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Analytical dispersive parameterization for elastic scattering of spinless particles (arXiv: 2206.15223)

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Introduction

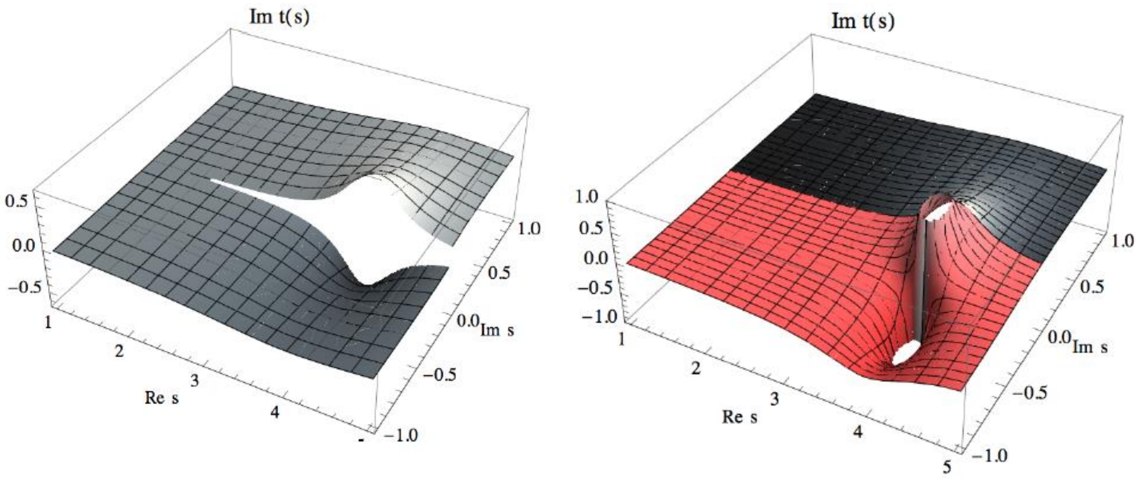
✓ The recent discoveries of exotic hadron resonances, done by LHCb, BESIII, COMPASS, Belle and other collaborations, excited a strong interest in hadron spectroscopy

✓ The significant progress in lattice QCD studies of poorly known hadronic states is also observed

The correct identification of resonance parameters requires the search for poles of the S-matrix in the complex plane.

It is particularly important when

- there is an interplay between several inelastic channels
- the pole is lying very deep in the complex plane



S-matrix constraints and Roy analysis

The fundamental constraints of the theory of S-matrix

- Analyticity (causality)
- Unitarity
- Crossing symmetry

Roy (Roy-Steiner) analysis - the most rigorous way to implement <u>all</u> constraints.

$$t^I_J = k^I_J(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4m_\pi^2}^{\infty} ds' K^{II'}_{JJ'}(s,s')\,\mathrm{Im}\, t^{I'}_{J'}(s')$$

