


Fig 1a: A phase space ellipse of a circular accelerator, defined by α, β, γ, ε

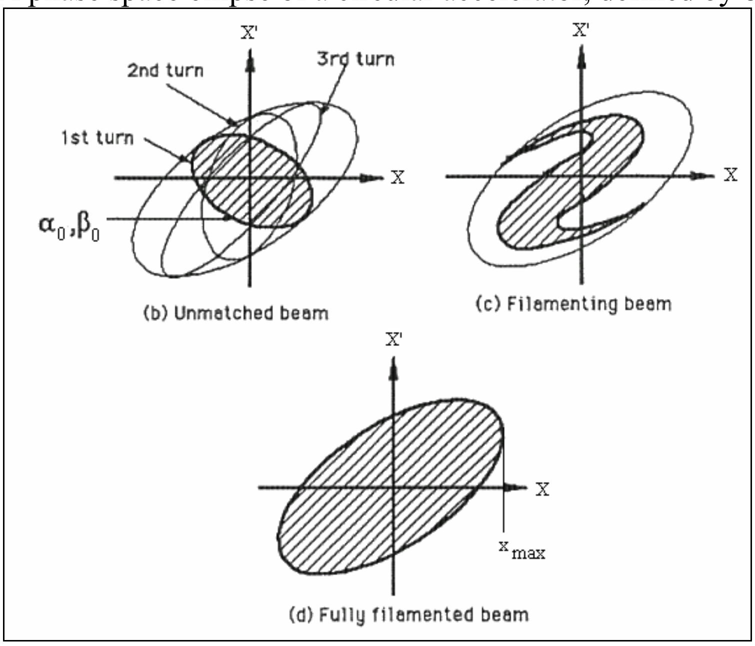


Fig. M1 b-d: Filamentation of an unmatched beam (from Ref. 2)

Assuming a beam is injected into the circular machine, defined by β_0 and α_0 (and therefore γ_0) with a given emittance ε_0. For each turn i in the machine the three optical parameters will be transformed by

$$\begin{pmatrix} \beta_{i+1} \\ \alpha_{i+1} \\ \gamma_{i+1} \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{pmatrix} \quad (Starting\ with\ i=0)$$

where C and S are the elements of the Twiss matrix ($\mu = 2\pi q$, q = tune):

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \cdot \sin\mu & \beta_0 \cdot \sin\mu \\ -\gamma_0 \cdot \sin\mu & \cos\mu - \alpha_0 \cdot \sin\mu \end{pmatrix} \qquad (1)$$

and $\gamma = (1+\alpha^2)/\beta$

Without any mismatch, the three parameters will be constant while a mismatch will result in an oscillation of the parameters.

Exercise M1: Show the beta-oscillation for the mismatch. What is the oscillation frequency? What are typical amplitudes?

$$\beta_{i+1} = C^2\beta_i - 2SC\alpha_i + S^2\gamma_i$$

with (1) and some transformations

$$\beta_{i+1} = \beta_i \cdot \cos^2\mu + (\alpha_0^2 \cdot \beta_i - 2\alpha_0\beta_0\alpha_i + \beta^2\gamma_i) \cdot \sin^2\mu + ((\alpha_0 \cdot \beta_i - \beta_0\alpha_i) \cdot 2 \cdot \sin\mu \cdot \cos\mu$$

with
sin μ · cos μ = ½ sin |**2**μ|, $\cos^2$ μ = ½ (1+cos**2μ**), $\sin^2$ μ = ½ (1-cos**2μ**)

one gets **twice the betatron tune.**

Fig. M2 shows the oscillations of β for 20 turns reaching values between 180 and 310:

