

Dispersive analysis of the $\gamma\gamma \rightarrow D\bar{D}$ data

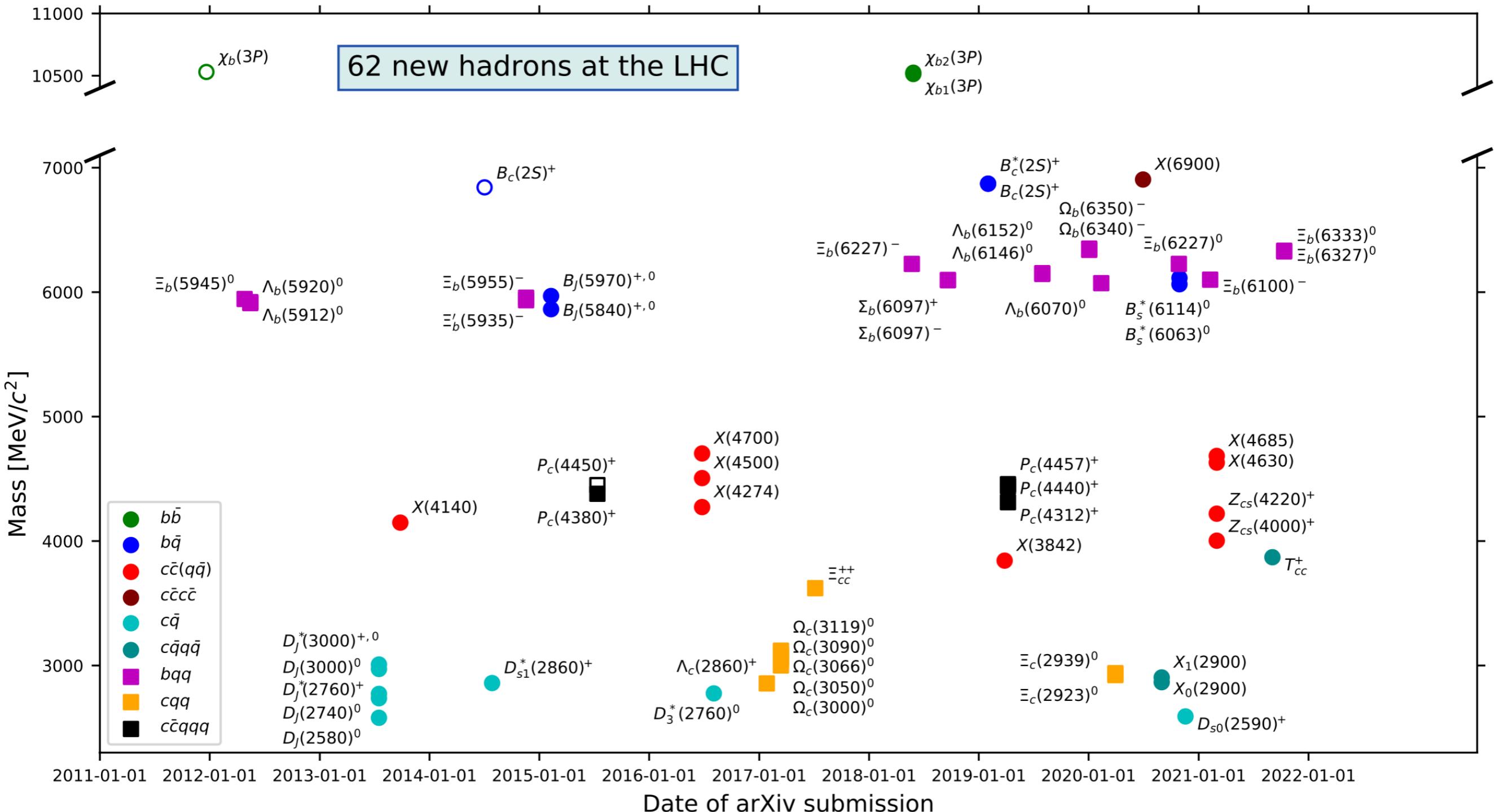
Igor Danilkin

in coll. with Oleksandra Deineka,
Marc Vanderhaeghen

Phys. Lett. B 827, 136982 (2022)

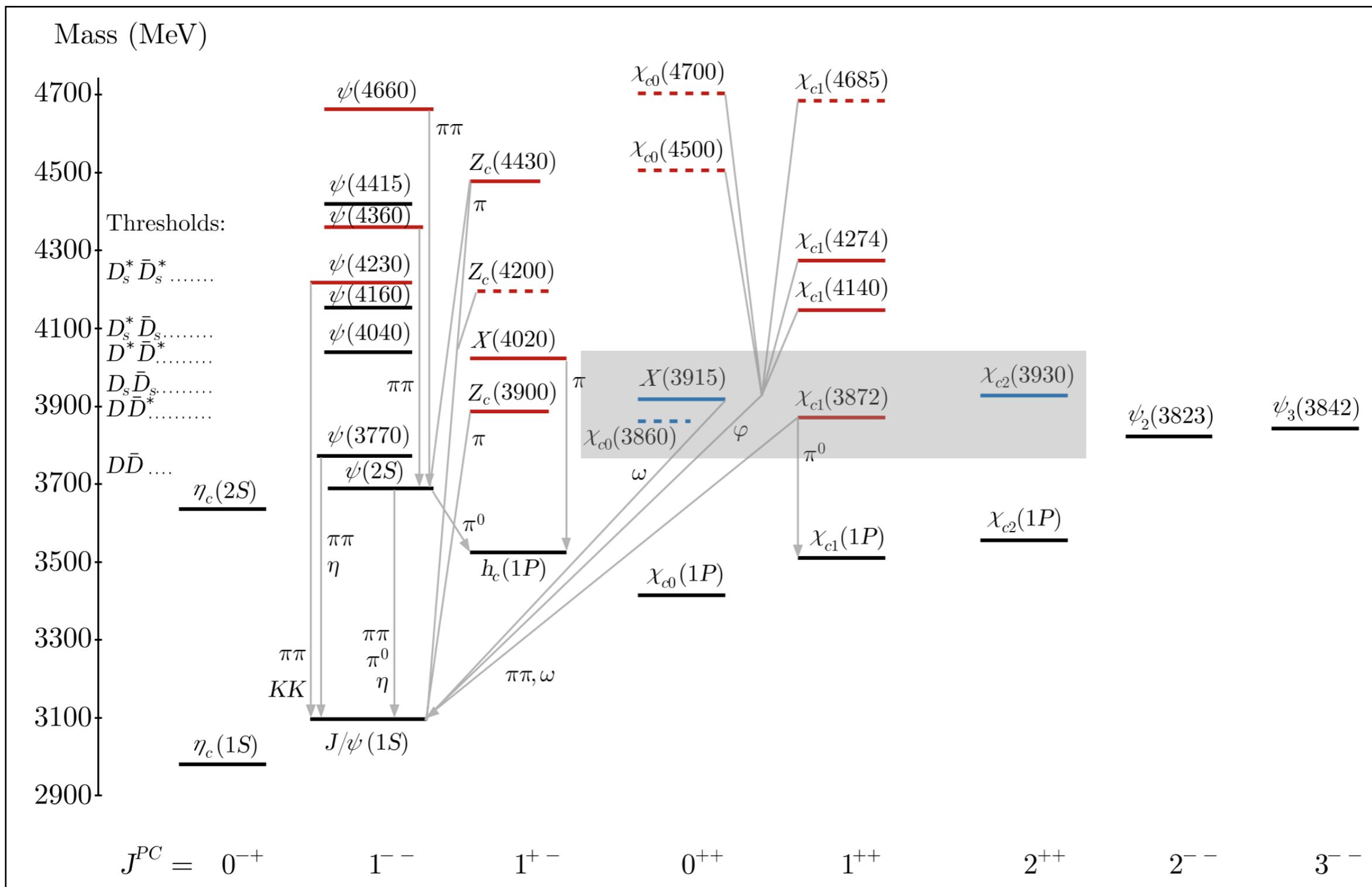
EXCITED QCD 2022, October 28, Sicily

New states



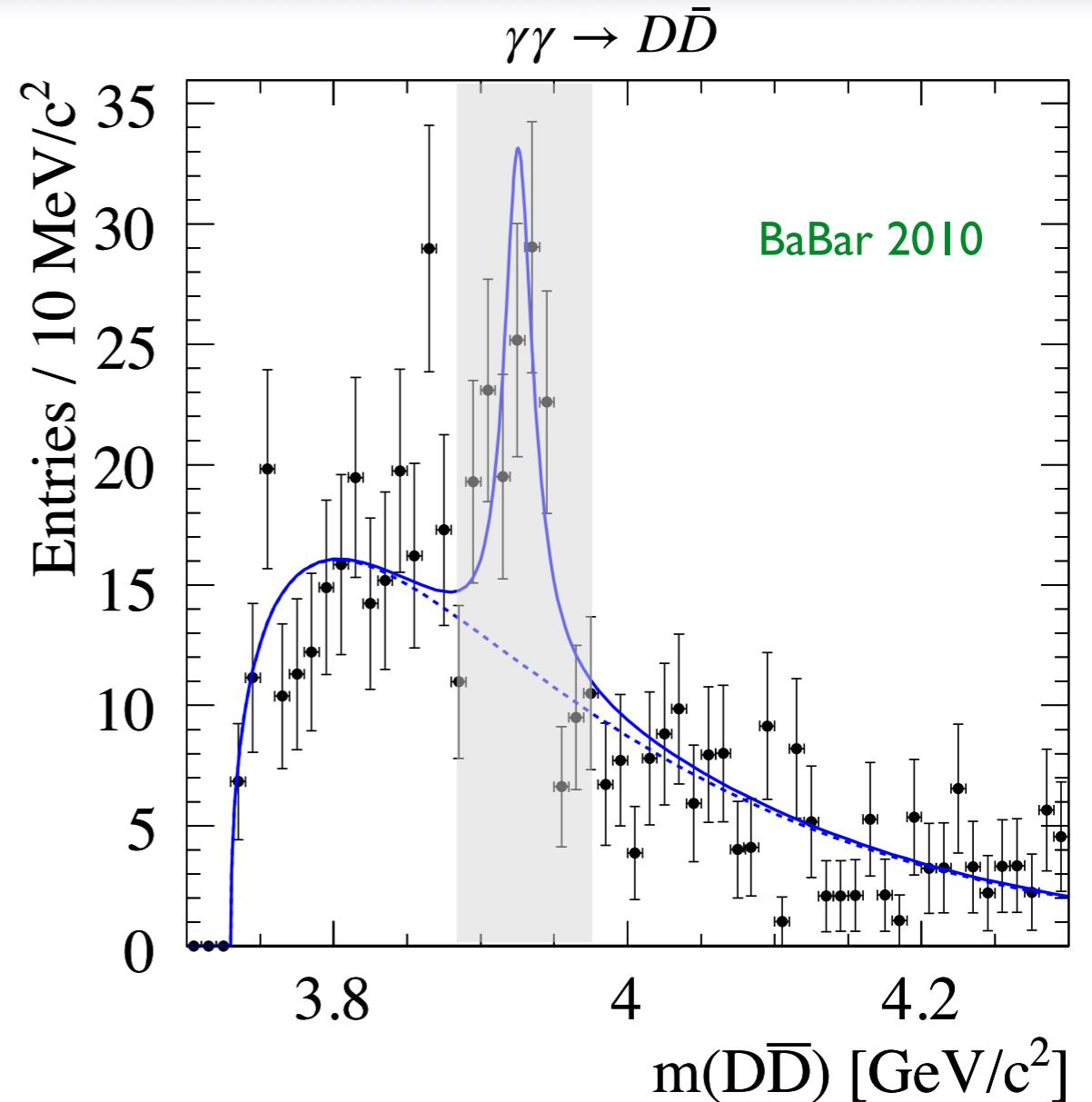
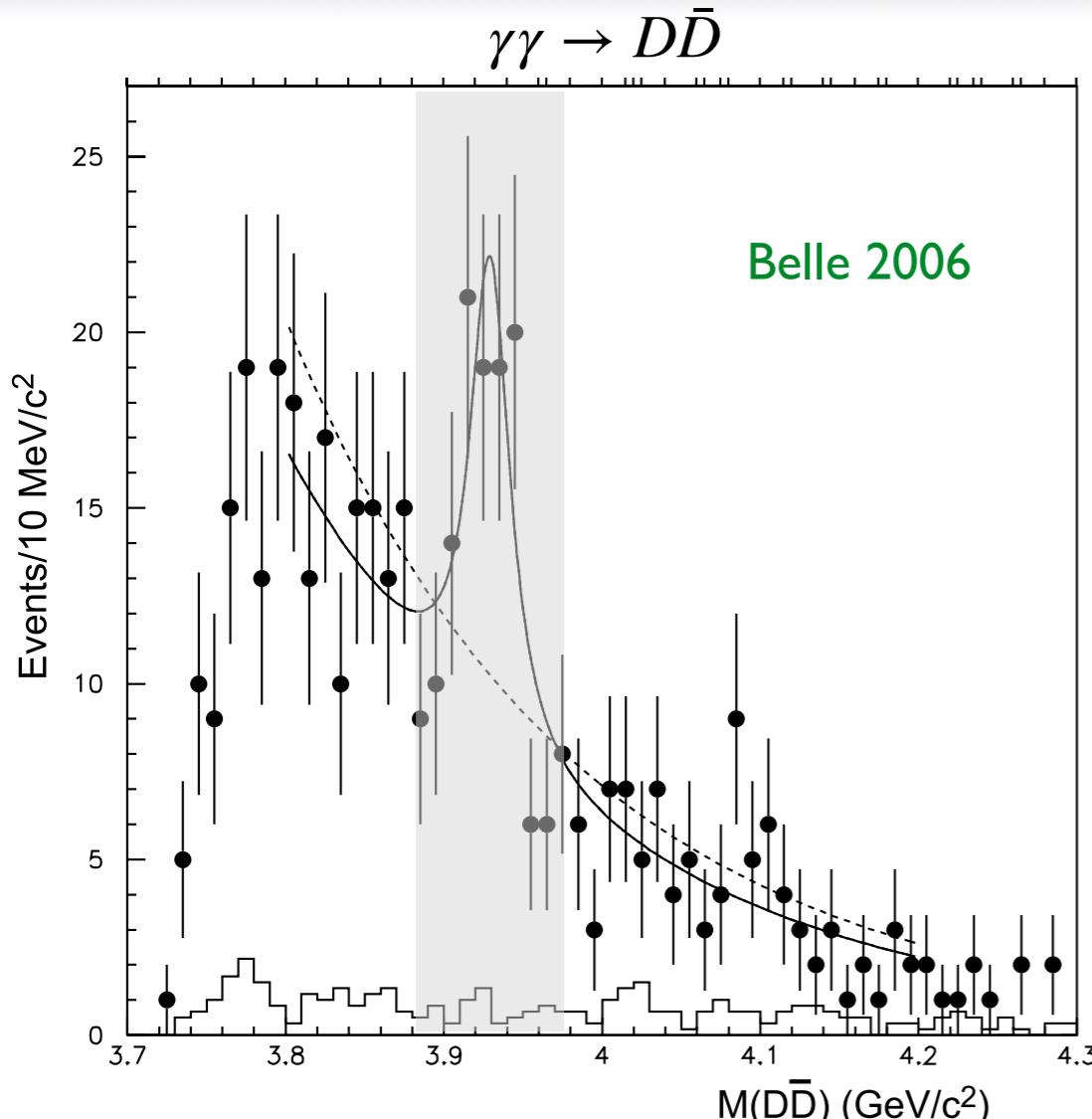
+ data from Babar, Belle, COMPASS, ...

What are we looking for?