subtraction polynomial

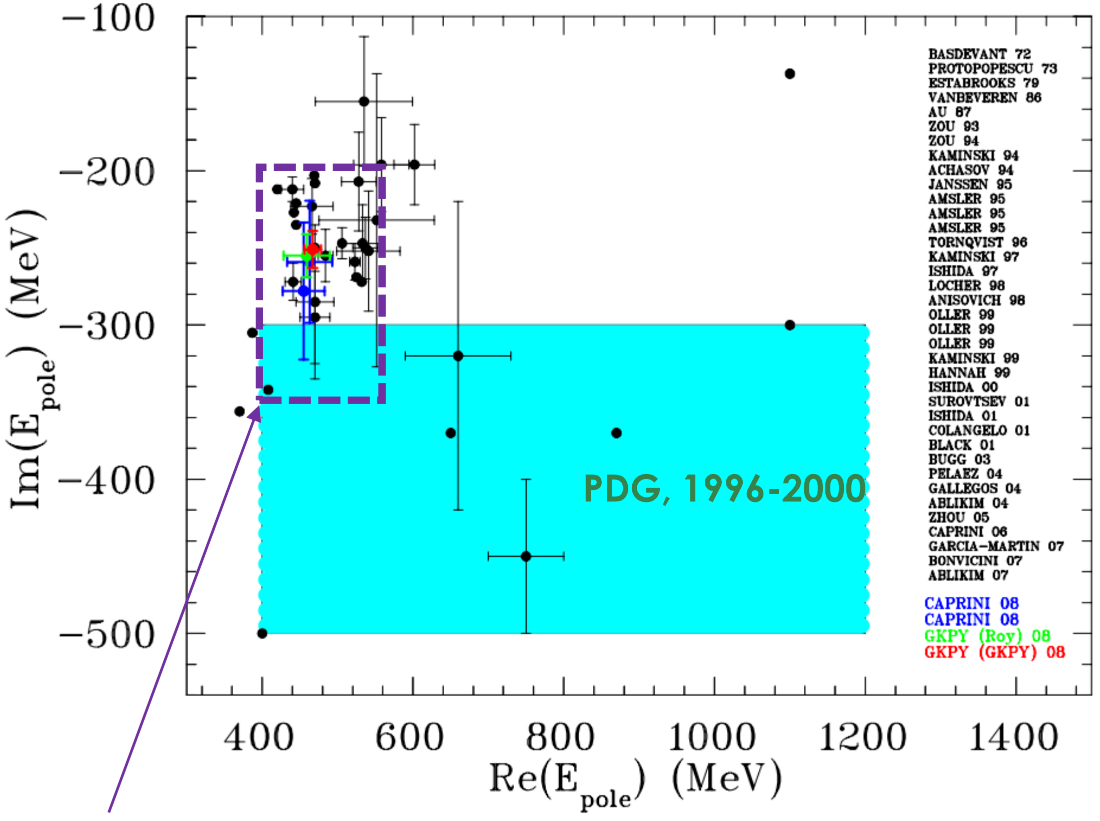
kernel functions *known analytically*

[Roy, 1971]
[Colangelo et al., 2001]
[Caprini et al., 2006]
[Garcia-Martin et al., 2011]

Limitations of Roy analysis:

- experimental knowledge of many partial waves in direct and crossed channels is required
- finite truncation limits the results to a given kinematical region
- coupled-channel treatment is very complicated

Physical σ-pole ($f_0(500)$ resonance) position:



[PDG 2022]

[taken from R. Kaminski talk]