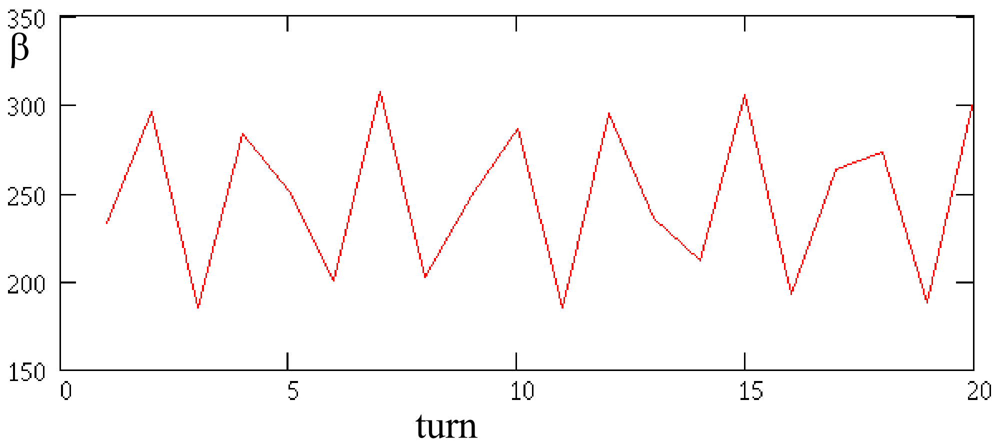


Fig. M2: β-oscillation amplitudes

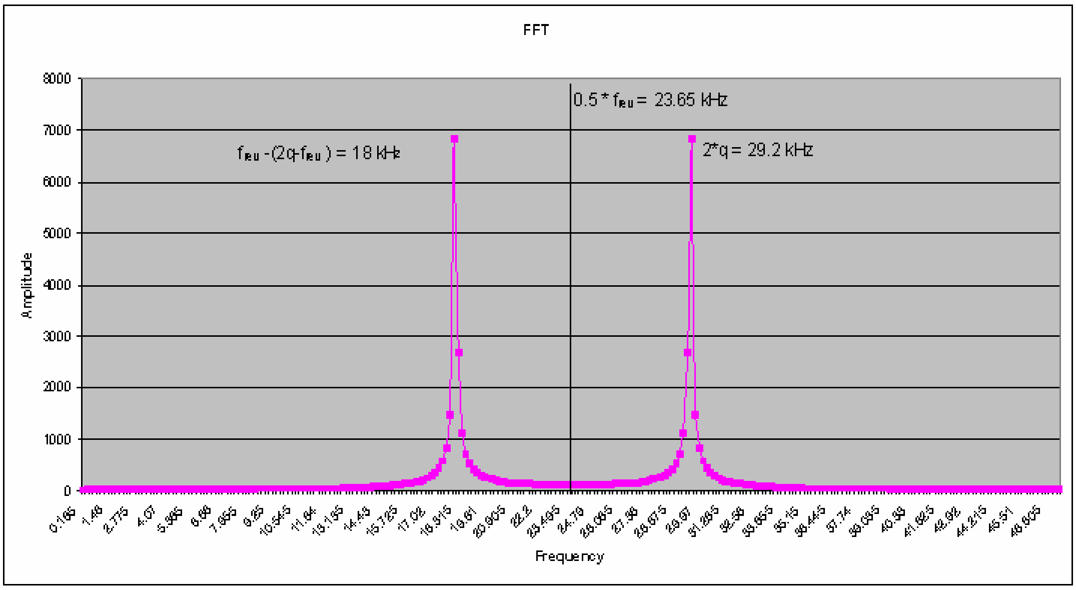
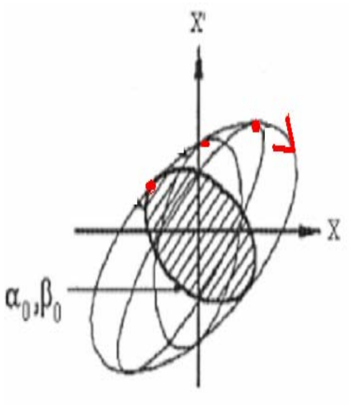


Fig. M2a: FFT spectrum of Fig. M2



During 1 turn the whole ellipse rotates with Q, but the projection on the x-axis oscillates with 2 Q. One turn gives two periods.

Discuss how to measure a 10% betatron mismatch at injection between a transport line and a storage ring , for example in the HERAp accelerator. Exercise M2: How large is the emittance blow-up?

Some important HERAp parameters
Circumference circ = 6.3 km, f_{rev} = 47.3 kHz
Tune q = 0.31 or f_{tune} = 14.6 kHz
Momentum E_p = 40 GeV/c at injection
Normalized emittance $\varepsilon_n = 20\ \pi$ mm mrad, $\varepsilon_0 = 5 \cdot 10^{-7}$
β_0 = 238 m, α_0 = -2.2, => γ_0 = 0.0245 at the injection point ($\beta\gamma - \alpha^2 = 1$). => ring
β = 214 m, $\alpha_0 = \alpha$, => γ = 0.0272 at the injection point. => injected beam

The emittance blow-up due to the betatron mismatch ($\alpha_0 = \alpha$) can be calculated with the following formula derived from Ref. 2, 3 (gaussian beams):

$$\varepsilon_{filamented} = \varepsilon_0 \cdot (1 + 0.5 \cdot |\det(\Delta J)|) \quad with \quad \Delta J = \begin{pmatrix} \alpha_0 - \alpha & \beta_0 - \beta \\ -(\gamma_0 - \gamma) & -(\alpha_0 - \alpha) \end{pmatrix}.$$

$|\det \Delta J| = (\alpha_0 - \alpha)^2 + (\gamma_0 - \gamma)\cdot(\beta_0 - \beta) = 0.066$
In this example a 10% mismatch leads to an emittance blow up
$\mathbf{\Delta\varepsilon = (\varepsilon_{filamented} - \varepsilon_0)/\ \varepsilon_0 \cdot 100\ \%}$ = 3.3%.