PDG 2021

Everything is fine with $\chi_{c2}(2P)$



LHCb, 2019: $pp \rightarrow D\bar{D} + \text{anything}$
 LHCb, 2020: $B^+ \rightarrow D^+ D^- K^+$

$X(3930)$
 $=$
 $\chi_{c2}(3930)$

$\chi_{c2}(3930)$

$\chi_{c2}(3930)$ MASS

$\chi_{c2}(3930)$ WIDTH

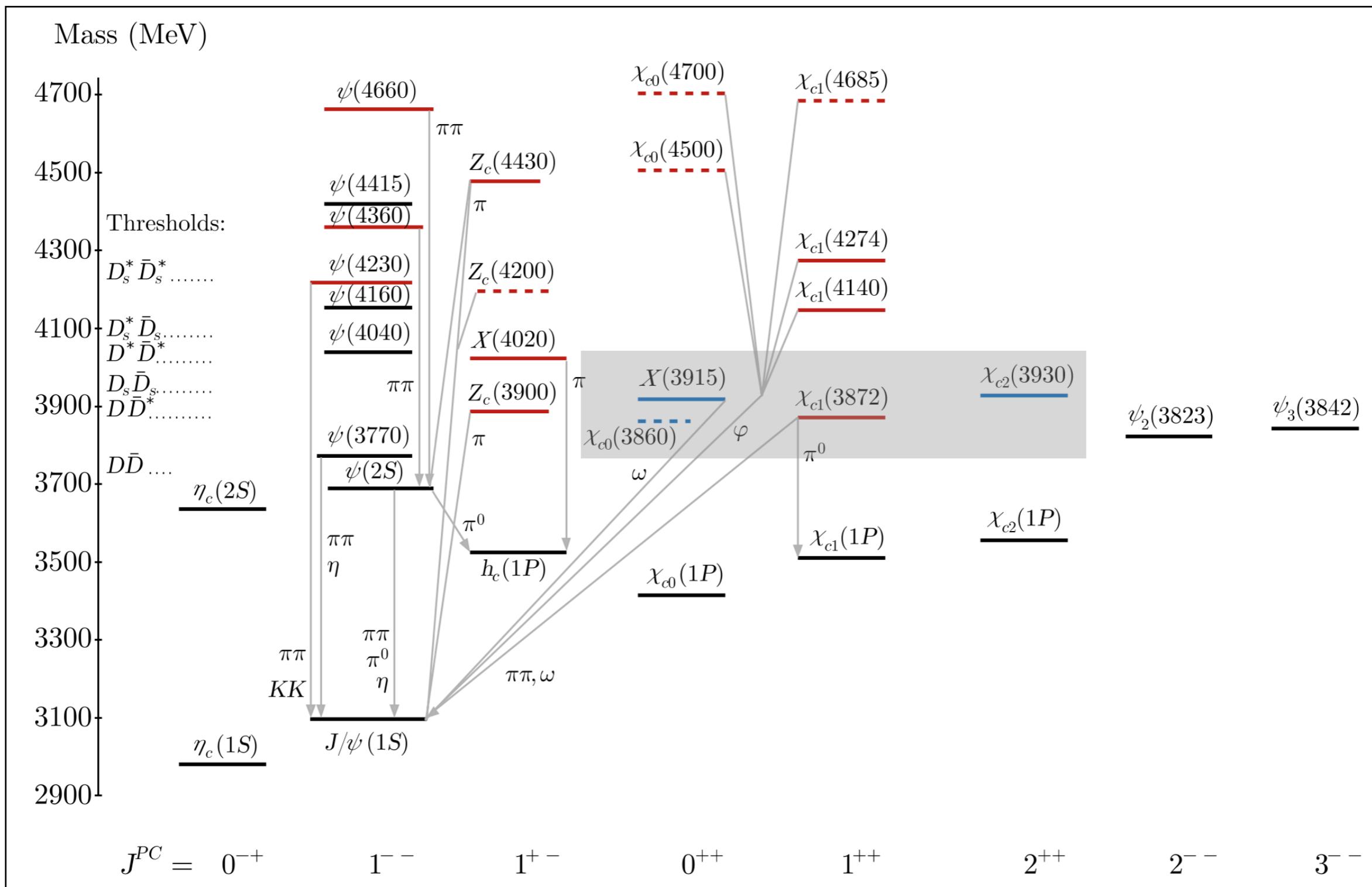
$I^G(J^{PC}) = 0^+(2^{++})$

PDG 2022

3922.5 ± 1.0 MeV ($S = 1.7$)

35.2 ± 2.2 MeV ($S = 1.2$)

What are we looking for?



is $X(3915)$ a $\chi_{c0}(2P)$?

[Belle 2005] $B \rightarrow J/\psi\omega K : X(3915)$:
later confirmed by [BaBar 2008, 2010]

[Belle 2010] $\gamma\gamma \rightarrow X(3915) \rightarrow J/\psi\omega$
[BaBar 2012] spin-parity analysis : $J^{PC} = 0^{++}$
(assuming helicity-2 dominance of tensor resonance)

Problems: [Brambilla et al. 2011, Olsen 2015, ...]

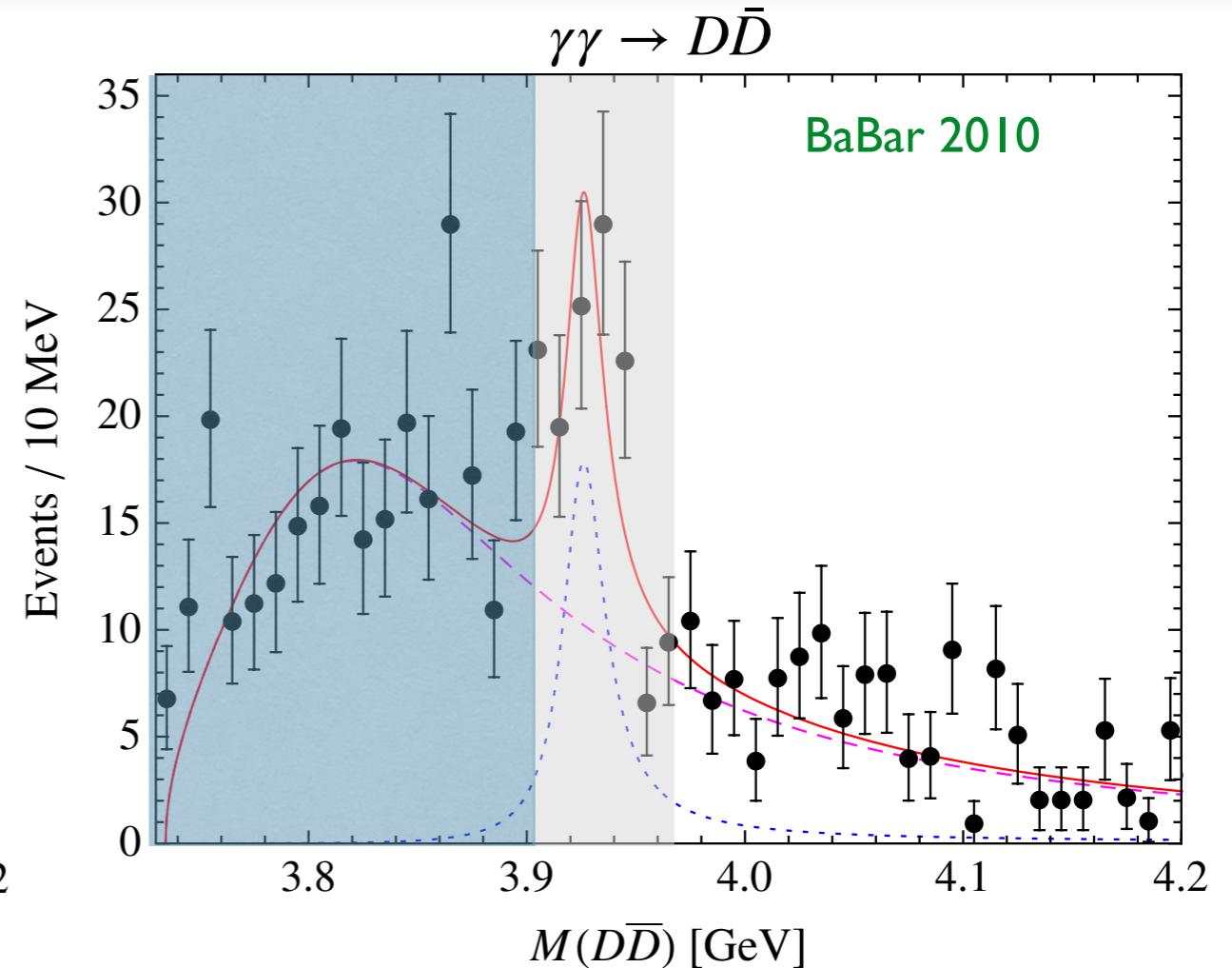
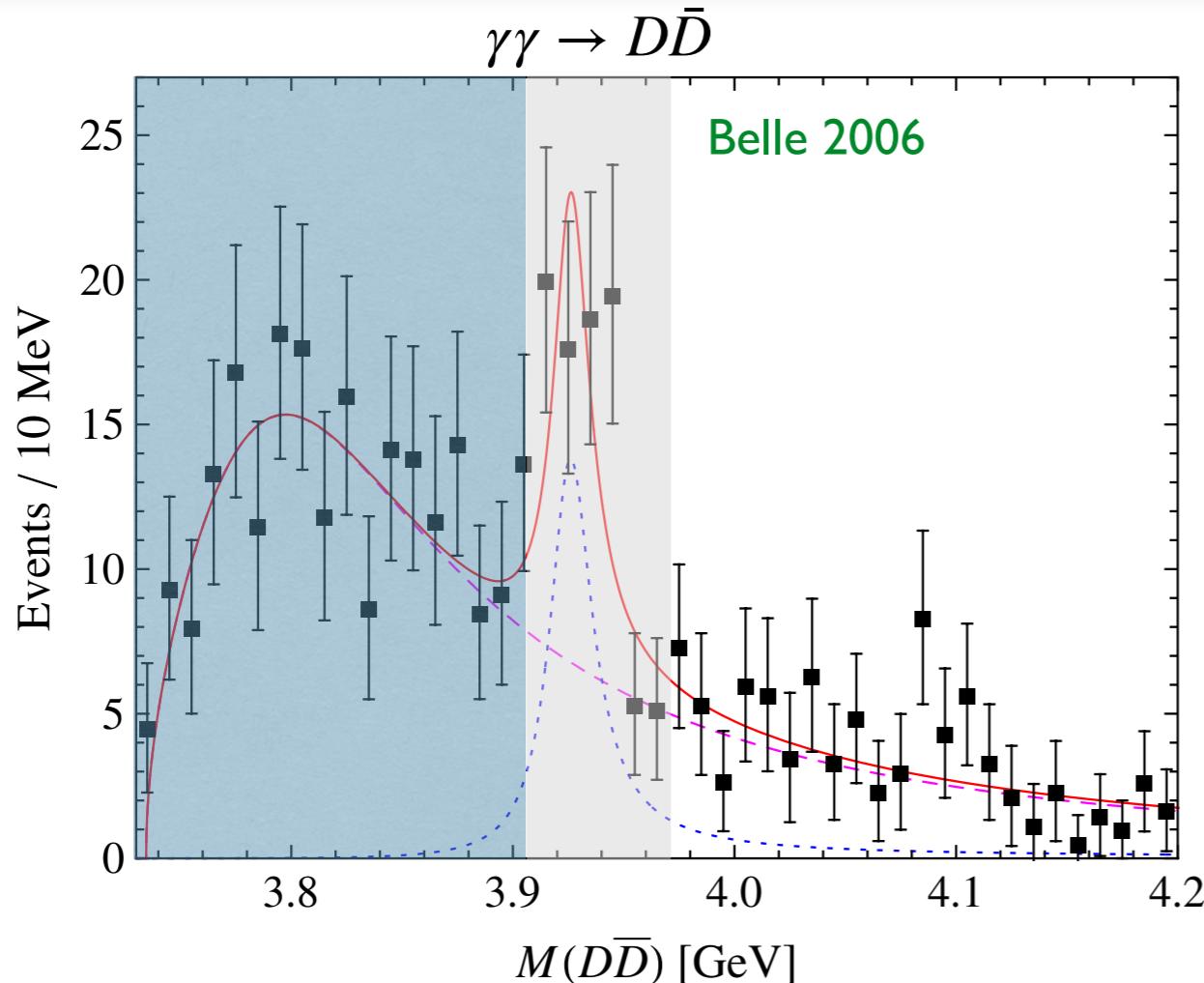
- No decay mode to DD (S-wave) was observed
- The $X(3915) \rightarrow J/\psi\omega$ decay should be OZI suppressed
- Narrow, width of ~ 20 MeV
- Small mass splitting with $\chi_{c2}(3930)$
- Might actually be the same state as $\chi_{c2}(3930)$ [Zhou et al. 2015]

[LHCb 2021] $B^+ \rightarrow D^+D^-K^+$
found narrow $J^{PC} = 0^{++}$ resonance around ~ 3.92 GeV
(amplitude analysis)

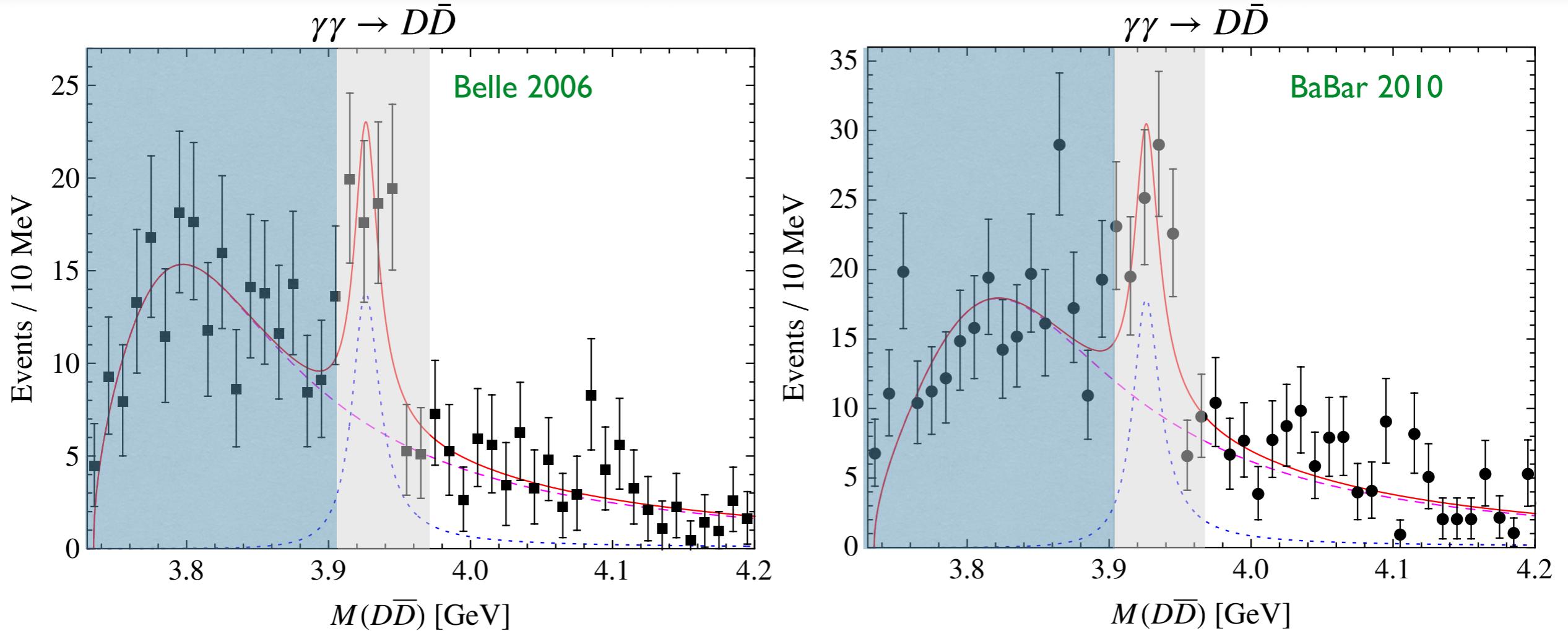
$X(3915)$
?
 $\chi_{c0}(3915)$

$\chi_{c0}(3915)$	$I^G(J^{PC}) = 0^+(0^{++})$	PDG 2022
$\chi_{c0}(3915)$ MASS		3921.7 ± 1.8 MeV
$\chi_{c0}(3915)$ WIDTH		18.8 ± 3.5 MeV

Another possibility for $\chi_{c0}(2P)$?



Another possibility for $\chi_{c0}(2P)$?