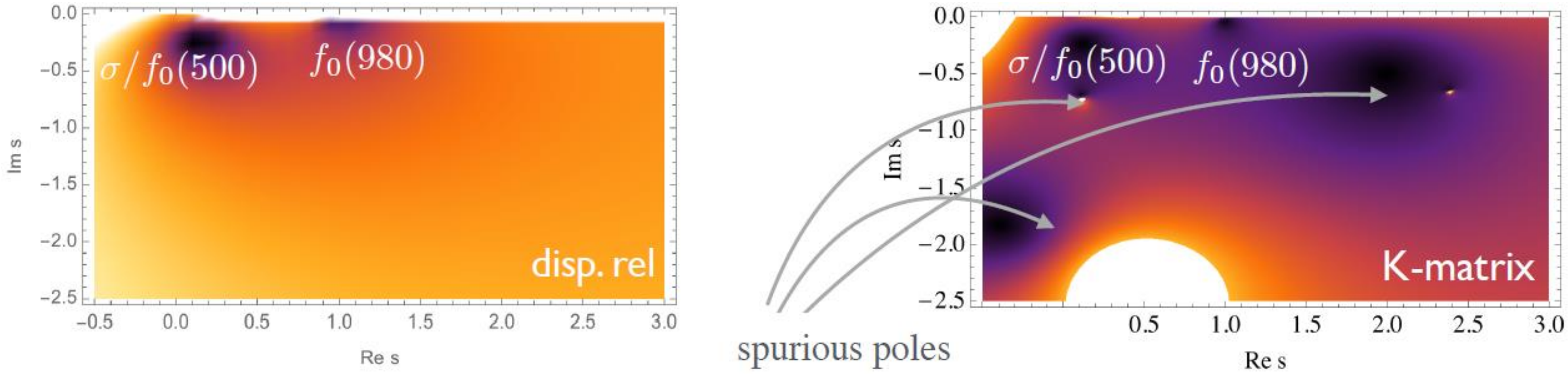
“Utilitarian” approaches

In practice, people prefer more simple approaches like

- ✓ Breit-Wigner parametrizations (with Blatt-Weisskopf factors)
- ✓ K-matrix – very popular in the analyses of lattice data
- ✓ Bethe-Salpeter-like equations
- ✓ and others…

Only unitarity is imposed

It causes the nonphysical behavior –
spurious poles can emerge!



$\boldsymbol{\sigma}$-pole in lattice QCD

$$[t_J(s)]^{-1} = \frac{1}{p(s)^{2J}} K^{-1}(s) + I(s)\,, \qquad \mathrm{Im}\, I(s) \;=\; -\rho(s).$$

[Supplemental material from Briceño et al., PRL 2017]

1) Chew-Mandelstam phase-space, $I(s)$

- a) $K(s) = \frac{g^2}{m^2-s} + c$
- b) $K(s) = \frac{g^2}{m^2-s}$
- c) $K(s) = \frac{g^2}{m^2-s} + as$

2) $K(s) = \frac{g^2}{m^2-s} + c, \;\; I(s) = -i\,\rho$

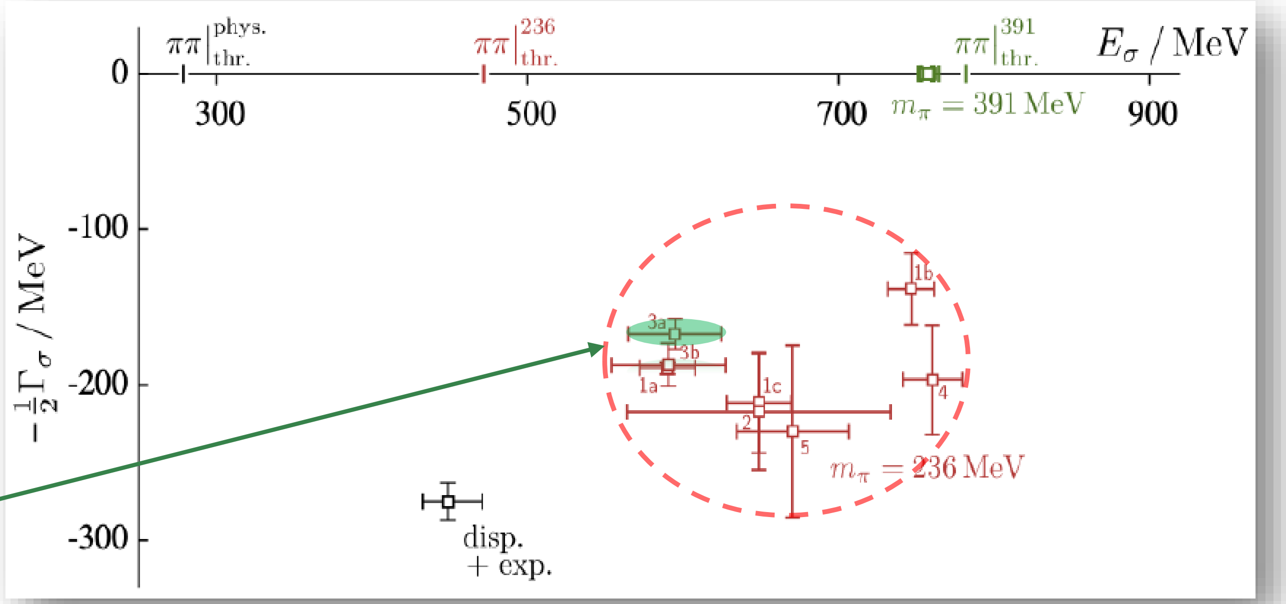
3) The K-matrix, weighted by a factor $(s - s_A)$ to take into account the Adler zero

