Inhomogeneous Phases in the QCD Phase Diagram

Theo F. Motta (JLU Gießen & TU Darmstadt)

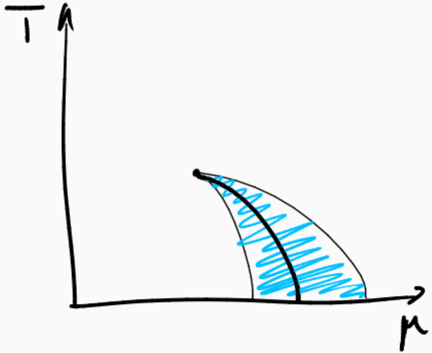
October 25, 2022

Excited QCD
Giardini-Naxos

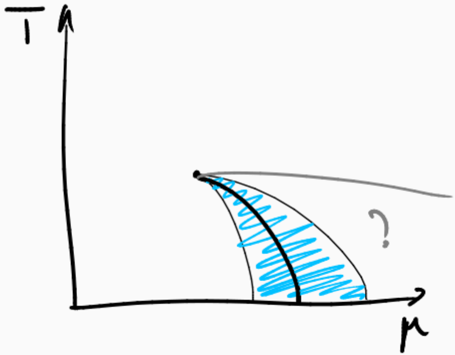
Overview of Inhomogeneous Phases

Inhomogeneous Phases

Inhomogeneous Phases



Inhomogeneous Phases



How do we Study *Inhomogeneous* Phases?

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- With Models of QCD:
    - Gross-Neveu
    - NJL
    - QM

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- By Stability Analysis

NJL

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left\{(\bar{\psi}\psi)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right\}$$

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$$\phi_S(\mathrm{x}) = \langle\bar{\psi}(\mathrm{x})\psi(\mathrm{x})\rangle, \quad \phi_P(\mathrm{x}) = \left\langle\bar{\psi}(\mathrm{x})i\gamma_5\tau^3\psi(\mathrm{x})\right\rangle$$

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$$\Omega_{\mathrm{MF}} = -\frac{T}{V}\,\mathrm{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\frac{1}{V}\int d^3x\left(\phi_S^2(\mathsf{x}) + \phi_P^2(\mathsf{x})\right)$$

Anzats

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- Chiral Density Wave:

$$\phi_S(\vec{x}) = -\frac{\Delta}{2G_S}\cos(\vec{q}\cdot\vec{x}), \qquad \phi_P(\vec{x}) = -\frac{\Delta}{2G_P}\sin(\vec{q}\cdot\vec{x})$$

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$$M(x) = \Delta\sqrt{\nu}\,\mathrm{sn}(\Delta x|\nu)$$

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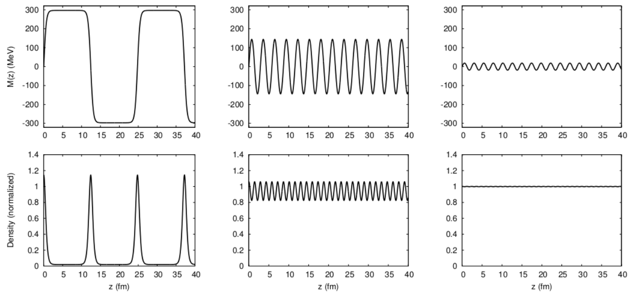

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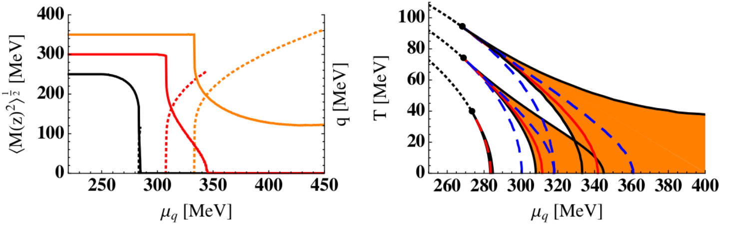
Inhomogeneous phases in the Nambu–Jona-Lasinio and quark-meson model

Dominik Nickel

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 10 July 2009; published 22 October 2009)



Stability Analysis

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$$\phi_S(\mathsf{x}) = \bar{\phi}_S + \delta\phi_S(\mathsf{x}), \quad \phi_P(\mathsf{x}) \equiv \delta\phi_P(\mathsf{x})$$