[Guo Meißner 2010]

two Breit-Wigner functions: mass and width of $\chi_{c2}(3930)$ is fixed, and $\chi_{c0}(2P)$ is fitted

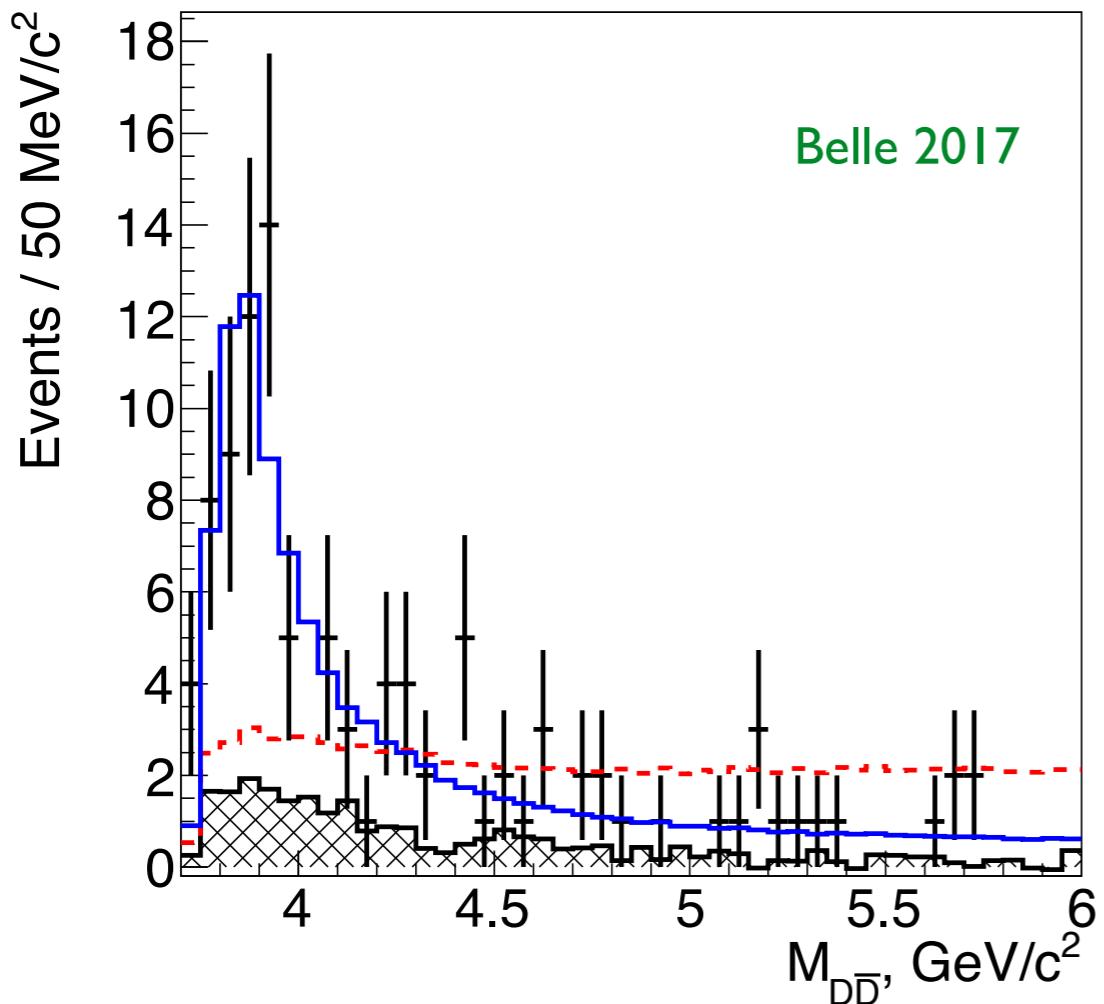
$$B_L(s) = \left(\frac{p(s)}{p(m_R^2)} \right)^{2L+1} \frac{m_R}{\sqrt{s}} \frac{F_L^2(s)}{(s - m_R^2)^2 + m_R^2 \Gamma^2(s)}, \quad \Gamma(s) = \Gamma_R \left(\frac{p(s)}{p(m_R^2)} \right)^{2L+1} \frac{m_R}{\sqrt{s}} F_L^2(s)$$



$$M_{\chi_{c0}(2P)} = 3837.6 \pm 11.5 \text{ MeV}, \quad \Gamma_{\chi_{c0}(2P)} = 221 \pm 19 \text{ MeV}$$

is $X(3860)$ a $\chi_{c0}(2P)$?

$e^+e^- \rightarrow J/\psi D\bar{D}$



Problems:

- The $e^+e^- \rightarrow J/\psi D\bar{D}$ statistics is rather limited
- LHCb 2021 amplitude analysis of $B^+ \rightarrow D^+D^-K^+$ see no evidence for broad $X(3860)$

$X(3860)$
?
 $\chi_{c0}(3860)$

$\chi_{c0}(3860)$

$\chi_{c0}(3860)$ MASS

$\chi_{c0}(3860)$ WIDTH

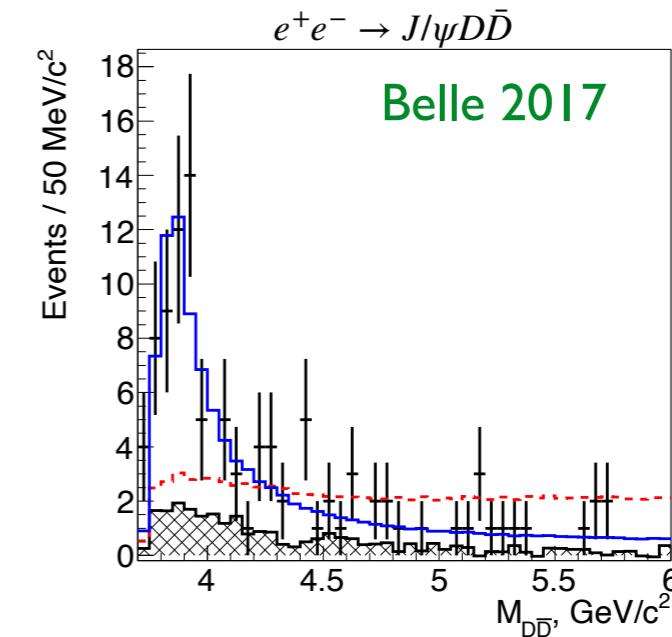
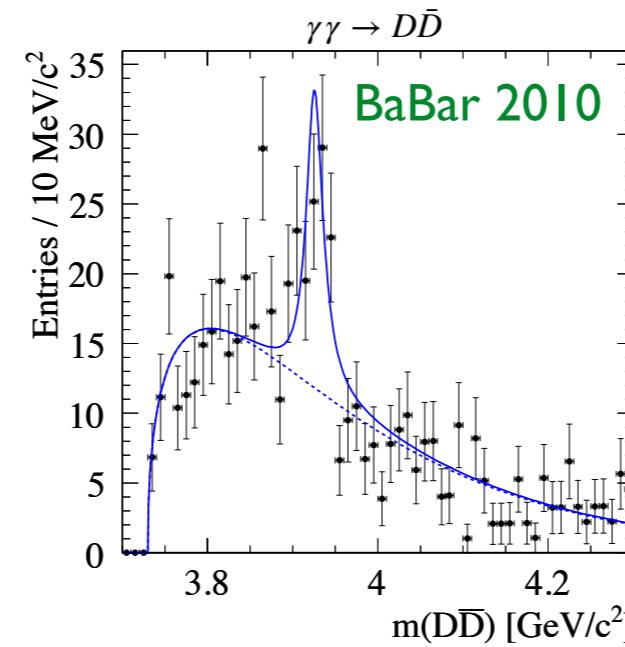
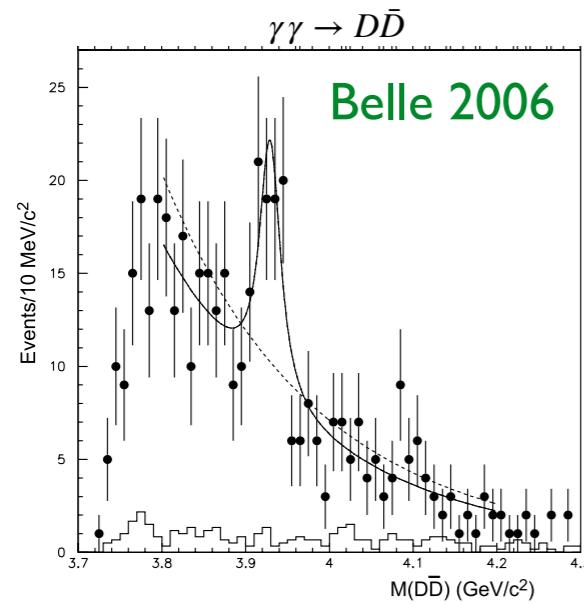
$I^G(J^{PC}) = 0^+(0^{++})$

PDG 2022

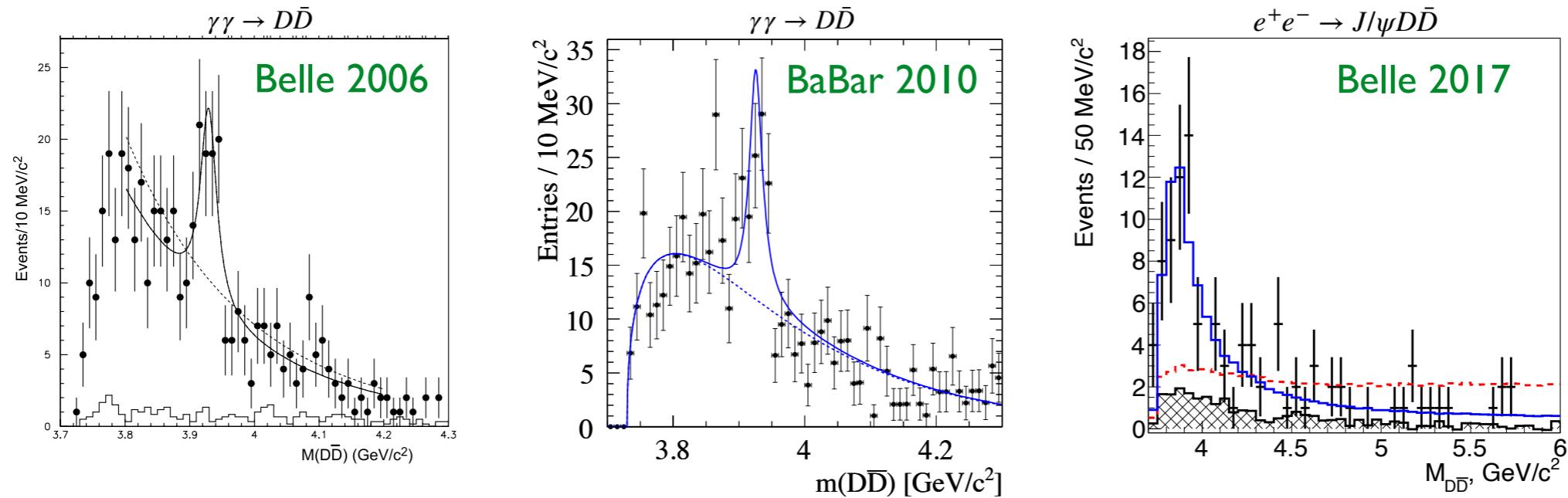
3862^{+50}_{-35} MeV

201^{+180}_{-110} MeV

Theory overview

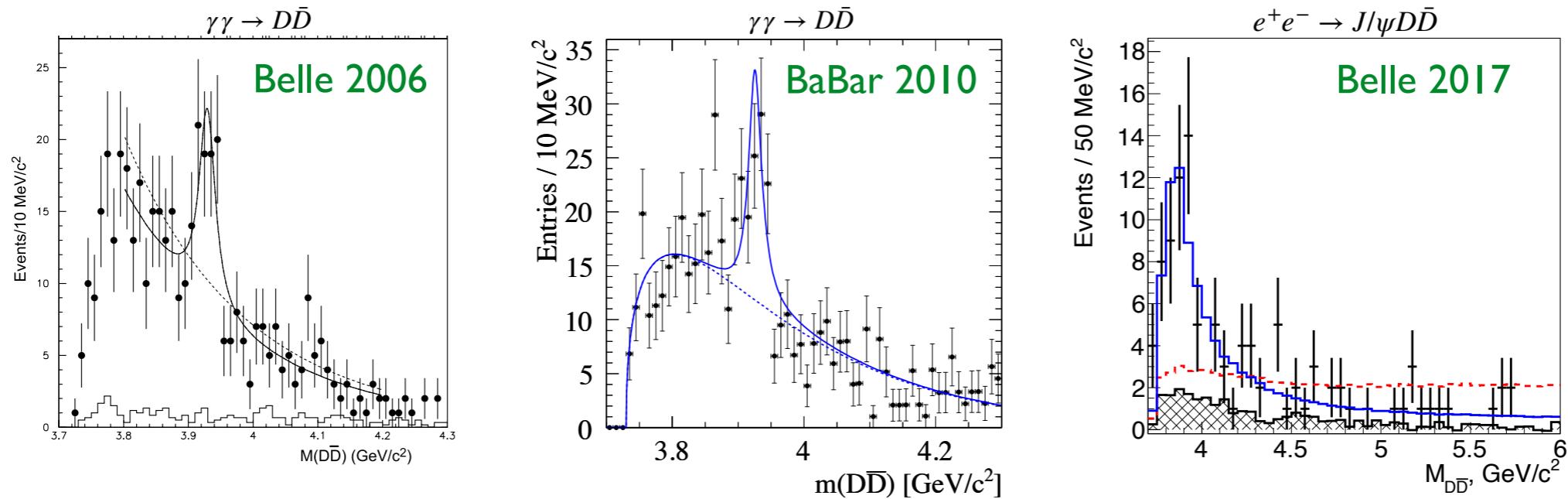


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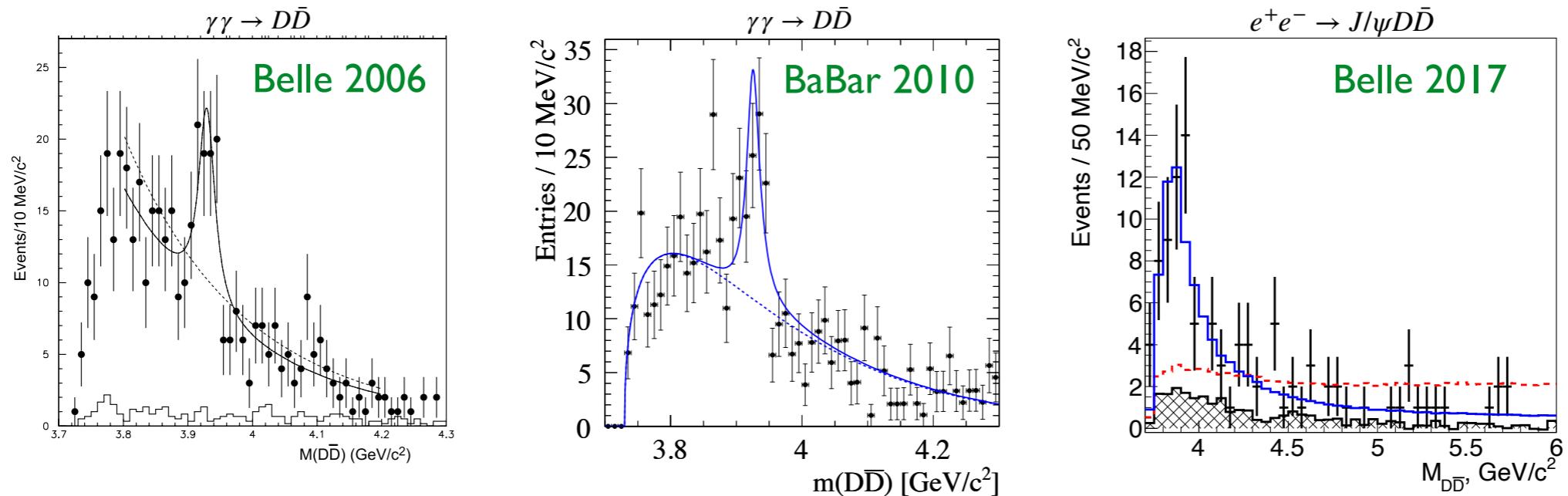
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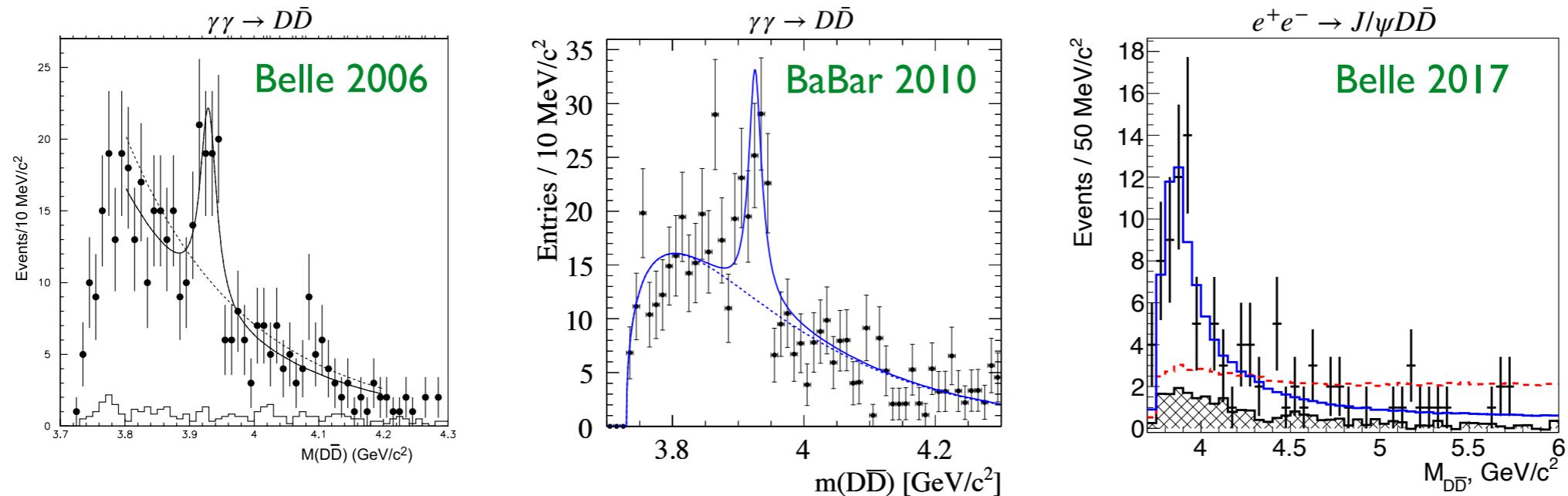
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- [Wang et al. 2021, Gemanian et al. 2007] Unitary approach (BSE) found that $\gamma\gamma \rightarrow D\bar{D}$ & $e^+e^- \rightarrow J/\psi D\bar{D}$ data do not contradict dynamics with the encoded **bound state**: no broad $\chi_{c0}(3860)$ is needed