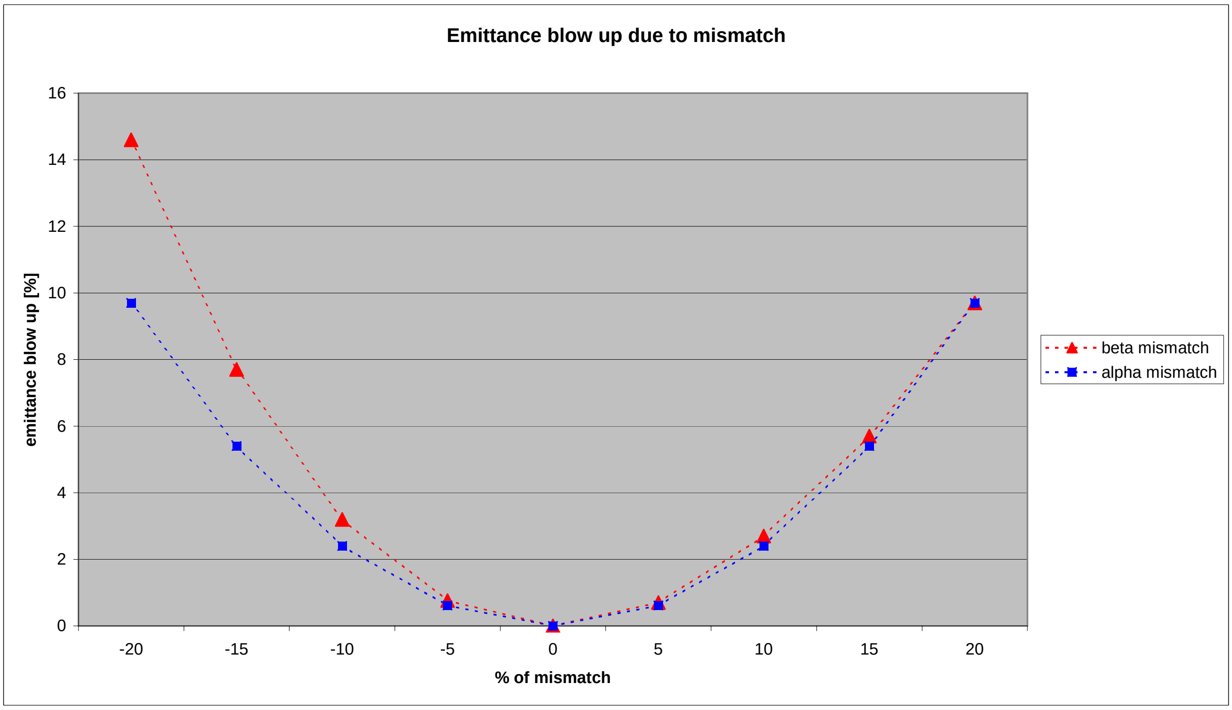


Fig. M3: Emittance blow up $\Delta\varepsilon = (\varepsilon_{filamented} - \varepsilon)/\ \varepsilon \cdot 100\ \%$ due to mismatch

Exercise M2a: What kind of measurement will you propose to determine the β-mismatch? Which monitor do you propose for this measurement?

A simple beam width measurement after filamentation at β_1=238 m results in:

$$\sigma_0 = \sqrt{\varepsilon_0 \cdot \beta_1} = 10.91\,mm$$

$$\sigma_{0\,filamented} = \sqrt{\varepsilon_{filamented} \cdot \beta_1} = 11.08\,mm$$

⇨ **The effect is to small for a precise measurement**

A mismatch of the betatron phase space will result in transverse shape oscillations, at least for some ten turns, before the different phases of the protons lead to a filamentation of the beam. Try to observe the oscillation at one location.

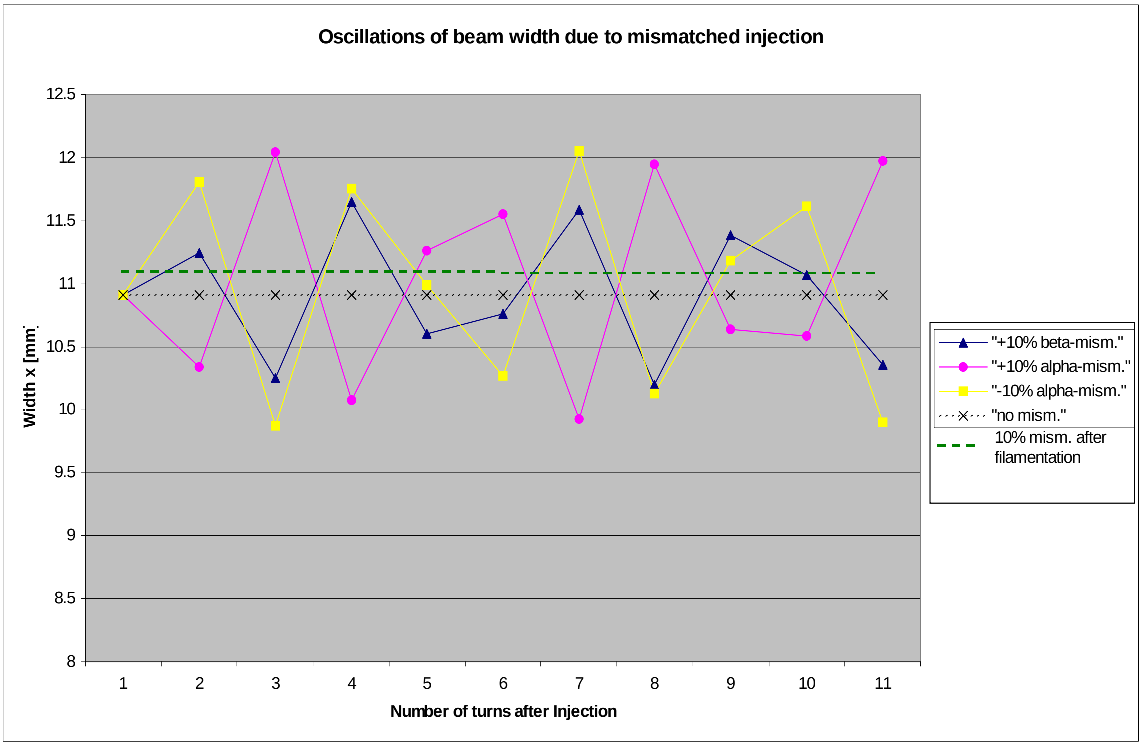


Figure M4: Oscillations of the beam width due to mismatched injection. Note also the small difference of the beam width after filamentation.

A measurement of width oscillations at injection is a very efficient method to detect an optical mismatch that increases the emittance in the circular accelerator.
Measurement of the turn-by-turn shape oscillation is possible with a fast (turn by turn) readout of:

1. Thin screen (OTR, Phosphor) (see for details Ref. 4)
2. SEM grids, (Ref. 5)
3. Ionization Profile Monitor (IPM), (Ref. 6)
4. Quadrupole (QP-) Pickup, (Ref. 7)
5. Synchrotron Radiation (SR) -Monitor (electrons). (Ref. 8)

Exersice M2b: What is the effect of the proposed monitor(s) on the beam?

- Screen/Grid: Emittance blow-up and losses
- IPM: Very small, a sufficient signal at each turn needs a pressure bump => emittance blow-up and losses
- QP-Pickup: None (see Rodri's talk), but very difficult to suppress the dipole mode.
- SR-Monitior: None, but no light from protons!