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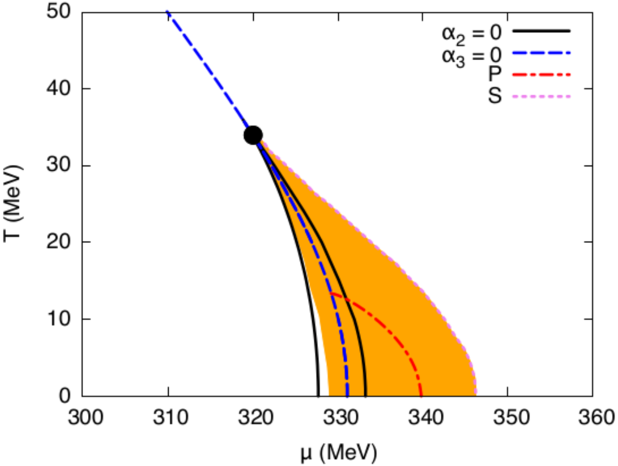
$$\Omega^{(2)} = 2G^2\sum_{\mathsf{q}_k}\left\{\left|\delta\phi_{S,\mathsf{q}_k}\right|^2\Gamma_S^{-1}\left(\mathsf{q}_k^2\right) + \left|\delta\phi_{P,\mathsf{q}_k}\right|^2\Gamma_P^{-1}\left(\mathsf{q}_k^2\right)\right\}$$

Inhomogeneous chiral phases away from the chiral limit

Michael Buballa[1] and Stefano Carignano[2]

[1] *Theoriezentrum, Institut für Kernphysik, Technische Universität Darmstadt, Schlossgartenstr. 2, D-64289 Darmstadt, Germany*

[2] *Departament de Física Quàntica i Astrofísica and Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Catalonia, Spain.*

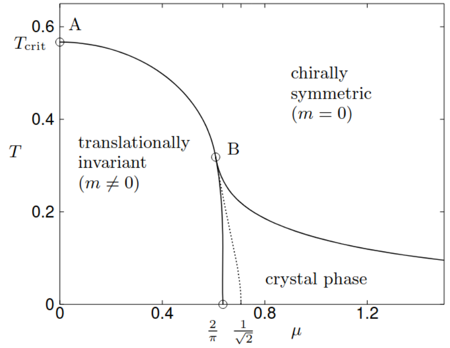


A plot twist? Gross-Neveu Model!

Revised Phase Diagram of the Gross-Neveu Model

Michael Thies and Konrad Urlichs
Institut für Theoretische Physik III
Universität Erlangen-Nürnberg
Staudtstraße 7
D-91058 Erlangen
Germany

(Dated: October 25, 2018)



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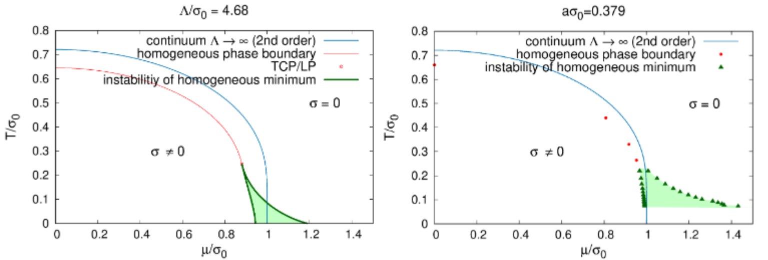
Regulator dependence of inhomogeneous phases in the 2+1-dimensional Gross-Neveu model

Michael Buballa[a,c], Lennart Kurth[a], Marc Wagner[b,c], Marc Winstel[b]



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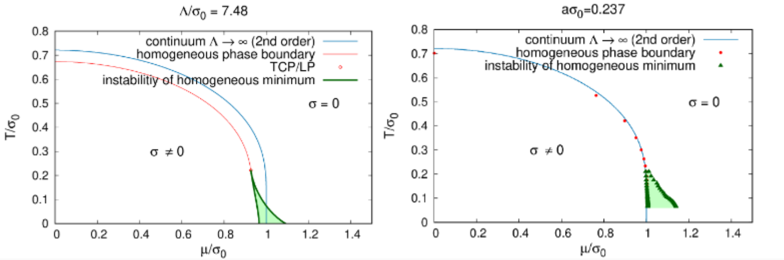