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- [Prelovsek et al. 2022] Lattice analysis of $\{D\bar{D}, D_s\bar{D}_s\}$ with $m_\pi = 280 \text{ MeV}$ found 3 states with 0^{++}
 - 1) $D\bar{D}$ bound state
 - 2) broad state likely related to $\chi_{c0}(3860)$
 - 3) $D_s\bar{D}_s$ quasi bound state which might be $\chi_{c0}(3915)$ or $X(3960)$

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In order to figure out what is going on with 0^{++} we employ a data driven dispersive analysis of $\gamma\gamma \rightarrow D\bar{D}$ and $e^+e^- \rightarrow J/\psi D\bar{D}$

Formalism

partial wave dispersion relation (S wave)

- Full p.w. dispersion relation (causality, crossing, unitarity)

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\operatorname{Im} t_{ab}(s')}{s' - s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} t_{ab}(s')}{s' - s}$$

- Unitarity relation

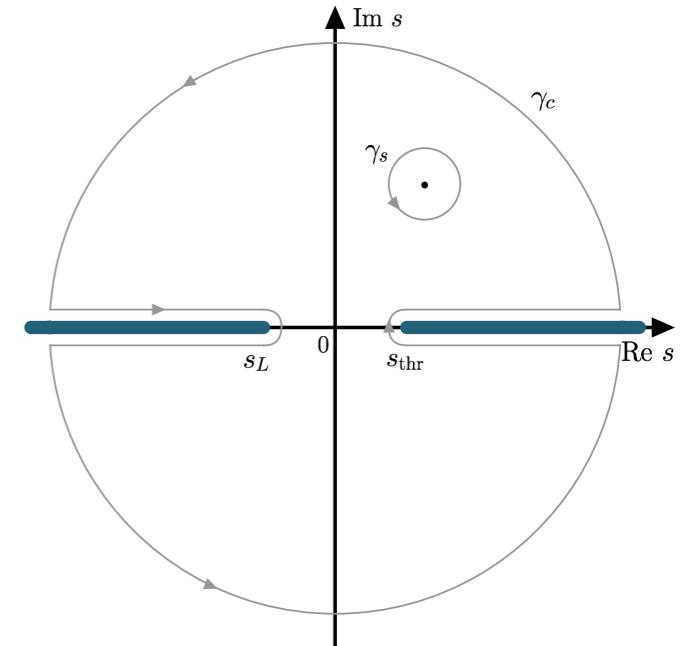
$$\operatorname{Im} t_{ab}(s) = \sum_c t_{ac}(s) \rho_c(s) t_{cb}^*(s)$$

$$-\frac{1}{2\rho_1} \leq \operatorname{Re} t_{11}(s) \leq \frac{1}{2\rho_1}, \quad 0 < \operatorname{Im} t_{11}(s) \leq \frac{1}{\rho_1}, \quad \dots$$

- Assuming $t(\infty) \rightarrow \text{const}$ we subtract the dispersion relation once

$$t_{ab}(s) = t_{ab}(0) + \underbrace{\frac{s}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\operatorname{Im} t_{ab}(s')}{s' - s}}_{U_{ab}(s)} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s') \rho_c(s') t_{cb}^*(s')}{s' - s}$$

(asymptotically bounded
unknown function)



partial wave dispersion relation (S wave)

- Once-subtracted p.w. dispersion relation

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can be solved using N/D method with input from $U_{ab}(s)$ **above threshold**

Chew, Mandelstam (1960)
Luming (1964)
Johnson, Warnock (1981)

$$t_{ab}(s) = \sum_c D_{ac}^{-1} N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ac}(s') \rho_c(s') (U_{cb}(s') - U_{cb}(s))}{s' - s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{N_{ab}(s') \rho_b(s')}{s' - s}$$

the obtained N/D solution can be checked
 that it **fulfills** the p.w. dispersion relation

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- Bound state case

$$\det(D_{ab}(s_B)) = 0$$

$$t_{ab}(s) = U_{ab}(s) + \frac{s}{s_B} \frac{g_a g_b}{s_B - s} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t_{ac}(s') \rho_c(s') t_{cb}^*(s')}{s' - s}$$

partial wave dispersion relation (S wave)

- $I = 0$ has $\{\gamma\gamma, \pi\pi, K\bar{K}, \dots, D\bar{D}\}$ channels, but the coupling of charmed $\{D\bar{D}\}$ with uncharmed $\{\pi\pi, K\bar{K}, \dots\}$ are strongly suppressed: separately focus on $\{\gamma\gamma, D\bar{D}\}$ and $\{\gamma\gamma, \pi\pi, K\bar{K}\}$

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- Neglect $\gamma\gamma$ intermediate states in the unitary relation and put $U_{\gamma\gamma \rightarrow \gamma\gamma} = 0$ ($\sim e^4$)
 => coupled-channel $\{\gamma\gamma, D\bar{D}\}$ equations reduce to the **hadronic part** and **photon-fusion part**

$$t_{DD}(s) = U_{DD}(s) + \frac{s}{\pi} \int_{4m_D^2}^{\infty} \frac{ds'}{s'} \frac{t_{DD}(s') \rho_D(s') t_{DD}^*(s')}{s' - s} = \frac{N_{DD}(s)}{D_{DD}(s)}$$

$$t_{\gamma\gamma,DD}(s) = U_{\gamma\gamma,DD}(s) + D_{DD}^{-1}(s) \left(-\frac{s}{\pi} \int_{4m_D^2}^{\infty} \frac{ds'}{s'} \frac{\text{Im } D_{DD}(s') U_{\gamma\gamma,DD}(s')}{s' - s} \right)$$

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- Using the known analytical structure of **left-hand cuts**, one can approximate $U_{ab}(s)$ as an expansion in a conformal mapping variable $\xi(s)$ **Gasparyan, Lutz (2010)**

to be determined from the fits

$$U_{DD}(s) \simeq \sum_{n=0}^{\infty} C_n \xi^n(s)$$

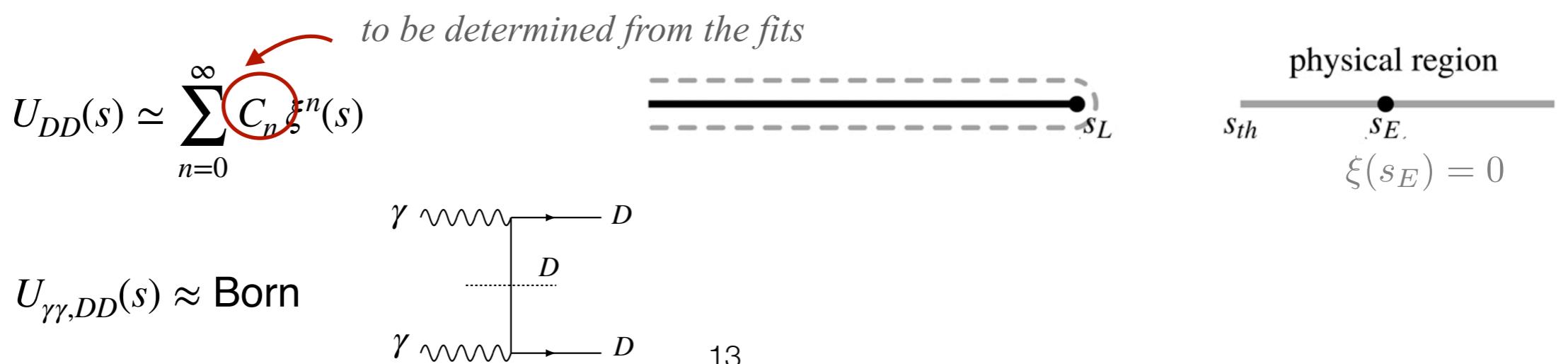

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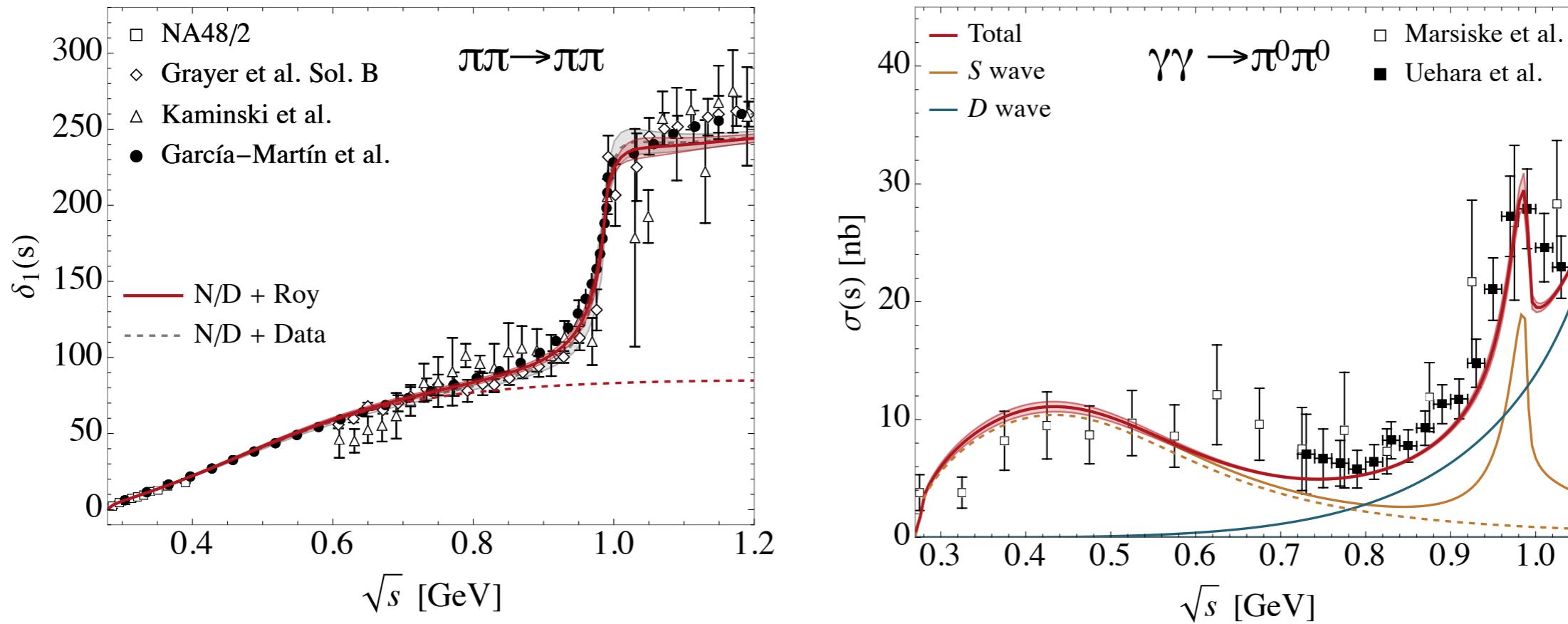
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How good is Born left-hand cut for $\{\gamma\gamma, \pi\pi, K\bar{K}\}$?



Input: experimental data/Roy analysis + threshold parameters NNLO + Adler zero NLO

	Our results		Roy-like analyses	
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i 256(9)^{+5}_{-8}$	$\gamma\gamma : 5.6(1)(1) \cdot 10^{-3}$ $\pi\pi : 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K} : 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i 275(15)$	$\gamma\gamma : 6.1(7) \cdot 10^{-3}$ $\pi\pi : 3.45^{+0.25}_{-0.29}$ $K\bar{K} : -$
$f_0(980)$	$993(2)^{+2}_{-1} - i 21(3)^{+2}_{-4}$	$\gamma\gamma : 4.0(8)^{+0.3}_{-1.1} \cdot 10^{-3}$ $\pi\pi : 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K} : 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i 25^{+11}_{-6}$	$\gamma\gamma : 3.8(1.4) \cdot 10^{-3}$ $\pi\pi : 2.3(2)$ $K\bar{K} : -$

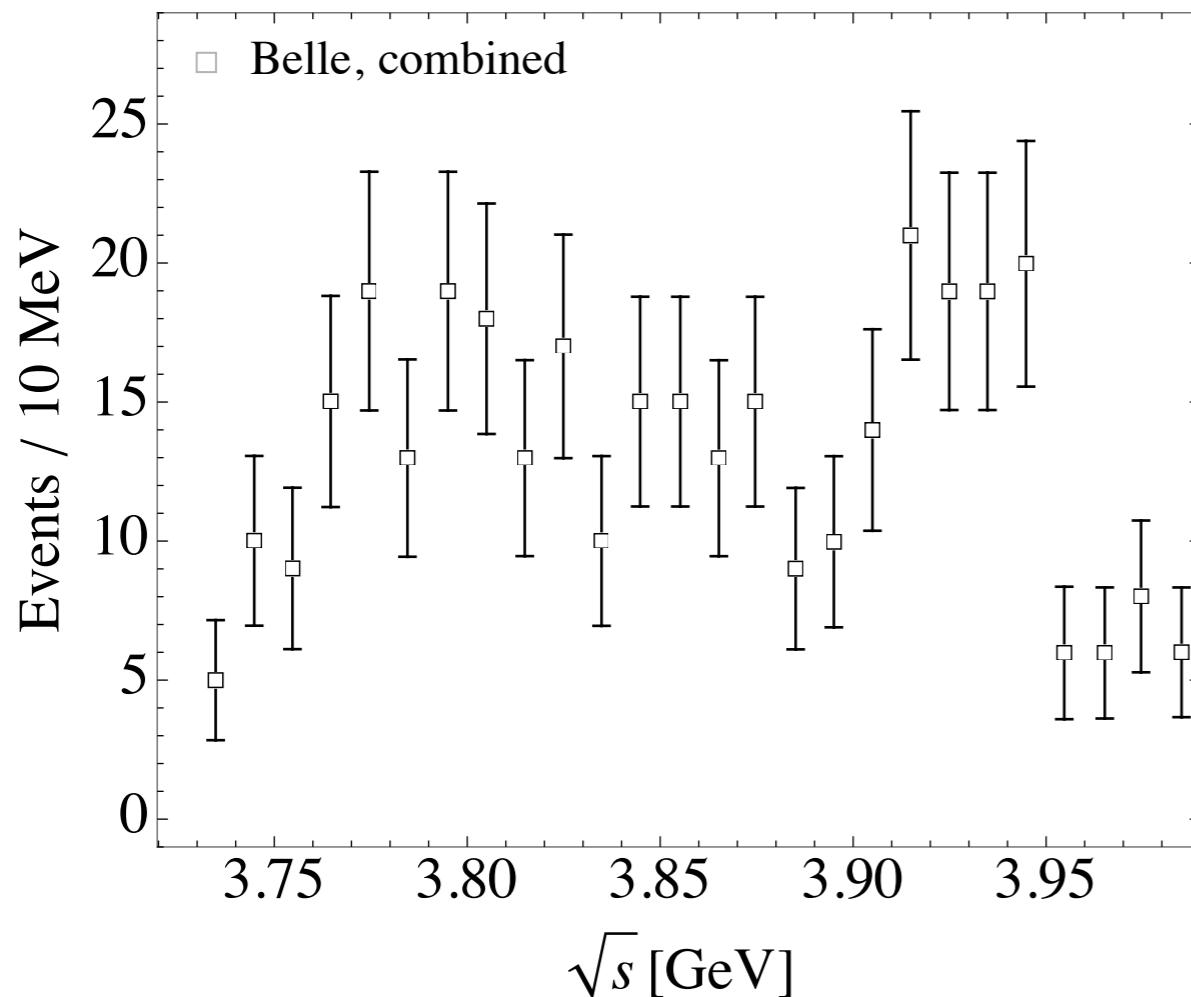
Naïve analysis of the combined $\gamma\gamma \rightarrow D\bar{D}$ data