Blow-up:
A screen/grid or IPM pressure bump will give an additional constant increase of the emittance, but it can easily be separated from the oscillation observation. The protons receive a mean kick at each traverse through a screen resulting in an additional angle θ.

$$\theta = \frac{0.014}{p \cdot \beta} \cdot Z \cdot \sqrt{\frac{d}{l_{rad}}} \left[1 + \frac{1}{9} \log_{10} \left(\frac{d}{l_{rad}} \right) \right] \quad \textit{in radians}$$

where p is the momentum in GeV/c and Z=1 the charge number of the proton, β = v/c the velocity, d the thickness of the foil and l_{rad} the radiation length of the material of the foil. This formula describes the gaussian approximation of the mean scattering angle of the protons after one traverse. The change of the emittance δε for every turn can be calculated by:,

$$\delta\varepsilon_{rms} = \sqrt{2 \cdot \pi} \cdot \theta^2 \cdot \beta$$

which adds quadratically to the 1 σ - emittance of the previous turn.
The emittance blow-up is shown in Fig. M5 for a 10 µm thick titanium foil as the source of OTR radiation. In addition a betatron mismatch of 10% is assumed. The figure shows a small growth of the beam width due to the foil, which does not affect the beam width oscillation. The growth rate is small compared to the oscillation amplitude. The faster growth rate in PETRA is a result of the smaller momentum of the injected protons and therefore a larger scattering angle in the foil. This angle will become much larger in DESY III (p=310 MeV/c, β = 0.3), so that the beam width will become unacceptably large within one turn and the loss rate will increase drastically (in Fig. M5 the line for DESY III extend the border of the figure within 3 turns even with a 1 µm screen).

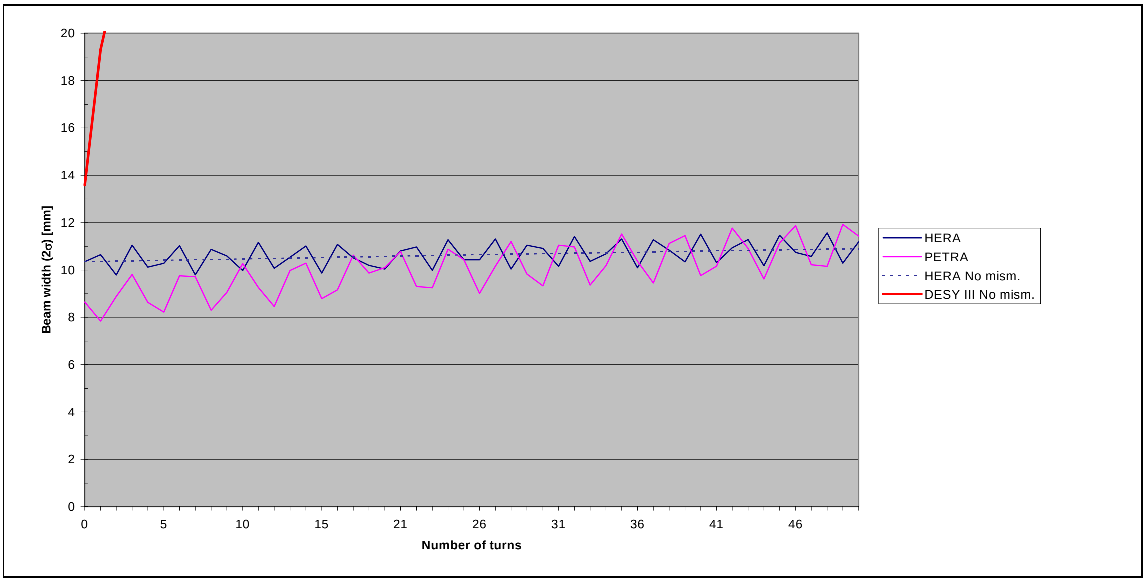


Fig. M5: Emittance growth due to a d = 10 µm Titanium foil at injection energy of HERA and PETRA (α = –2 m, β = 40 m, ε_n = 14 π mm mrad, q = 0.14, p = 7.5 GeV/c) and DESY III (with a foil-thickness of d = 1 µm (!))

The emittance blow up in DESY III due to a thin foil is much too large. A harp of thin wires produces less emittance blow up. Assuming a harp of 20 µm titanium wires at a separation of 1 mm, the blowup can be calculated like a 0.2 µm foil. Fig. 9 shows the beam oscillation due to a 10% mismatch in DESY III together with the blowup due to these wires. The secondary emission (SEM) current created in the wires can be read out by fast ADCs turn by turn (315 kHz). Such a readout schema is applied in the PS-Booster at CERN (Ref.4).