- a) $K(s) = (s - s_A)\frac{g^2}{m^2-s}$, with Chew-Mandelstam phase-space
- b) $K(s) = (s - s_A)\frac{g^2}{m^2-s}$, with $I(s) = -i\,\rho$

4) A relativistic Breit-Wigner formula

5) An effective range expansion

σ-pole from lattice at $m_\pi = 236$ MeV



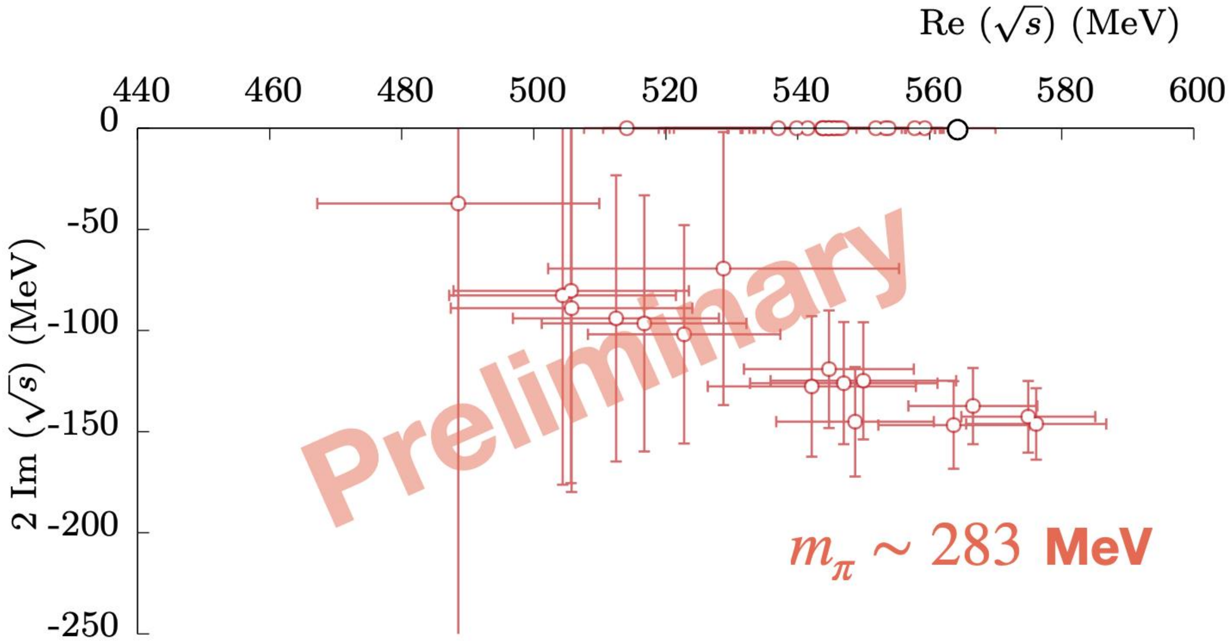
only one parametrization has a dispersive ground:
this fact can remarkably reduce the uncertainty!

$\boldsymbol{\sigma}$- and $\boldsymbol{\kappa}$-poles in lattice QCD

$$[t_J(s)]^{-1} = \frac{1}{p(s)^{2J}} K^{-1}(s) + I(s)\,, \qquad \mathrm{Im}\, I(s) \;=\; -\rho(s).$$

- Ongoing analysis of $\boldsymbol{\sigma}$-pole by HadSpec

[talk Rodas, Lattice 2022]



- Ongoing analysis of $\boldsymbol{\kappa}$ -pole by Mohler et al.