S-wave: $I = 0$ with dispersive rescattering, $I = 1$ only Born

D-wave: $\chi_{c2}(3930)$ as a Breit-Wigner hel-2 with PDG mass/width

\Rightarrow 2 parameters from N/D
+ normalisation ratio (S/D wave)

$$\gamma\gamma \rightarrow D\bar{D} : \sigma_c(s) + \sigma_n(s) = \sigma_{I=0} + \sigma_{I=1}$$



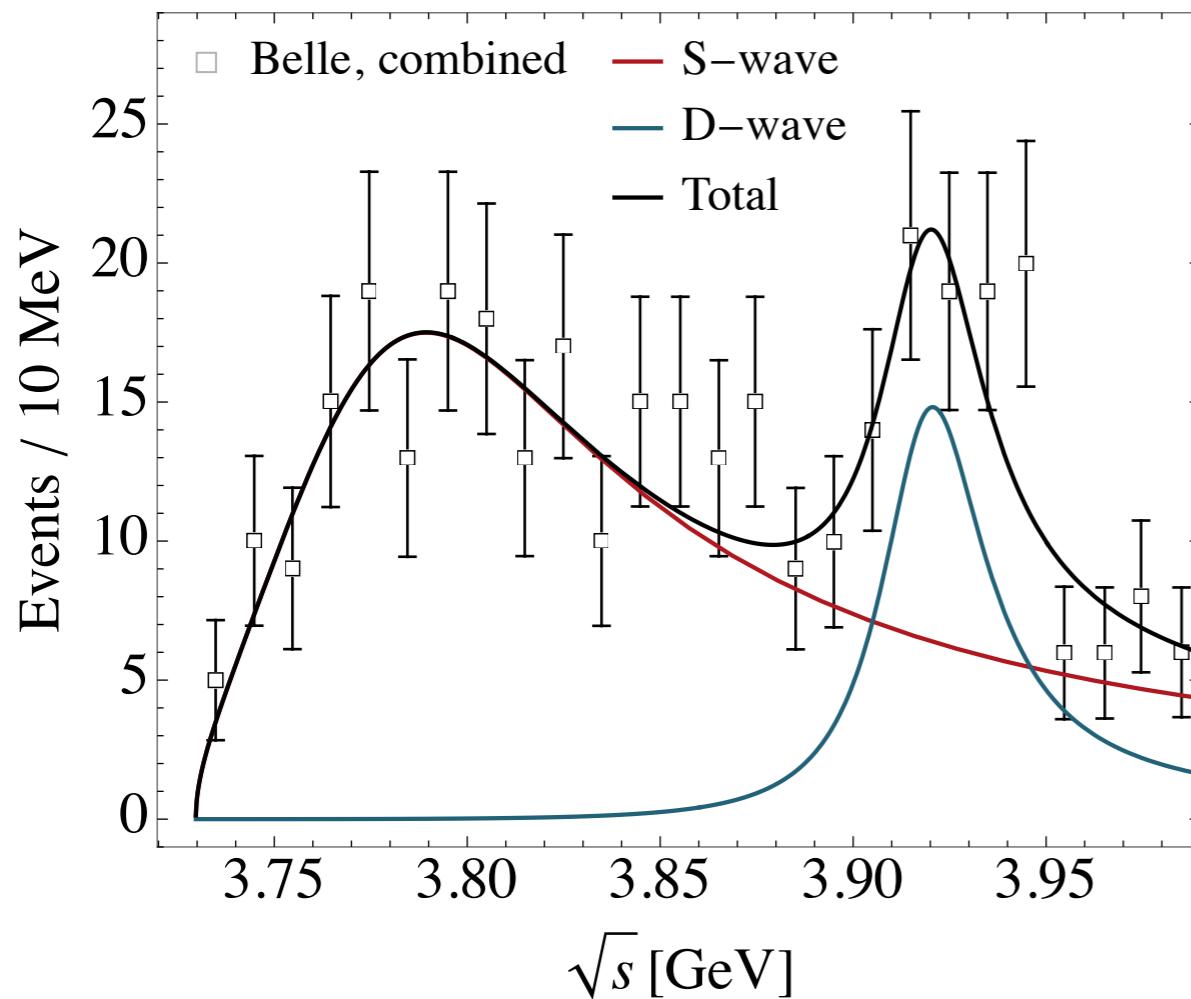
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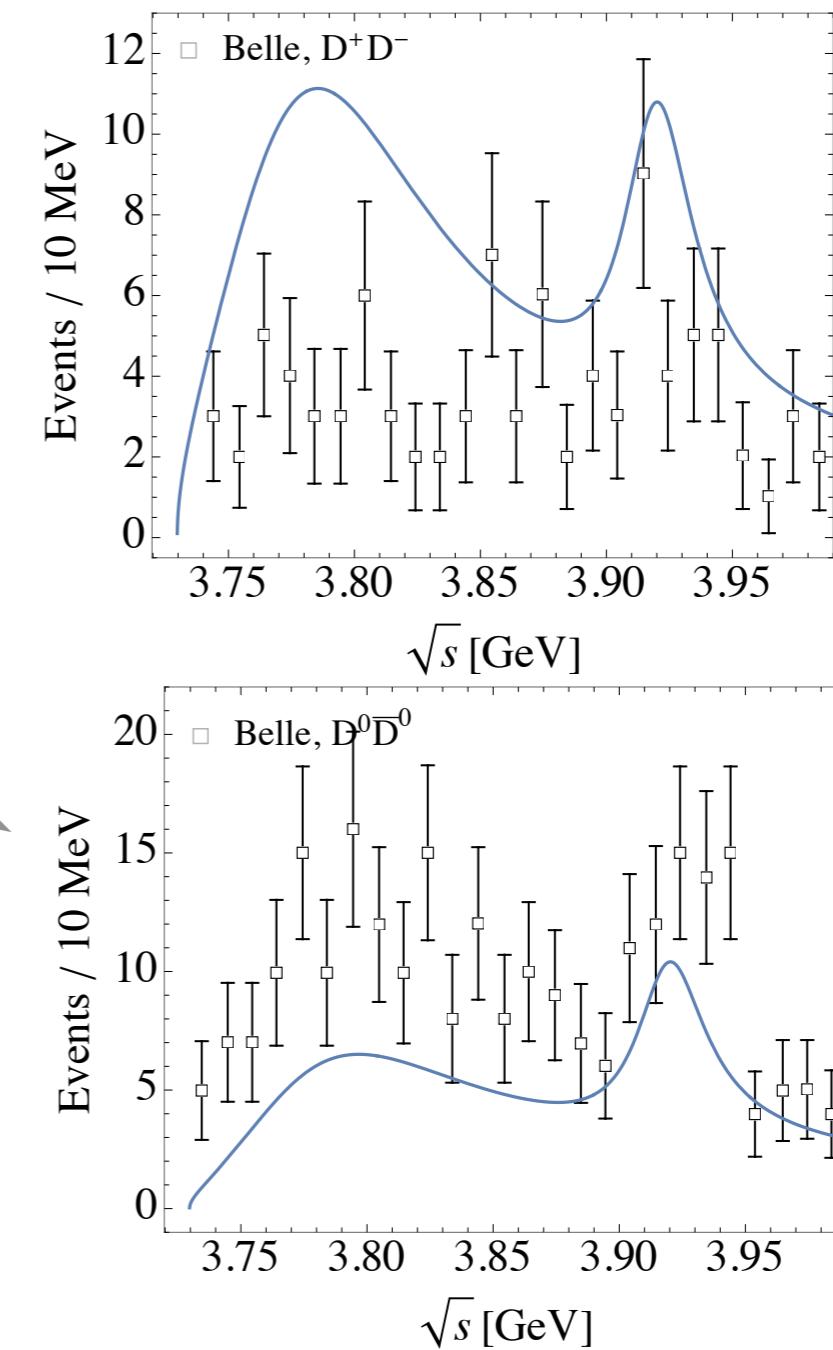
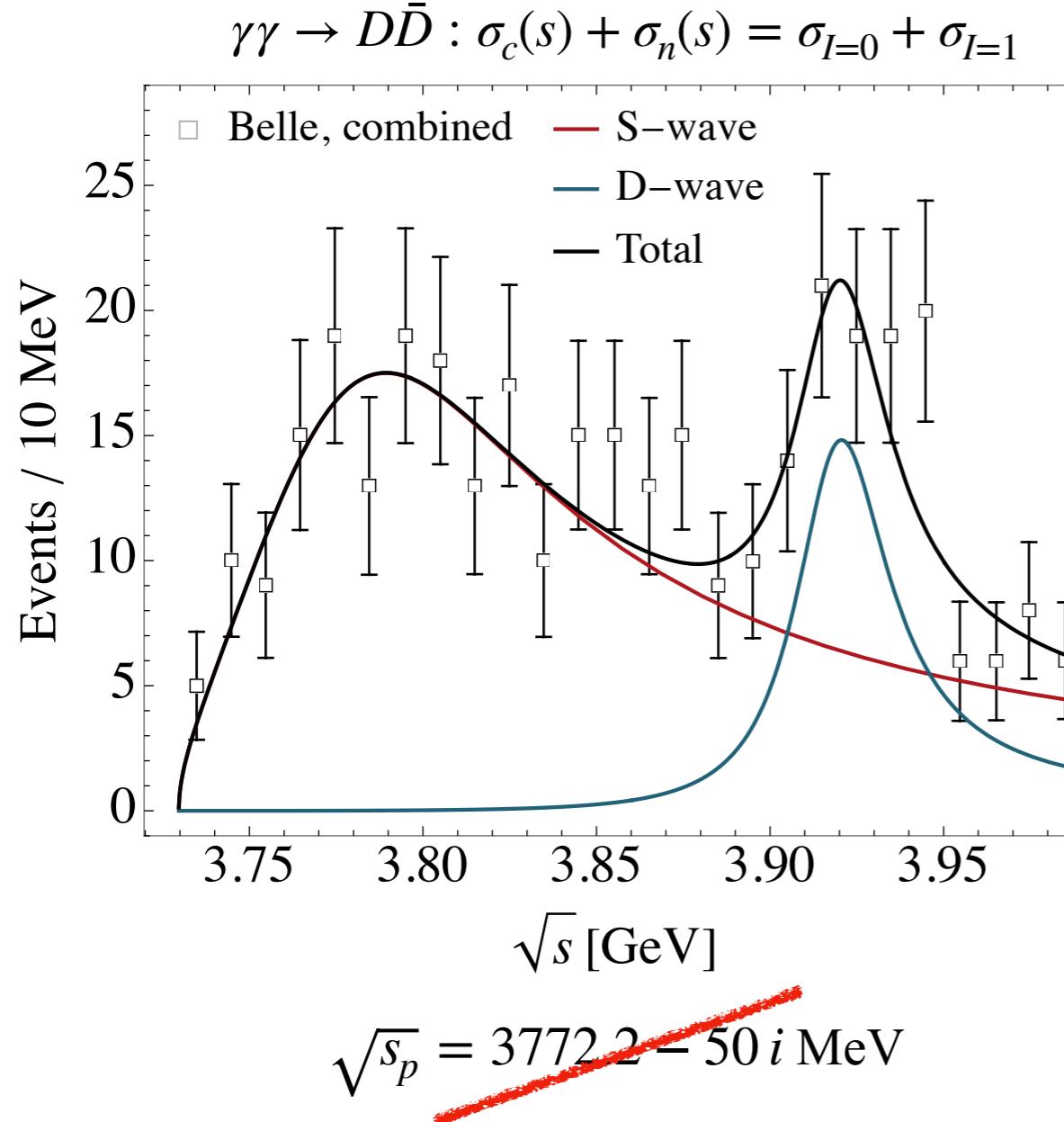
$$\sqrt{s_p} = 3772.2 - 50i \text{ MeV}$$