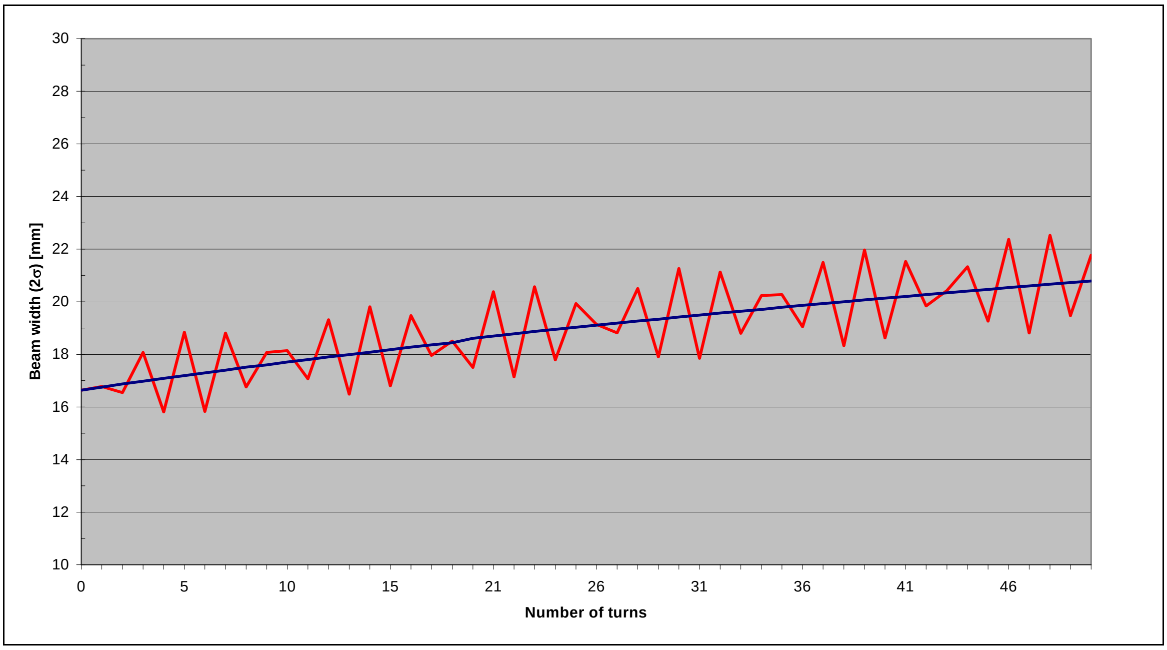


Fig.M6: Simulation of the beam width versus turns as measured by SEM grid with and without a +10% beta mismatch in DESY III (α = -1.7 m, β = 14.3 m, ε_n = 6 π mm mrad, q = 0.28, p = 310 MeV/c)

Losses:

The relative proton losses per turn dN/N0 in the foil (thickness d) is given by the nuclear interaction length L_{nuc}:

$$\frac{dN}{N_0} = \frac{d}{L_{nuc}} \qquad \text{with} \qquad L_{nuc} = \frac{A}{\rho \cdot N_A \cdot \sigma_{nuc}}$$

L_{nuc} depends on the total nuclear cross section of the nuclear interaction σ_{nuc}, the density ρ of the foil and the Avogadro constant $N_A = 6.0225 \cdot 10^{23}\ mol^{-1}$. The nuclear cross section σ_{nuc} depends on the proton momentum and on the material of the foil and is shown for different materials in Tab. 1 between a momentum of $0.3 < p < 40$ GeV/c:

Material A [g/mol] ρ [g/cm³]	Momentum [GeV/c]	σ_{nuc} [mb]	L_{nuc} [cm]	relative loss/turn $dN/N_0 \cdot 100$ [%] with d = 10 µm
Carbon	0.3	280	31.5	$3 \cdot 10^{-3}$
12.01	7.5	360	24.5	$4 \cdot 10^{-3}$
2.26	40	330	22.5	$4.4 \cdot 10^{-3}$
Aluminum	0.3	550	30.2	$3.3 \cdot 10^{-3}$
26.98	7.5	700	38.4	$2.6 \cdot 10^{-3}$
2.70	40	640	35.1	$2.8 \cdot 10^{-3}$
Copper	0.3	950	12.4	$8.1 \cdot 10^{-3}$
63.546	7.5	1350	17.6	$5.7 \cdot 10^{-3}$
8.96	40	1260	16.4	$6.1 \cdot 10^{-3}$

Tab. 2: Nuclear total cross sections, interaction length and particle losses

The loss rate is negligible small at the injection energies of the DESY proton machines and will not influence the mismatch measurement.

Some notes to the readout:

The optical readout of screens/IPM is slow. A turn by turn observation needs a 100 kHz (3 km) data collection of the whole image. Line sensors with a larger pixel size (for better sensitivity) may have a readout frequency of 15 MHz/pixel. Assuming 128 pixel will give a maximum readout frequency of 117 kHz for a 1 dim image.

IL-P3	IL-C6
Single O/P, PPD, 5V clocks	Tall Pixels, High Dynamic Range
512 / 1024 / 2048	2048
73 / 37.8 / 19.2kHz	7.2kHz
40MHz	15MHz
14µm x 14µm	13µm x 500µm

SMD-64K1M

- One milliom frames per second
- 256 x 256 pixels
- 16 consecutive frames

Silicon Mountain Design's SMD-64K1M digital camera provides 256 x 256 images at up to one million frames per second (fps) with true 12 bit dynamic range. The SMD-64K1M is a solid state CCD camera using a progressive scan CCD to achieve outstanding resolution and gray scale characteristics.

The SMD-64K1M's 12 bit RS-422 digital signal output is perfectly suited for interfacing with external image processing systems. Special interface cables are available for connecting directly to Datacube image processing system.

A SEM signal as well as the QP-Pickup signal (R. Jones's talk) can be picked up with very high frequencies, even bunch by bunch (100 MHz) and is therefore preferred for smaller ring diameters with a higher revolution frequency.

References

Ref. 1: P.J. Bryant; Beam Transfer Lines, Proc. 5th CAS, 7-18. Sept. 1992, CERN 94-01 Vol. 1.
Ref. 2: M.J. Syphers, T. Sen; Notes on Amplitude Function Mismatch. SSCL-604-mc, Oct 1992. 19pp.
Ref. 3: M. Syphers, T. Sen, D. Edwards; Amplitude Function Mismatch.. SSCL-PREPRINT-438, Published in IEEE PAC 1993 (attached)
Ref. 4: Syphers, M.J.; Injection mismatch and phase space dilution. Fermilab-FN-458, June 1987

Monitors:

Ref. 4: FIRST RESULTS FROM BETATRON MATCHING MONITORS INSTALLED IN THE CERN PSB AND SPS.
By C. Bovet, R. Colchester, C. Dutriat, G. Ferioli, J.J. Gras, R. Jung, P. Knaus, U. Raich, J.M. Vouillot (CERN). CERN-SL-98-037-BI, CERN-SL-98-37-BI, Jun 1998. 4pp. 6th European Particle Accelerator Conference (EPAC 98), Stockholm, Sweden, 22-26 Jun 1998.