$m_\pi \;=\; 200, 280$ MeV

Dispersive representation for the S-wave amplitudes

- Unitarity:

$$\mathrm{Im}\, t_J(s) = \rho(s)\, |t_J(s)|^2\, \theta(s - s_{th})\,,$$
$$\mathrm{Im}\, [t_J(s)]^{-1} = -\rho(s)\, \theta(s - s_{th})\,,$$

$$\Rightarrow$$

$$-\frac{1}{2\rho(s)} \le \mathrm{Re}\, t_J(s) \le \frac{1}{2\rho(s)},$$
$$0 \le \mathrm{Im}\, t_J(s) \le \frac{1}{\rho(s)}$$

- Maximal analyticity: the partial-wave amplitude should satisfy the dispersion relation

$$t_0(s) = \boxed{t_0(s_M) + \frac{s - s_M}{\pi} \int_L \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t_0(s')}{s' - s}} + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{\rho(s')\, |t_0(s')|^2}{s' - s} + \frac{s - s_M}{s_B - s_M} \frac{g_B^2}{s_B - s}\,.$$

can be approximated by the conformal variable series:

$$\boxed{\sum_{n=0}^{\infty} C_n\, \omega^n(s)}$$

The good

- The correct and straight way to implement all the fundamental principles
- Can be easily extended to the couple-channel case

The bad

- Adler zero imposes a constraint on the conformal variable series: $t(s_A) = 0$

The ugly

- One needs to solve integral equation
- CDD ambiguity

Dispersive representation for the S-wave inverse amplitudes

- Due to the same cut structure (in a single-channel scattering case only!), the inverse partial-wave amplitude should also satisfy the similar dispersion relation:

$$[t_0(s)]^{-1} = \boxed{[t_0(\tilde{s}_M)]^{-1} + \frac{s-\tilde{s}_M}{\pi}\int_L \frac{ds'}{s'-\tilde{s}_M}\frac{\mathrm{Im}\ [t_0(s')]^{-1}}{s'-s}} + \frac{s-\tilde{s}_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-\tilde{s}_M}\frac{-\rho(s')}{s'-s} + \frac{s-\tilde{s}_M}{s_A-\tilde{s}_M}\frac{g_A}{s-s_A}$$

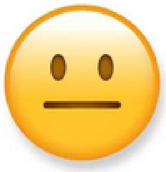
can be approximated by the conformal variable series as well:

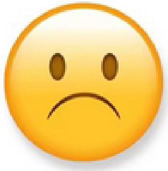
$$\boxed{\sum_{n=0}^{\infty} C_n\, \omega^n(s)}$$

Possible pole term (Adler zero)

The good

- Simple formula – no needs to solve integral equation
- Easy implementation of the Adler zeros