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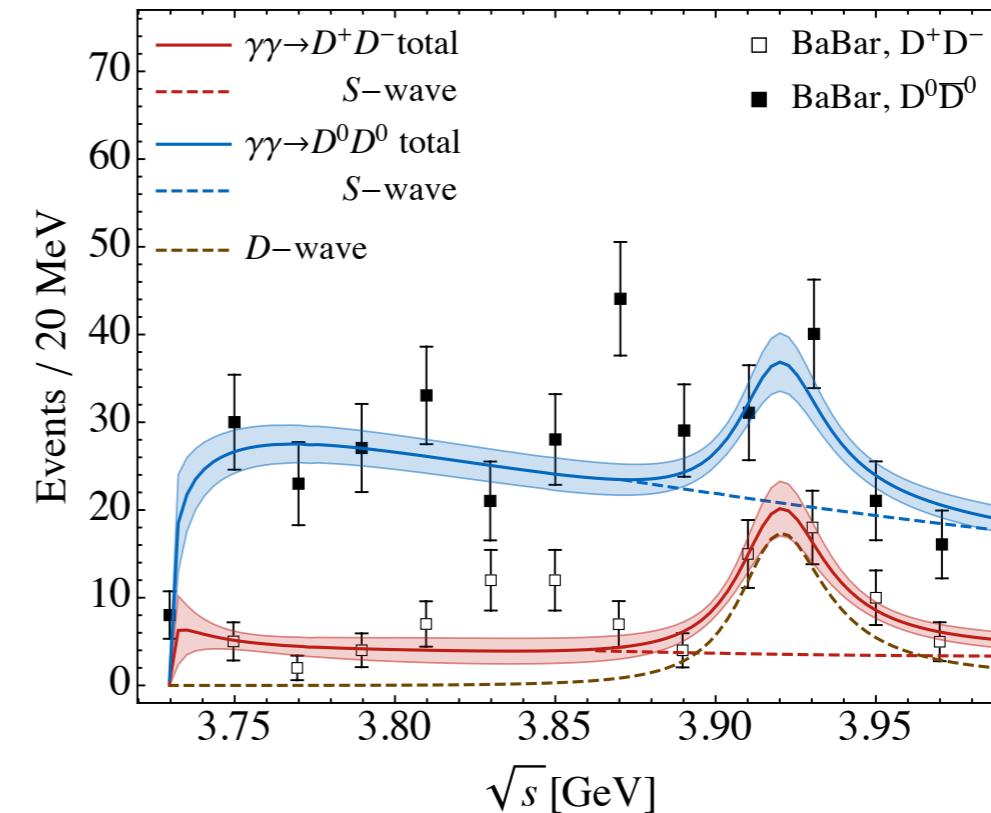
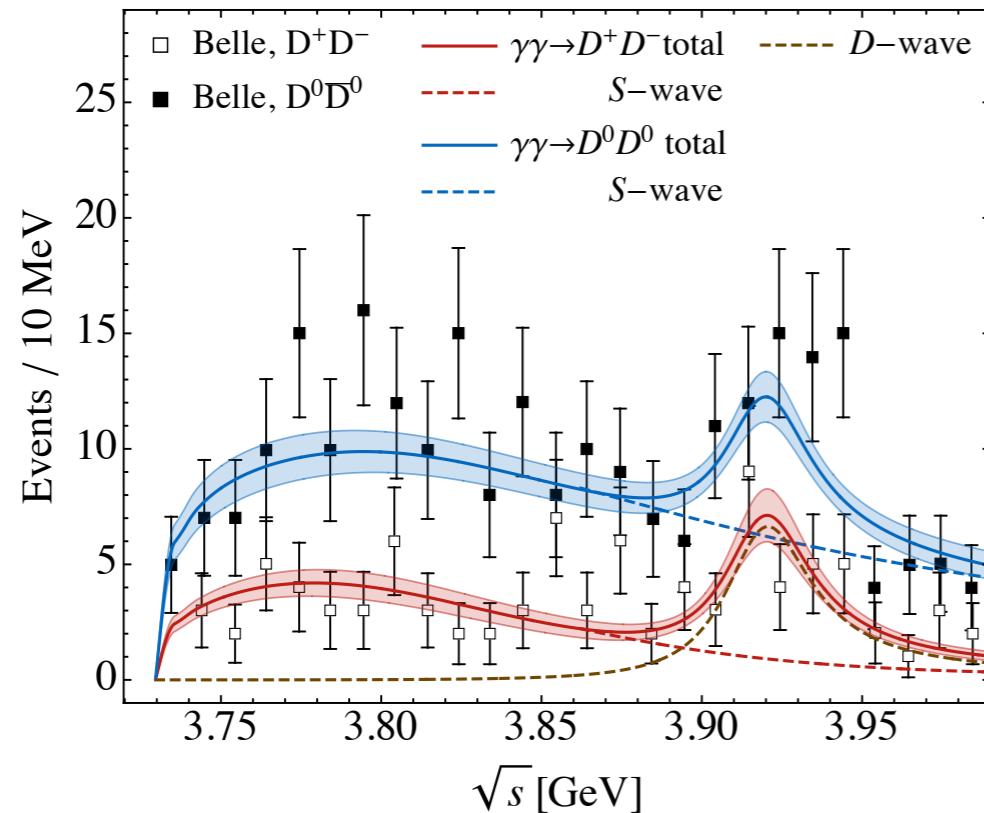
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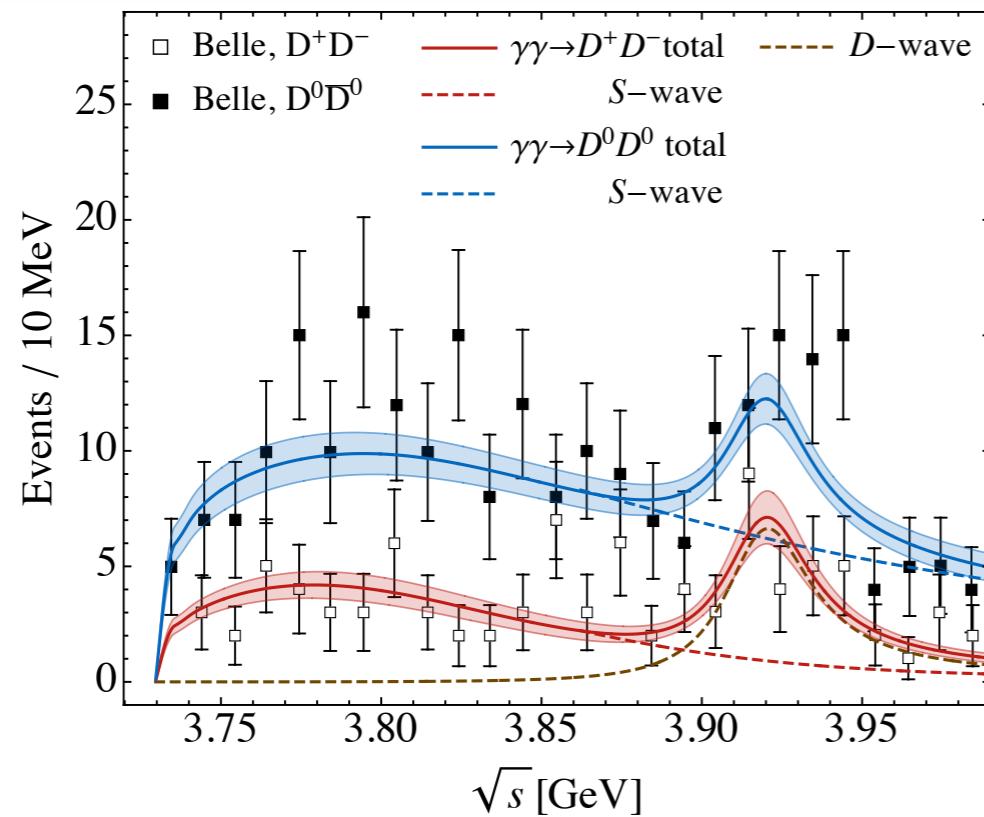


The fit to combined $\sigma_c(s) + \sigma_n(s)$ data **does not** describe individually $\sigma_c(s), \sigma_n(s)$

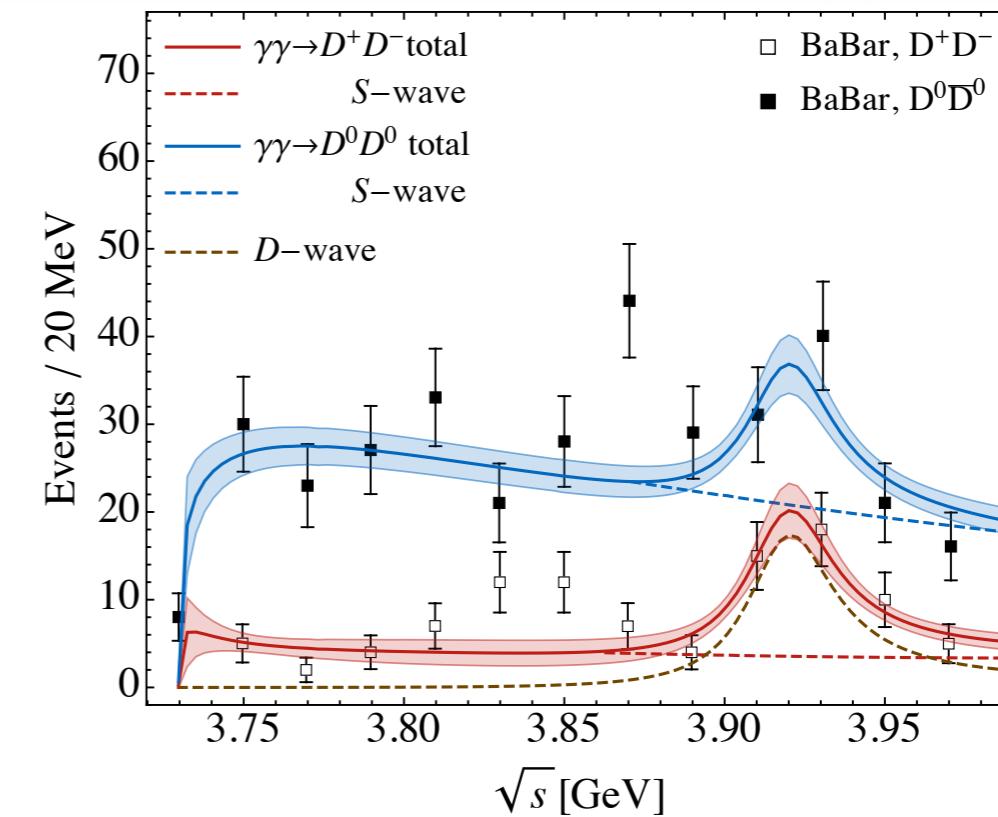
Analysis of $\gamma\gamma \rightarrow D^+D^-$ and $\gamma\gamma \rightarrow D^0\bar{D}^0$ data



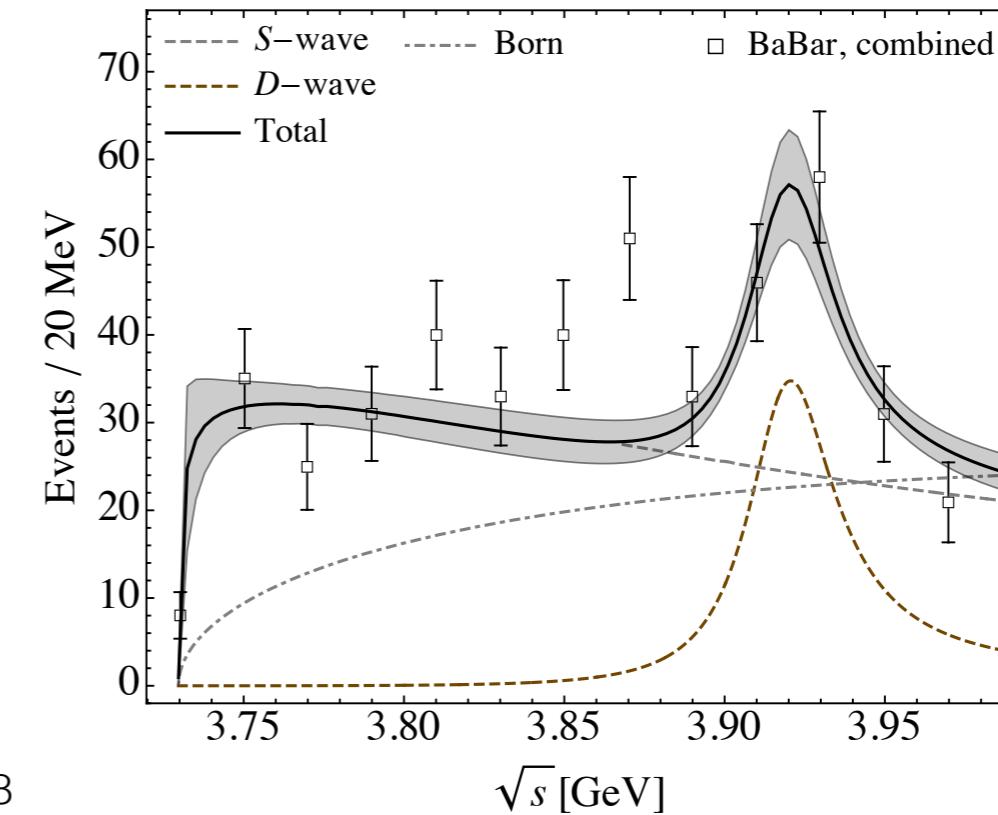
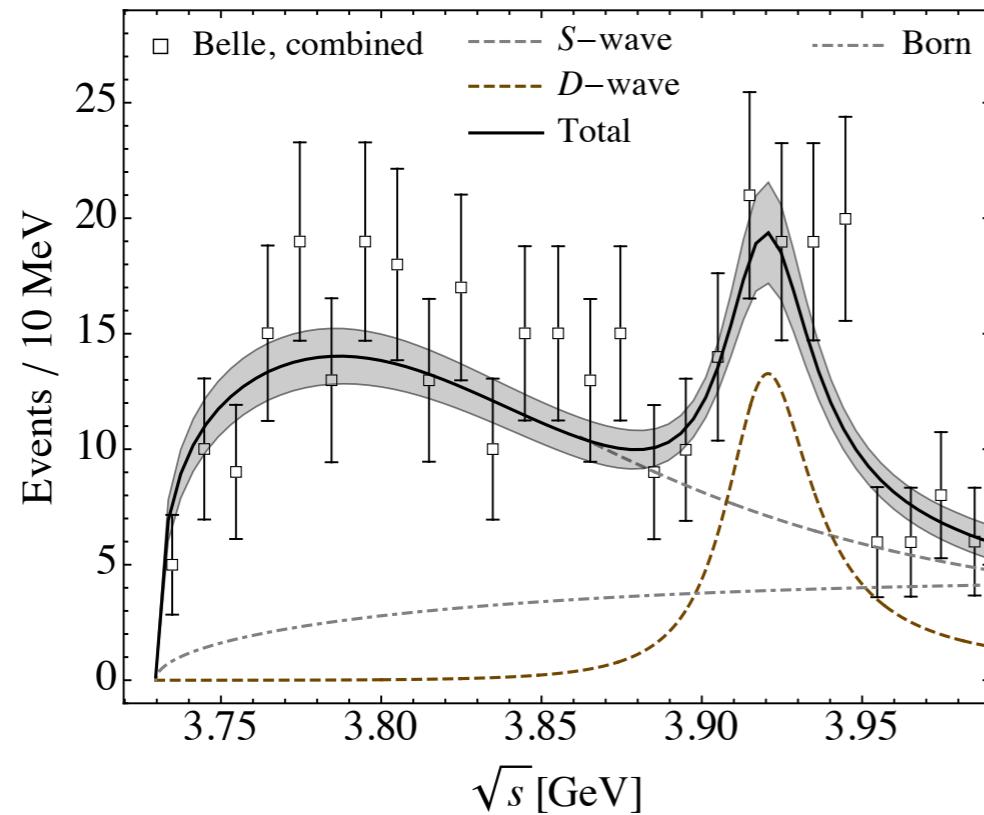
Analysis of $\gamma\gamma \rightarrow D^+D^-$ and $\gamma\gamma \rightarrow D^0\bar{D}^0$ data



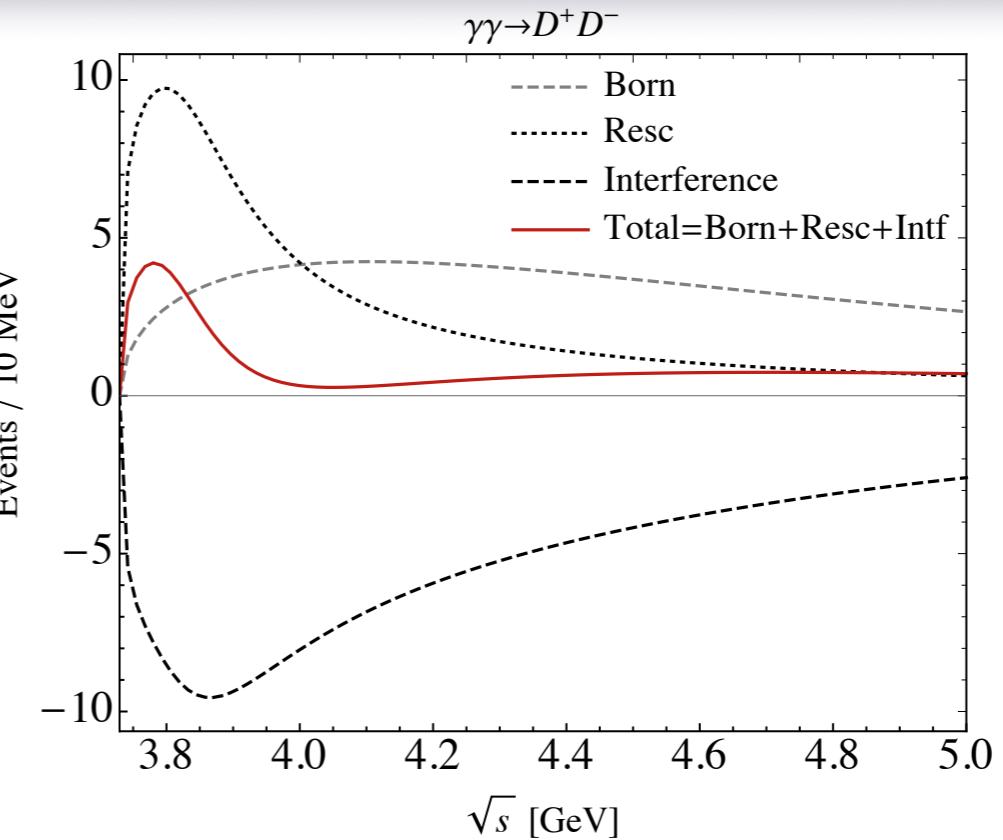
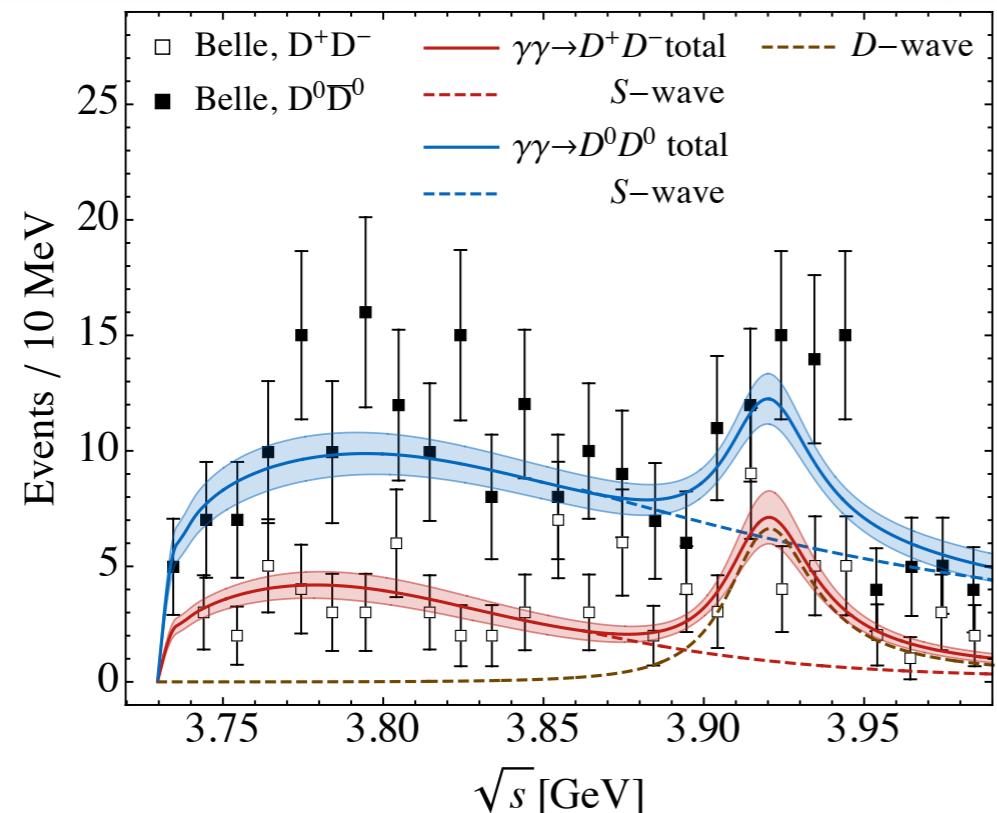
Bound state: $\sqrt{s_B} = 3695(4)$ MeV