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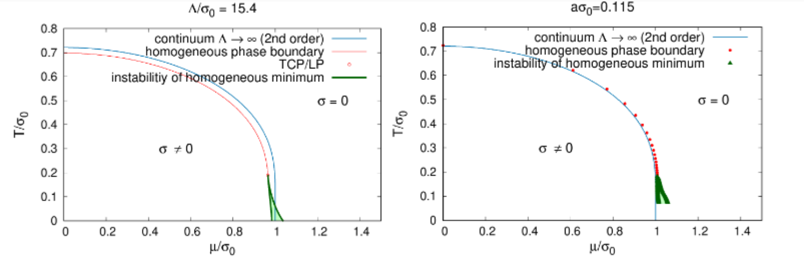


Quantum Chromodynamics

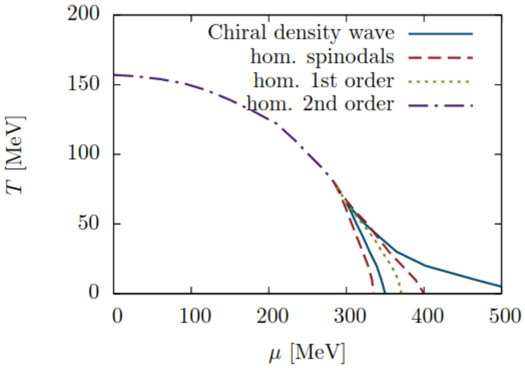
How about QCD?

How about QCD?

Dyson-Schwinger study of chiral density waves in QCD

D. Müller[a], M. Buballa[a], J. Wambach[a,b]

[a] *Institut für Kernphysik (Theoriezentrum), Technische Universität Darmstadt, Germany*
[b] *GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany*



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$$S^{-1}\left(p,p'\right) = \Big[- i\left(\omega_n + i\mu\right)\gamma_4 C(p) - ip_3\gamma_3 E(p) - i\vec{p}_\perp A(p) \\ - i\left(\omega_n + i\mu\right)\gamma_5\gamma_4 C_5(p) - ip_3\gamma_5\gamma_3 E_5(p) - i\gamma_5\vec{p}_\perp A_5(p)\Big]\delta\left(p-p'\right) \\ + \left(B\left(p,p'\right) - i\gamma_4\gamma_3 F\left(p,p'\right) - i\gamma_4\frac{\vec{p}_\perp}{|\vec{p}_\perp|}G\left(p,p'\right) - i\gamma_3\frac{\vec{p}_\perp}{|\vec{p}_\perp|}H\left(p,p'\right)\right)\frac{(\mathrm{I}-\gamma_5)}{2}\delta\left(p-p'+Q\right) \\ + \left(B\left(p,p'\right) + i\gamma_4\gamma_3 F\left(p,p'\right) + i\gamma_4\frac{\vec{p}_\perp}{|\vec{p}_\perp|}G\left(p,p'\right) + i\gamma_3\frac{\vec{p}_\perp}{|\vec{p}_\perp|}H\left(p,p'\right)\right)\frac{(\mathrm{I}+\gamma_5)}{2}\delta\left(p-p'-Q\right).$$

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& + \left(B\left(p, p'\right) - i\gamma_4\gamma_3 F\left(p, p'\right) - i\gamma_4 \frac{\vec{p}_\perp}{|\vec{p}_\perp|} G\left(p, p'\right) - i\gamma_3 \frac{\vec{p}_\perp}{|\vec{p}_\perp|} H\left(p, p'\right)\right) \frac{(\mathrm{I} - \gamma_5)}{2}\delta\left(p - p' + Q\right) \\
& + \left(B\left(p, p'\right) + i\gamma_4\gamma_3 F\left(p, p'\right) + i\gamma_4 \frac{\vec{p}_\perp}{|\vec{p}_\perp|} G\left(p, p'\right) + i\gamma_3 \frac{\vec{p}_\perp}{|\vec{p}_\perp|} H\left(p, p'\right)\right) \frac{(\mathrm{I} + \gamma_5)}{2}\delta\left(p - p' - Q\right).
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- Then you solve the DSE **and**, in theory, you must calculate whether or not this solution is favoured!