Ref. 5: INJECTION MATCHING STUDIES USING TURN BY TURN BEAM PROFILE MEASUREMENTS IN THE CERN PS
M. Benedikt, Ch. Carli, Ch. Dutriat, A. Jansson, M. Giovannozzi, M. Martini, U. Raich, CERN, Geneva, Switzerland, DIPAC 2001

Ref. 6: **SENSITIVITY STUDIES WITH THE SPS REST GAS PROFILE MONITOR**
G. Ferioli, C. Fischer, J. Koopman, M. Sillanoli, CERN, Geneva, Switzerland
DIPAC 2001 Proceedings - ESRF, Grenoble

Ref. 7: Noninvasive single-bunch matching and emittance monitor
PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS, VOLUME 5, 072803, A. Jansson; CERN, CH-1211 Geneva 23, Switzerland

Ref. 8: BETA-MATCHING AND DAMPING OBSERVATION BY SR MONITOR AT ATF DR
T. Naito, Y. Hashimoto, H. Hayano, K. Kubo, M. Muto, J. Urakawa
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Amplitude Function Mismatch

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Abstract

We develop the general equation of motion of an amplitude function mismatch in an accelerator lattice and look at its solution for some interesting cases. For a free β-wave oscillation the amplitude of the mismatch is written in terms of the determinant of a single matrix made up of the difference between the new Courant-Snyder parameters and their ideal values. Using this result, once one calculates the mismatch of the amplitude function and its slope at one point in the lattice (at the end of a *nearly* matched insertion, for example), then the maximum mismatch downstream can be easily computed. The formalism is also used to describe emittance growth in a hadron synchrotron caused by amplitude function mismatches at injection.

While most of the content of this paper is not new to the accelerator physics community, we thought it would be useful to place this important, basic information all in one place. Besides the classic work of Courant and Snyder, our sources include other papers, internal reports, and numerous discussions with our colleagues. More details may be found in a related paper.[1]

I. A STARTING POINT

The general solution for linear betatron oscillations in one transverse degree of freedom can be written as[2] $x(s) = A\sqrt{\beta(s)}\cos[\psi(s)+\delta]$ where A and δ are constants given by the particle's initial conditions. The phase advance $\psi(s)$ and the amplitude function $\beta(s)$ satisfy the differential equations $\psi' = \frac{1}{\beta}$, $2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$, where $K = e(\partial B_y/\partial x)/p$, with e = charge, p = momentum, $\partial B_y/\partial x$ = magnetic field gradient, and $\beta' = d\beta/ds$, etc. When one considers the periodic solution of the amplitude function, the motion through a single repeat period can be described in terms of the Courant-Snyder parameters $\beta(s)$, $\alpha(s) \equiv -(d\beta(s)/ds)/2$, and $\gamma(s) \equiv (1+\alpha^2)/\beta$, using the matrix

$$\begin{pmatrix} \cos\psi_C + \alpha\sin\psi_C & \beta\sin\psi_C \\ -\gamma\sin\psi_C & \cos\psi_C - \alpha\sin\psi_C \end{pmatrix} \quad (1)$$

which operates on the state vector X, with $X = (x, x')^T$. Here, the phase advance is $\psi_C = 2\pi\nu = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}$, where C is the repeat distance of the hardware, which may be the circumference of the accelerator, and ν is the *tune* of the synchrotron.

The matrix of Equation 1 is often written in compact form as $M = I\cos\psi_C + J\sin\psi_C$ where

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \quad (2)$$

The amplitude function and its slope propagate through an accelerator section according to

$$J_2 = M(s_1 \to s_2) J_1 M(s_1 \to s_2)^{-1}, \quad (3)$$

where J_1 and J_2 contain Courant-Snyder parameters corresponding to points 1 and 2, and $M(s_1 \to s_2)$ is the transport matrix between these two points.

II. PROPAGATION OF A THIN GRADIENT ERROR

We wish to see how the amplitude function downstream of a thin gradient error is altered. If $J_0(s_0)$ is the matrix of unperturbed Courant-Snyder parameters at the location of the error and $J_0(s)$ contains the unperturbed parameters at a point downstream, then, using Equation 3,

$$\Delta J(s) = M(s_0 \to s)\Delta J(s_0) M(s_0 \to s)^{-1}, \quad (4)$$

where

$$\Delta J(s) = J(s) - J_0(s) = \begin{pmatrix} \alpha - \alpha_0 & \beta - \beta_0 \\ -(\gamma - \gamma_0) & -(\alpha - \alpha_0) \end{pmatrix}; \quad (5)$$

β is the new value of the amplitude function at s, β_0 is the unperturbed value, etc. Through a thin quad, $\Delta\alpha = q\beta_0$, $\Delta\beta = 0$, and $\Delta\gamma = 2\alpha q + \beta q^2$ and so

$$\begin{aligned} \frac{\Delta\beta(s)}{\beta_0(s)} &= -(\beta_i q)\sin 2\psi_0(s-s_0) \\ &\quad + \frac{1}{2}(\beta_i q)^2 [1 - \cos 2\psi_0(s-s_0)] \end{aligned} \quad (6)$$

where $\psi_0(s-s_0)$ is the unperturbed phase advance between points s_0 and s and $\beta_i \equiv \beta_0(s_0)$. The amplitude function perturbation oscillates at twice the betatron frequency and for $(\beta_i q)$ sufficiently small, the perturbation describes simple harmonic motion. The change in α also propagates at

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twice the betatron frequency, it being given by

$$\begin{aligned}\Delta\alpha(s) &= \beta_i q\,[\cos 2\psi_0(s-s_0) - \alpha_0(s)\sin 2\psi_0(s-s_0)] \\ &\quad -\frac{1}{2}(\beta_i q)^2[\sin 2\psi_0(s-s_0) \\ &\quad -\alpha_0(s)(1-\cos 2\psi_0(s-s_0))] \ . \qquad (7)\end{aligned}$$