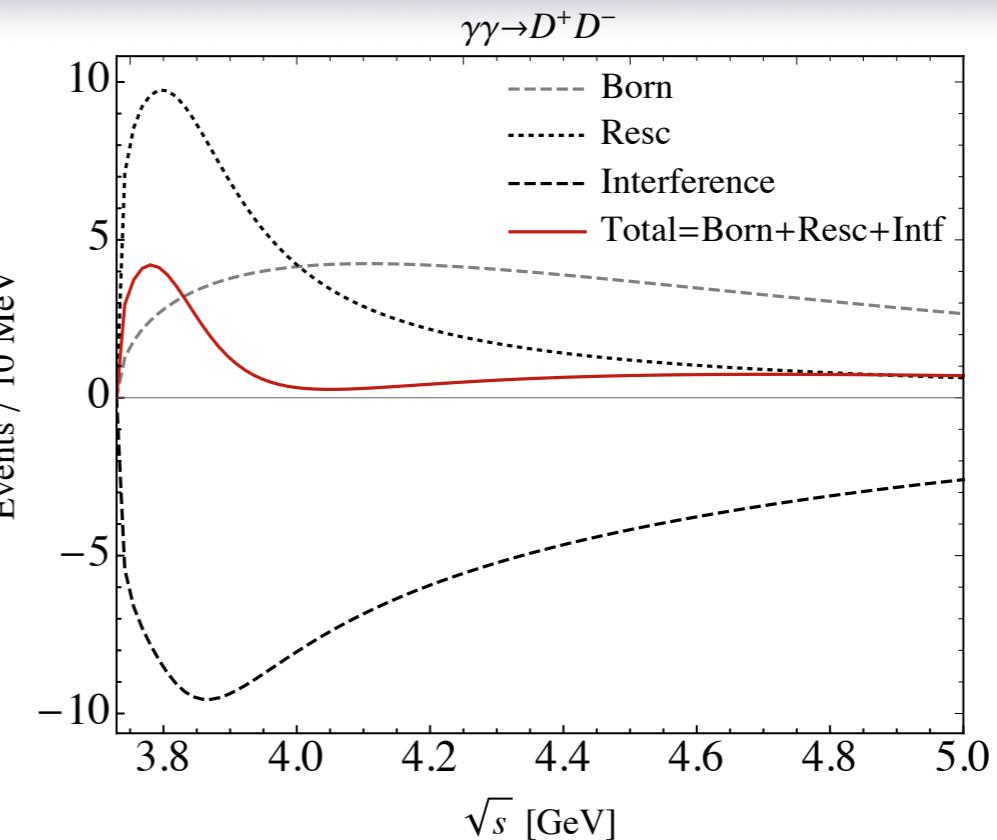
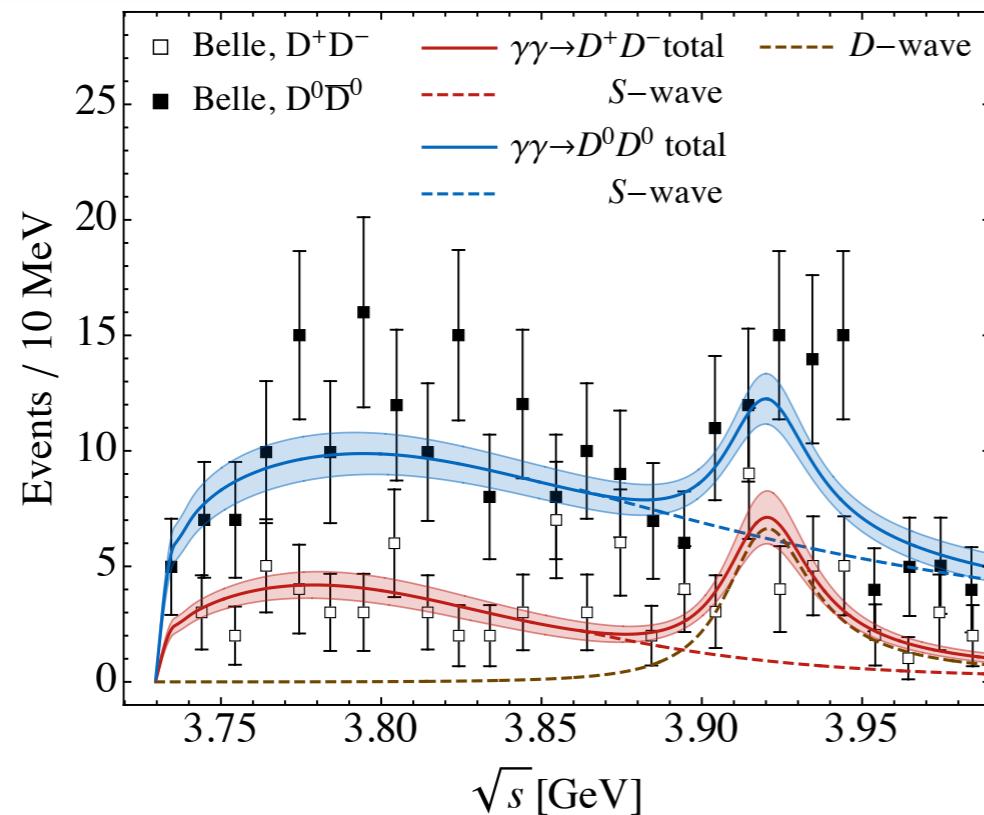
Bound state: $\sqrt{s_B} = 3669(18)$ MeV



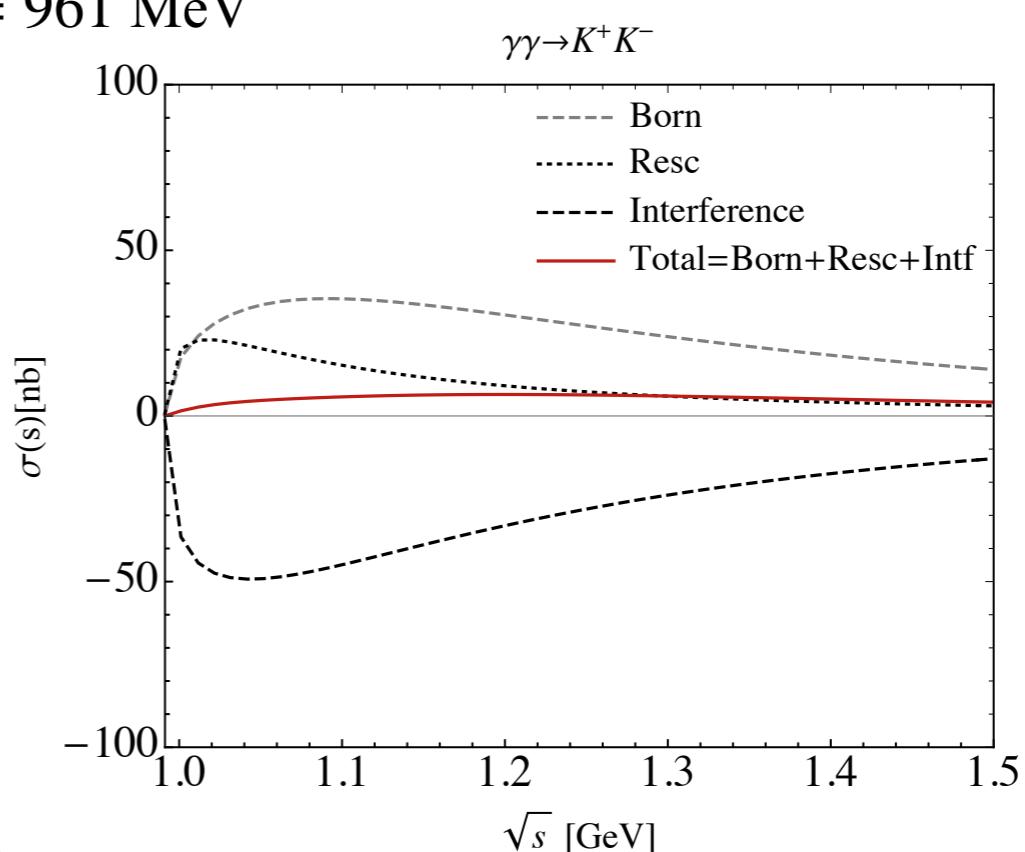
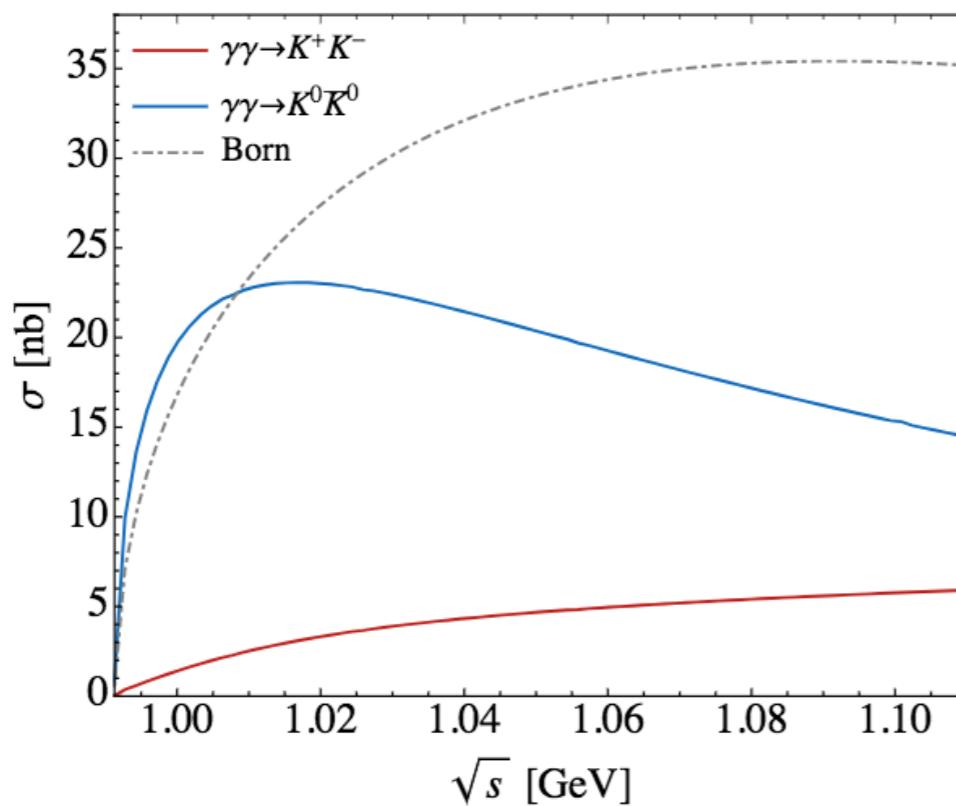
Destructive interference



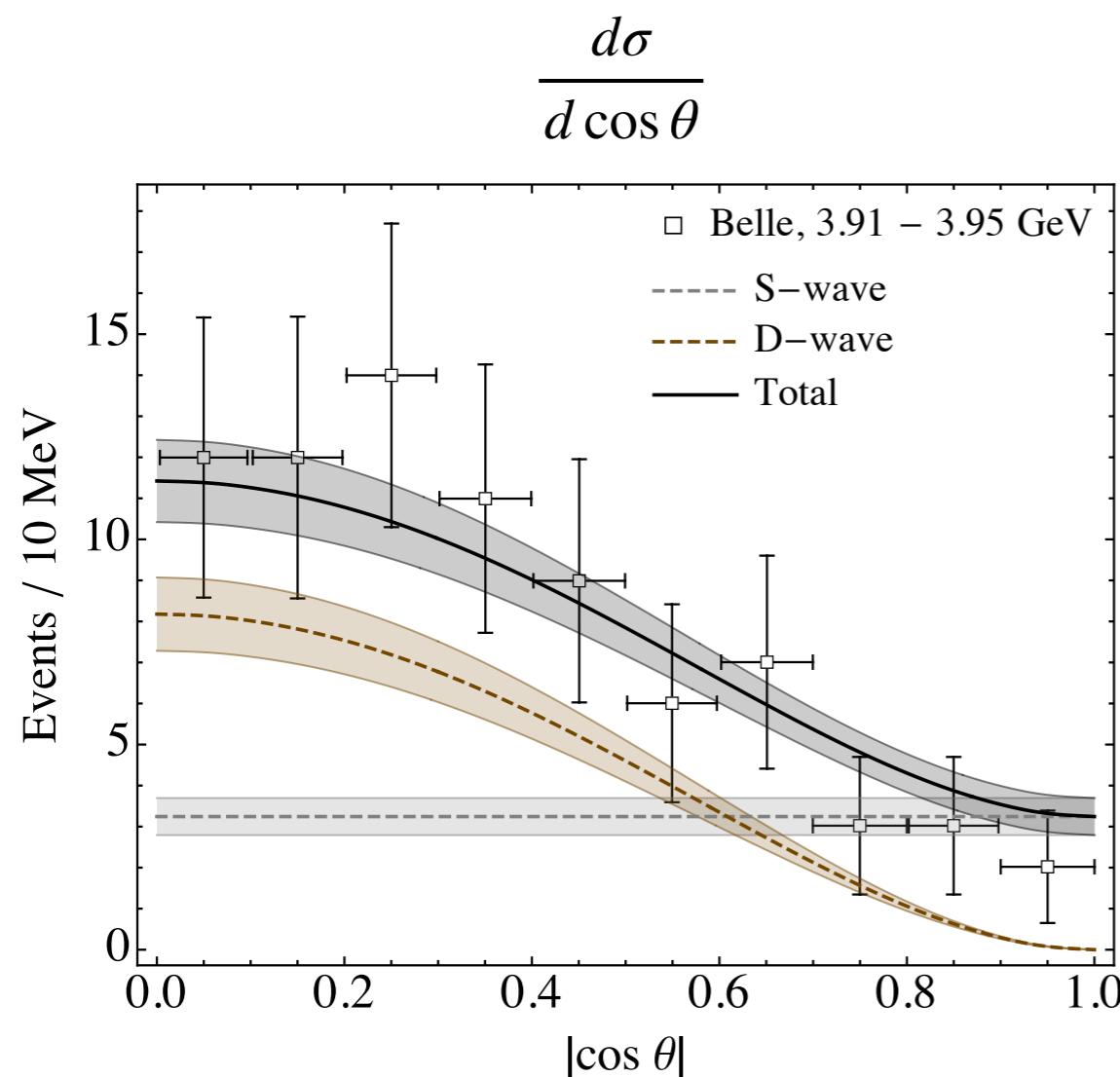
Destructive interference



Consider again $\{\pi\pi, K\bar{K}\}$ coupled channel system and switch off $\pi\pi$ channel
 $\implies f_0(980)$ becomes a $K\bar{K}$ bound state with $\sqrt{s_B} = 961$ MeV