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- Then you solve the DSE **and**, in theory, you must calculate whether or not this solution is favoured!
- They calculated a necessary condition for stability, namely

$$\frac{d\Gamma}{dQ} > 0.$$

How about QCD?

Dyson-Schwinger study of chiral density waves in QCD

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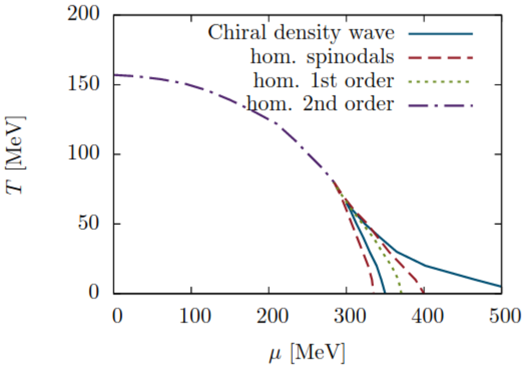

Stability Analysis

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- We chose to start from a 2PI effective action

$$\Gamma = \mathrm{Tr}\log\left[S^{-1}\right] - \mathrm{Tr}\left[1 - S_0^{-1}S\right] + \Gamma_{2PI}$$

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- Fundamentally, we want a Taylor expansion

$$\begin{aligned} F[\varphi(u)] =& F[\varphi_0(u)] + \int \left.\frac{\delta F[\varphi(u)]}{\delta\varphi(u')}\right|_{\varphi=\varphi_0} \times (\varphi(u) - \varphi_0(u'))du' \\ &+ \frac{1}{2!}\int\int \left.\frac{\delta^2 F[\varphi(u)]}{\delta\varphi(u')\delta\varphi(u'')}\right|_{\varphi=\varphi_0} \times (\varphi(u) - \varphi_0(u')) \times (\varphi(u) - \varphi_0(u''))du'du'' + \cdots \end{aligned}$$

Stability Analysis

- So zero-th order we get what we should

$$\Gamma^{(0)} = -\mathrm{Tr}\log[\bar{S}] - \mathrm{Tr}\left[\mathbb{1} - S_0^{-1}\bar{S}\right] + \Gamma_{2\mathrm{PI}}[\bar{S}]$$

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$$\Gamma^{(1)} = \mathrm{Tr}\left[\left(\bar{S}^{-1} - S_0^{-1} - \overline{\frac{\delta\Gamma_{2PI}}{\delta S}}\right)\delta S\right]$$

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$$\Gamma^{(1)} = \mathrm{Tr}\left[\left(\bar{S}^{-1} - S_0^{-1} - \overline{\frac{\delta\Gamma_{2\mathrm{PI}}}{\delta S}}\right)\delta S\right]$$

- Second order is the leading order

$$\Gamma^{(2)} = \frac{1}{2}\mathrm{Tr}[(\bar{S}^{-1}\delta S)^2] + \frac{1}{2}\mathrm{Tr}\left[\overline{\frac{\delta^2\Gamma_{2\mathrm{PI}}}{\delta S_{12}\delta S_{34}}}\delta S_{12}\delta S_{34}\right]$$

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  - Second term $\delta S(k_2 + q, k_1 + q)$ which prevents us writing a stability condition

$$\Omega^{(2)}_{\mathrm{NJL}} = 2G^2 \sum_{\mathbf{q}_k} \left\{ \left|\delta\phi_{S,\mathbf{q}_k}\right|^2 \Gamma_S^{-1}\left(\mathbf{q}_k^2\right) + \left|\delta\phi_{P,\mathbf{q}_k}\right|^2 \Gamma_P^{-1}\left(\mathbf{q}_k^2\right) \right\}$$

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$$\Omega_{\mathrm{NJL}}^{(2)} = 2G^2 \sum_{\mathbf{q}_k} \left\{ \left|\delta\phi_{S,\mathbf{q}_k}\right|^2 \Gamma_S^{-1}\left(\mathbf{q}_k^2\right) + \left|\delta\phi_{P,\mathbf{q}_k}\right|^2 \Gamma_P^{-1}\left(\mathbf{q}_k^2\right) \right\}$$