Introducing this quad error also changes the phase advance across the lattice. The new phase advance $\psi(s-s_0)$ across this section may be calculated using $\sin\psi(s-s_0) = \bar{M}(s_0 \to s)_{12}/\sqrt{\beta_i\beta(s)}$ where $\bar{M}(s_0 \to s)_{12}$ is the (1,2) element of the new ring matrix and $\beta(s)$ is the new amplitude function at s. Using Equation 6, we obtain

$$\sin\psi(s-s_0) = [1 - \beta_i q \sin 2\psi_0(s-s_0)$$

$$+ (\beta_i q)^2 \sin^2\psi_0(s-s_0)]^{-1/2} \sin\psi_0(s-s_0) \ . \qquad (8)$$

An explicit result for the change in the phase advance may be obtained perturbatively in orders of the quad error q from the above exact expression. To second order in q, we find that the change $\Delta\psi \equiv \psi(s-s_0) - \psi_0(s-s_0)$ is

$$\Delta\psi = \beta_i q \sin^2\psi_0(s-s_0)$$

$$-(\beta_i q)^2 \sin 2\psi_0(s-s_0)\sin^2\psi_0(s-s_0) + O(q^3) \ . \ (9)$$

To first order in q, at a point $\pi/2$ away from the location of the error, there is no change in the β function while the change in phase advance is at its maximum value of $\beta_i q$.

III. EQUATION OF MOTION OF β-WAVE

The equation of motion for an amplitude function mismatch is nonlinear when s is taken as the independent variable. A more congenial equation can be developed by using the the reduced phase $\phi \equiv \psi/\nu$ as the independent variable. For betatron oscillations the Floquet transformation, where the other variable is $\zeta = x/\sqrt{\beta}$, produces the equation of motion $\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = 0$ which is pure simple harmonic motion with frequency (tune) ν. For the amplitude function mismatch, we need to define the reduced phase in terms of the unperturbed functions. That is, let $\phi \equiv \psi_0/\nu_0$, where $d\psi_0/ds = 1/\beta_0$, and ν_0 is the unperturbed tune. The equation of motion for $[\beta(\phi) - \beta_0(\phi)]/\beta_0(\phi) \equiv \Delta\beta/\beta_0$ in the absence of gradient errors is then

$$\begin{aligned}\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0} + (2\nu_0)^2\frac{\Delta\beta}{\beta_0} &= -2\nu_0^2 \det\Delta J \\ &= 2\nu_0^2[\Delta\alpha^2 - \Delta\beta\Delta\gamma] \quad (10)\end{aligned}$$

where $\Delta\alpha = \alpha(\phi) - \alpha_0(\phi)$, etc. The quantity $\det\Delta J$ is an invariant in portions of the lattice without gradient perturbations as can be seen with the aid of Equation 3.

So, the free amplitude function distortion oscillates with twice the betatron tune and with a constant offset given by the determinant of the ΔJ matrix at any point. This offset must be there since $\beta > 0$ and hence $\Delta\beta/\beta$ must always be greater than -1.

Rewritten in terms of the Courant-Snyder parameters,

$$\det\Delta J = -\frac{\left(\frac{\Delta\beta}{\beta_0}\right)^2 + \left(\Delta\alpha - \alpha_0\frac{\Delta\beta}{\beta_0}\right)^2}{1 + \Delta\beta/\beta_0} < 0. \qquad (11)$$

Thus, $|\det\Delta J|^{1/2}$ can be interpreted as the amplitude of the β mismatch for small perturbations.

The solution to Equation 10 is just simple harmonic motion with a constant term added:

$$\frac{\Delta\beta}{\beta_0}(\phi) = A\cos 2\nu_0\phi + B\sin 2\nu_0\phi + \frac{1}{2}|\det\Delta J|. \qquad (12)$$

The constants A and B are found from the initial conditions:

$$A = \frac{\Delta\beta}{\beta_0}(0) - \frac{1}{2}|\det\Delta J|, \qquad (13)$$

$$B = \alpha_0\frac{\Delta\beta}{\beta_0}(0) - \Delta\alpha(0). \qquad (14)$$

Thus, the maximum value of $\Delta\beta/\beta_0$ downstream of our starting point $\phi = 0$ is given by

$$\begin{aligned}\left(\frac{\Delta\beta}{\beta_0}\right)_{max} &= \sqrt{A^2+B^2} + \frac{1}{2}|\det\Delta J| \\ &= \frac{|\det\Delta J|}{2} + \sqrt{|\det\Delta J| + \left(\frac{|\det\Delta J|}{2}\right)^2} \quad (15)\end{aligned}$$

where use has been made of Equation 11. The maxima occur at phases where

$$\tan 2\nu_0\phi = \left(\frac{\alpha_0\frac{\Delta\beta}{\beta_0} - \Delta\alpha}{\frac{\Delta\beta}{\beta_0} - |\det\Delta J|/2}\right)_0 \ . \qquad (16)$$

The usefulness of the above result is, of course, that once one calculates the mismatch of the amplitude function and its slope at one point in the lattice (at the end of a ***nearly*** matched insertion, for example), then the maximum mismatch downstream can be computed immediately.

If we look once again at the perturbation downstream of a thin quadrupole error, we see that just after the quad,

$$\det\Delta J = \begin{vmatrix} q\beta_i & 0 \\ -\Delta\gamma & -q\beta_i \end{vmatrix} = -(q\beta_i)^2 \qquad (17)$$

where $\beta_i = \beta_0$ at the location of the quadrupole. Then,

$$\begin{aligned}\left(\frac{\Delta\beta}{\beta_0}\right)_{max} &= q\beta_i\sqrt{1+(q\beta_i)^2/4} + \frac{1}{2}(q\beta_i)^2 \quad (18) \\ &\approx q\beta_i = \sqrt{|\det\Delta J|} \qquad (19)\end{aligned}$$

where the last line is valid for small perturbations.