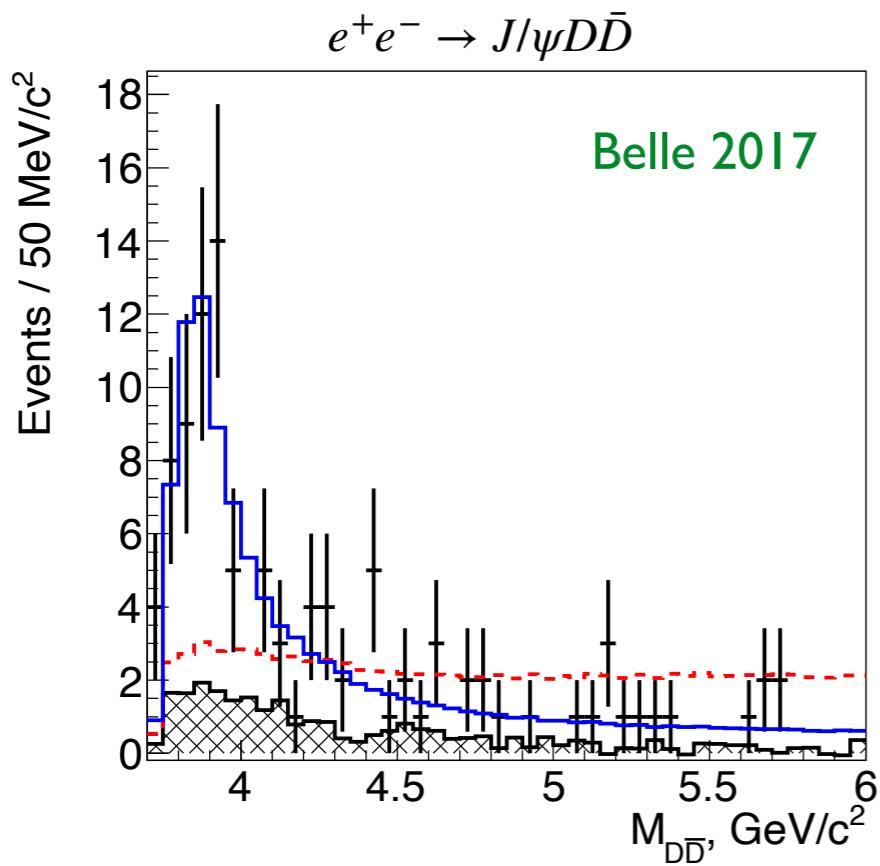


$\gamma\gamma \rightarrow D\bar{D}$ angular distribution

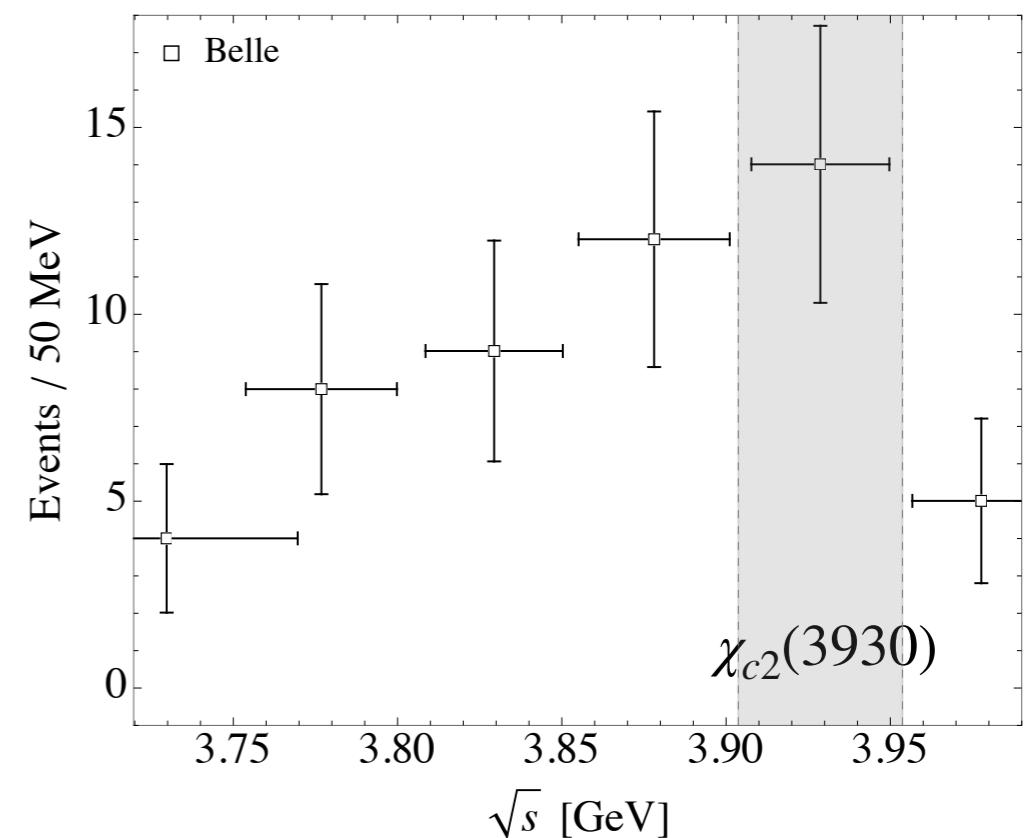


Angular distribution is mainly D-wave:
however, one cannot exclude an additional
small S-wave contribution from $\chi_{c0}(3915)$

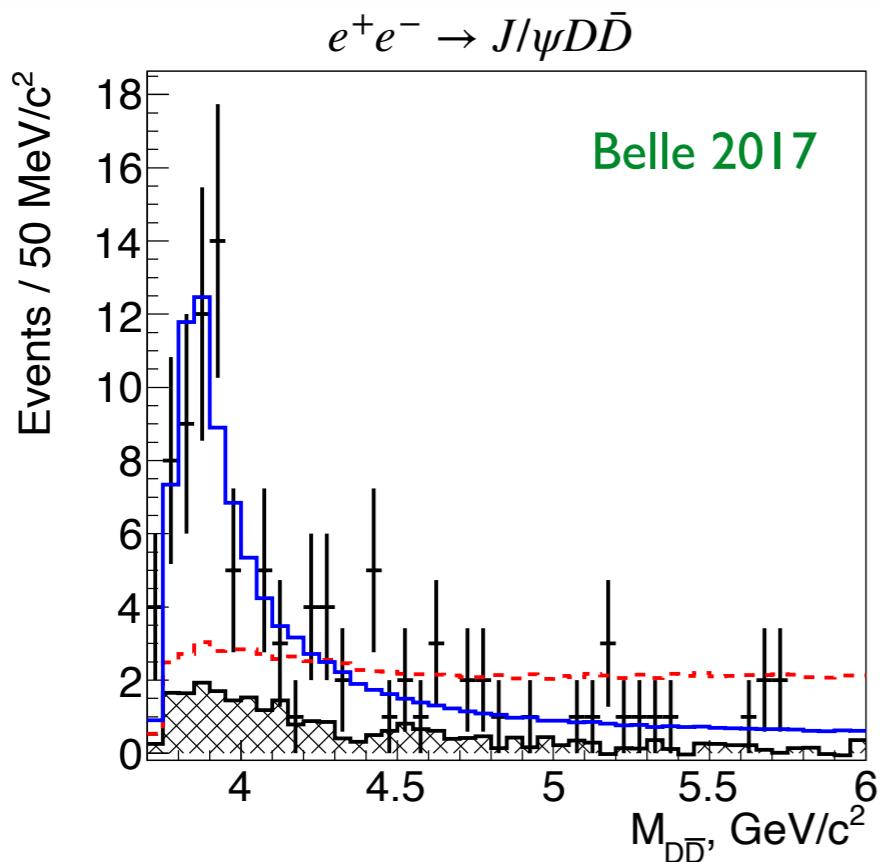
$e^+e^- \rightarrow J/\psi D\bar{D}$ data



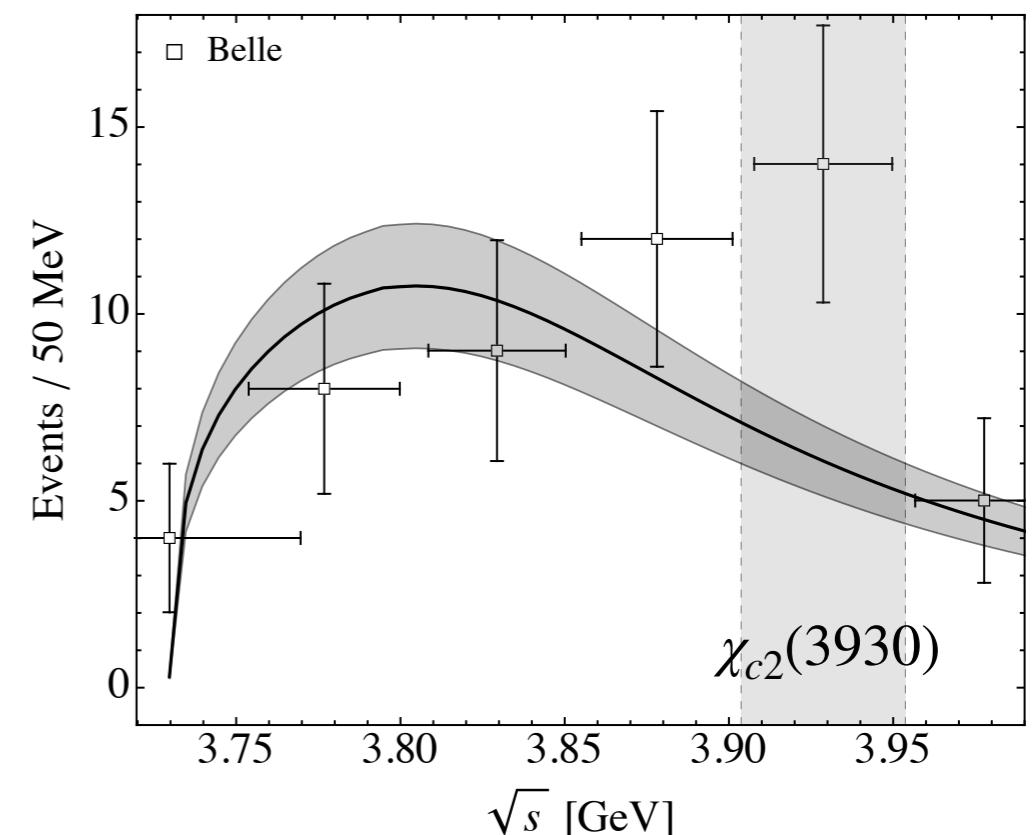
only 6 data points
below 4 GeV



- No realistic estimates can be done from this data alone; full experimental dataset is needed



only 6 data points
below 4 GeV



- No realistic estimates can be done from this data alone; full experimental dataset is needed
- Post-diction based on the obtained $D\bar{D}$ Omnes function shows a good description of the S-wave region

$$\frac{d\sigma}{d\sqrt{s}} = N \frac{\lambda^{1/2}(s, q^2, m_{J/\psi}^2) \lambda^{1/2}(s, m_D^2, m_D^2)}{q^6 \sqrt{s}} \left| D_{D\bar{D}}^{-1}(s) \right|^2$$

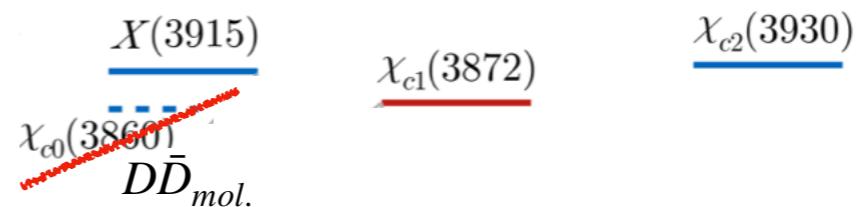
the only fitting parameter

- Our results based on $\gamma\gamma \rightarrow D\bar{D}$ data are consistent with $e^+e^- \rightarrow J/\psi D\bar{D}$ data

Conclusion and outlook

- Dispersive analysis of the $\gamma\gamma \rightarrow D^+D^-, D^0\bar{D}^0$ data, consistency check with the $e^+e^- \rightarrow J/\psi D\bar{D}$ data
- No broad resonance corresponding to $X(3860)$ found
- Bound state below the $D\bar{D}$ threshold, $\sim D\bar{D}$ molecule
- More data is needed: Belle II $\gamma\gamma \rightarrow D\bar{D}$, BESIII decay $\psi(3770) \rightarrow D\bar{D}\gamma$, PANDA ...

Still an open question about $\chi_{c0}(2P)$:
is it a $D\bar{D}$ molecular or it is a $X(3915)$?



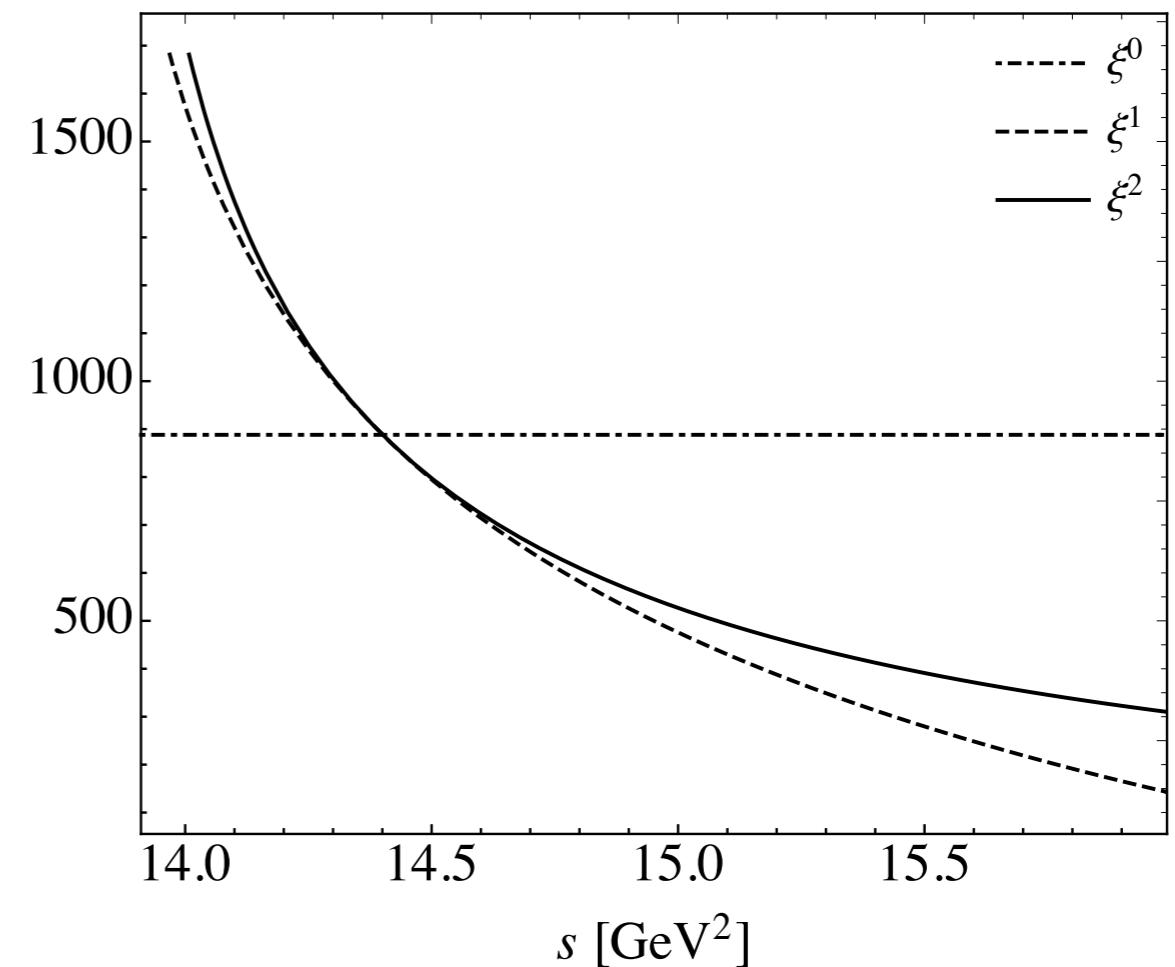
- Dispersion theory can be applied to many other exciting processes
- Light: $\gamma\gamma \rightarrow \pi^0\eta, K\bar{K}$; $(g-2)_\mu$ HLBL contributions
- $\{J/\psi J/\psi, J/\psi \psi(2S)\}$ scattering LHCb

Thank you!

Extra slides

Convergence of conformal expansion

$$U_{DD}(s) \simeq \sum_{n=0}^{\infty} C_n \xi^n(s)$$



Good convergence with 3 parameters in
conformal mapping expansion \implies
there is no need for more parameters