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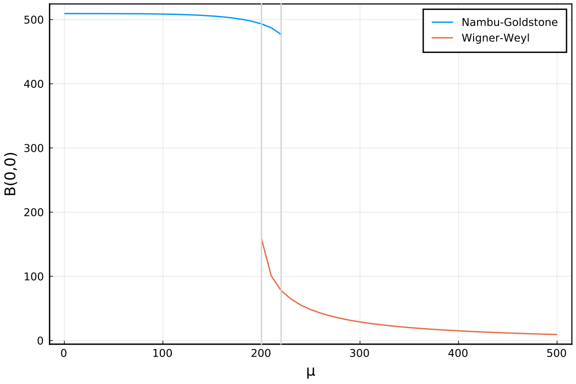

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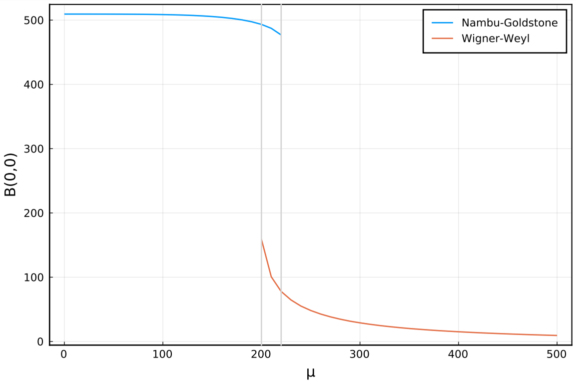
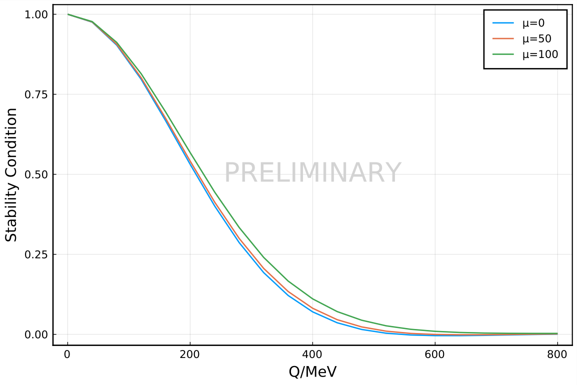
Stability Analysis



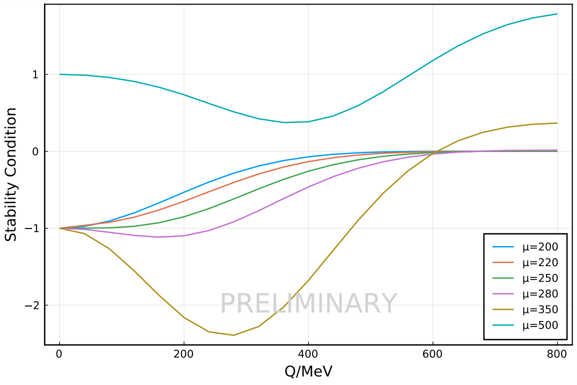
- Take a fixed temperature slice of the phase diagram.
- $\delta S(k_1, k_2) = \bar{S}(k_1)H(k_1, k_2)F(k_1 - k_2)\bar{S}(k_2)$

Stability Analysis



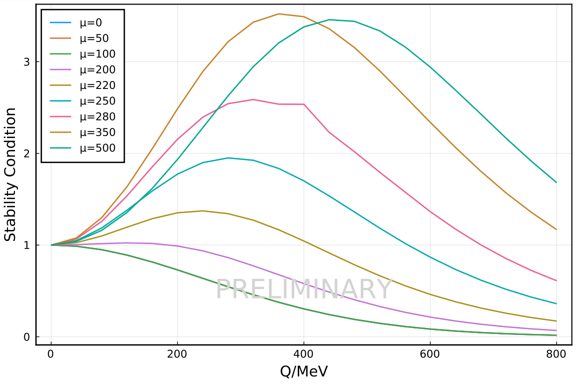
- The normalised stability condition is always positive for $\mu < \mu_c$ (Nambu-Goldstone)

Stability Analysis



- The Wigner-Weyl solution can show instabilities!

Stability Analysis



- However, not for every simplification of δS.

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Thanks!

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