IV. GENERAL EQUATION OF MOTION

To include the driving terms due to gradient errors in the equation of motion for $\Delta\beta/\beta_0$, we let β_0 satisfy the differential equation $K\beta_0 = \gamma_0 + \alpha_0'$, and let β satisfy $(K + k)\beta = \gamma + \alpha'$, where $\beta = \beta_0 + \Delta\beta$, etc. Then, the relative β error satisfies

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi)$$

$$= -2\nu_0^2\left[\beta_0^2(\phi)\,k(\phi)\left(1 + \frac{\Delta\beta}{\beta_0}(\phi)\right) + \det\Delta J(\phi)\right]. \quad (20)$$

Here, in general, $\det\Delta J(\phi)$ is not invariant as it is altered by gradient perturbations:

$$\frac{d}{d\phi}\det\Delta J(\phi) = \beta_0^2 k \frac{d}{d\phi}\frac{\Delta\beta}{\beta_0} \quad (21)$$

For small perturbations we can drop quantities which are second order in the small quantities, e.g. $k\Delta\beta$. This reduces Equation 20 to

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta}{\beta_0}(\phi) + (2\nu_0)^2\frac{\Delta\beta}{\beta_0}(\phi) = -2\nu_0^2\,\beta_0^2\,k(\phi) \quad (22)$$

as appears in Courant and Snyder.[2]

Noting that $\Delta\alpha - \alpha_0(\Delta\beta/\beta_0) = -(1/2\nu_0)d(\Delta\beta/\beta_0)/d\phi$, one can easily exhibit Equation 20 entirely in terms of $\Delta\beta/\beta_0$ and its derivatives with respect to ϕ. Differentiating this resulting equation one obtains a *linear* differential equation for $\Delta\beta/\beta_0$:

$$\frac{d^3}{d\phi^3}\frac{\Delta\beta}{\beta_0} + (2\nu_0)^2(1 + \beta_0^2 k)\frac{d}{d\phi}\frac{\Delta\beta}{\beta_0}$$

$$+ 2\nu_0^2\frac{d}{d\phi}[\beta_0^2 k](1 + \frac{\Delta\beta}{\beta_0}) = 0. \quad (23)$$

V. INJECTION MISMATCH

It is also of interest to look at the effects of mismatches of amplitude functions upon entrance to an accelerator. The treatment below may be followed in more detail in [3] and [4]. A beam which is described by Courant-Snyder parameters that are not the periodic parameters of the accelerator into which it is injected will tend to filament due to nonlinearities and hence have its emittance increased. Suppose β and α are the Courant-Snyder parameters as delivered by the beamline to a particular point in an accelerator, and β_o, α_o are the periodic lattice functions of the ring at that point. A particle with trajectory (x, x') can be viewed in the $(x, \beta x' + \alpha x) \equiv (x, \eta)$ phase space corresponding to the beamline functions, or in the $(x, \beta_0 x' + \alpha_0 x) \equiv (x, \eta_0)$ phase space corresponding to the lattice functions of the ring. If the phase space motion lies on a circle in the beamline view, then the phase space motion will lie on an ellipse in the ring view. The equation of the ellipse in the "ring" system will be

$$\frac{(1 + \Delta\alpha_r^2)}{\beta_r}x^2 + 2\Delta\alpha_r\, x\eta_0 + \beta_r\eta_0^2 = \beta_0 A^2. \quad (24)$$

where $\beta_r \equiv \beta/\beta_0$ and $\Delta\alpha_r \equiv \alpha - \alpha_0(\beta/\beta_0)$.

If the phase space coordinate system were rotated so that the cross-term in the equation of the ellipse were eliminated, the ellipse would have the form $x_e^2/b_r + b_r\eta_{oe}^2 = \beta_0 A^2$ where $b_r \equiv F + \sqrt{F^2 - 1}$ and F is given by

$$F \equiv \frac{1}{2}\left[\beta_0\gamma + \gamma_0\beta - 2\alpha_0\alpha\right]. \quad (25)$$

Note that if $\Delta\alpha_r = 0$, then $b_r = \beta_r$.

There is a physical significance to the quantity b_r; it is the ratio of the areas of two circumscribed ellipses which have shapes and orientations given by the two sets of Courant-Snyder parameters found in the matrices J and J_0. This might suggest that a beam contained within the smaller ellipse upon injection into the synchrotron (whose periodic functions give ellipses similar to the larger one) will have its emittance increased by a factor b_r. However, this would be an over-estimate of the increase of the average of the emittances of all the particles.

If in the beamline view the new phase space trajectory is $x^2 + \eta^2 = b_r R^2$, then in the synchrotron view, the equation of the ellipse would be $\frac{x^2}{b_r R^2} + \frac{\eta_0^2}{R^2/b_r} = 1$. A particle with initial phase space coordinates x_i and η_{oi} will commence describing a circular trajectory of radius a in phase space upon subsequent revolutions about the ring. The equilibrium distribution will have variance in the x coordinate

$$\sigma^2 = \langle x^2\rangle = \frac{\langle a^2\rangle}{2} = \frac{b_r^2 + 1}{2b_r}\sigma_0^2 = F\,\sigma_0^2, \quad (26)$$

where σ_0^2 is the variance in the absence of a mismatch. This expression can be rewritten in terms of $\det\Delta J$ which we found in Section III.:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left|\det(\Delta J)\right|. \quad (27)$$

For the case where the slope of the amplitude function is matched and equal to zero, we have

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left(\frac{\Delta\beta/\beta_0}{\sqrt{1 + \Delta\beta/\beta_0}}\right)^2. \quad (28)$$

This says that a 20% β mismatch at injection, for example, would cause only a 2% increase in the rms emittance.

VI. REFERENCES

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