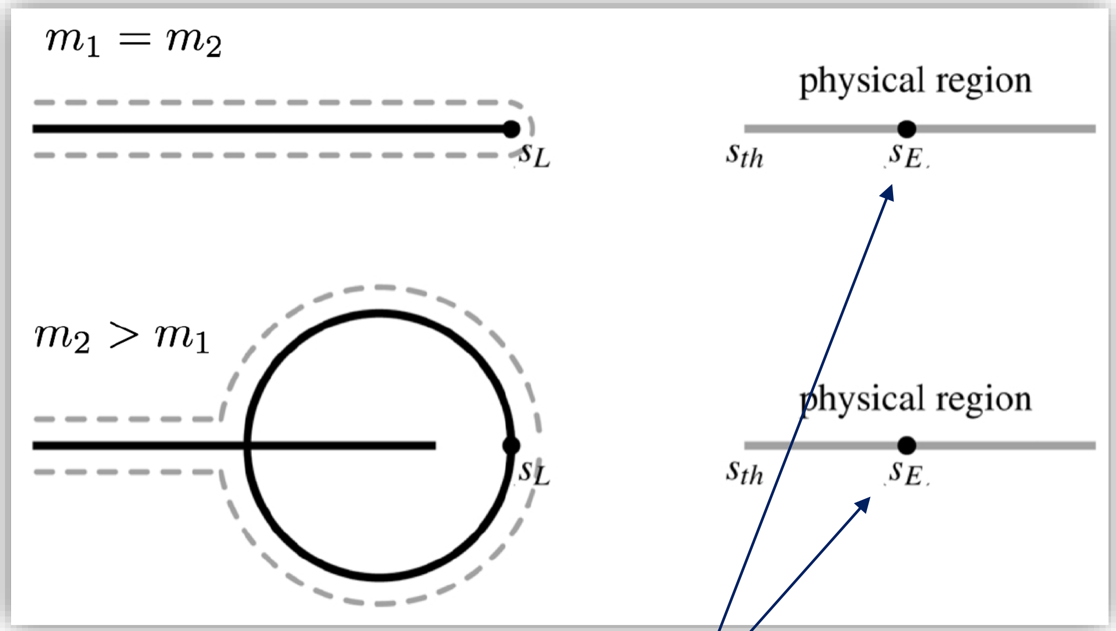
The bad and the ugly

- Cannot be extended to the coupled-channel case (left and right cuts mix)
- Inclusion of inelastic contributions can be made only phenomelogically

Left-hand cuts as conformal variable series

The idea is to express the subtraction constant together with the integrated left-hand cut discontinuity via the series of suitably constructed conformal variable $\omega(s)$

$$[t_0(\tilde{s}_M)]^{-1} + \frac{s-\tilde{s}_M}{\pi}\int_L \frac{ds'}{s'-\tilde{s}_M}\,\frac{\text{Im}\ [t_0(s')]^{-1}}{s'-s} \longrightarrow \sum_{n=0}^{\infty} C_n\,\omega^n(s)$$



$$\omega(s) = \frac{\sqrt{s-s_L} - \sqrt{s_E-s_L}}{\sqrt{s-s_L} + \sqrt{s_E-s_L}}\,,$$

$$s_L = 4\,m^2 - t_x\,,$$

the lowest threshold in the crossed t or u channels

$$\omega(s) = -\frac{\left(\sqrt{s}-\sqrt{s_E}\right)\left(\sqrt{s}\,\sqrt{s_E}+s_L\right)}{\left(\sqrt{s}+\sqrt{s_E}\right)\left(\sqrt{s}\,\sqrt{s_E}-s_L\right)}\,,$$

$$s_L = m_2^2 - m_1^2\,.$$

expansion point $\sqrt{s_E} = \frac{1}{2}\left(\sqrt{s_{th}} + \sqrt{s_{max}}\right)$

Dispersive Inverse Amplitude (DIA) method for J=0

Gathering all the ingredients together, we arrive to the following formula for inverse partial-wave amplitude:

$$[t_0(s)]^{-1} \simeq \sum_{n=0}^{\infty} C_n\, \omega^n(s) + \frac{s-\tilde{s}_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s'-\tilde{s}_M} \frac{-\rho(s')}{s'-s} + \frac{s-\tilde{s}_M}{s_A-\tilde{s}_M} \frac{g_A}{s-s_A}$$

The main features:

- ✓ Dispersive (Chew-Mandelstam) expression for the phase-space
- ✓ Additive pole term which is responsible for the Adler zero
- ✓ The conformal variable series is responsible for single-channel left-hand cut only

Let us compare it with the common parametrizations available in the literature…

DIA and common K-matrix parametrizations

$$[t_0(s)]^{-1} \simeq \sum_{n=0}^{\infty} C_n\,\omega^n(s) + \frac{s-\tilde{s}_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-\tilde{s}_M}\,\frac{-\rho(s')}{s'-s} + \frac{s-\tilde{s}_M}{s_A-\tilde{s}_M}\,\frac{g_A}{s-s_A}$$

vs.

$$[t_0(s)]^{-1} = \frac{1}{p(s)}K^{-1}(s) + I(s)$$

$$\mathrm{Im}\,I(s) = -\rho(s)$$

There are different variants in the literature how one parametrizes the K-marix:

1) Standard implementation

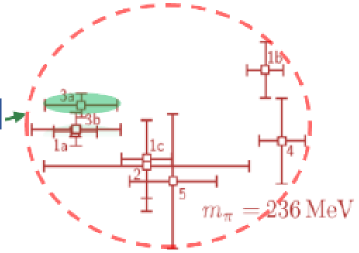
$$K(s) = \frac{g}{m^2-s} + \sum_n \gamma_n\, s^n$$

[PDG,2020]

2) Standard implementation + Adler zero:

$$K(s) = (s-s_A)\left(\frac{g}{m^2-s} + \sum_n \gamma_n\, s^n\right)$$

[Briceño et al., 2017]



3) Standard implementation
+ Adler zero, + left-hand cut

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\frac{2s_A}{m_\pi\sqrt{s}} + \sum_{n=0}^{\infty} C_n\,\omega^n(s)\right)$$

[Yndurain et al., 2007]

alternatively, $K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\sum_{n=0}^{\infty} C_n\,\omega^n(s)\right)$

[Caprini et al., 2008]
[Pelaez et al., 2016]

All of the above implementations are, in general, non-dispersive!

DIA and mIAM method

$$\left[t_0^{\mathrm{mIAM}}(s)\right]^{-1} = \left[t_0^{\mathrm{mIAM}}(\tilde{s}_M)\right]^{-1} + \frac{s-\tilde{s}_M}{s_A-\tilde{s}_M}\frac{g_A}{s-s_A} + \frac{s-\tilde{s}_M}{\pi}\int_{L,R}\frac{ds'}{s'-\tilde{s}_M}\frac{\mathrm{Im}\left[t_0^{\mathrm{mIAM}}(s')\right]^{-1}}{s'-s}$$

$$g_A = \left(\frac{dt_0^{\mathrm{mIAM}}(s)}{ds}\bigg|_{s=s_A}\right)^{-1}$$

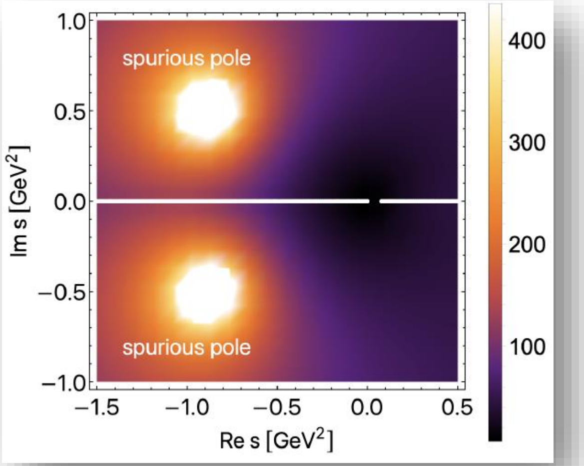
1) S-wave

$$t_0^{\mathrm{mIAM}}(s) = \frac{\left[t_0^{\mathrm{LO}}(s)\right]^2}{t_0^{\mathrm{LO}}(s) - \left[t_0^{\mathrm{NLO}}(s) - t_0^{\mathrm{LO}}(s)\right] + A^{\mathrm{mIAM}}(s)}$$

- satisfies the D.R. for inverse amplitude

- does NOT satisfiy the D.R. for direct amplitude – it has spurious pole on the first Riemann sheet!

$$t_0^{\mathrm{mIAM}}(s) \neq t_0^{\mathrm{mIAM}}(s_M) + \frac{s-s_M}{\pi}\int_{L,R}\frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,t_0^{\mathrm{mIAM}}(s')}{s'-s}$$



2) P-wave

$$t_1^{\mathrm{IAM}}(s) = \frac{\left[t_1^{\mathrm{LO}}(s)\right]^2}{t_1^{\mathrm{LO}}(s) - \left[t_1^{\mathrm{NLO}}(s) - t_1^{\mathrm{LO}}(s)\right]},$$

- satisfies both dispersion relations, for inverse and for direct amplitude as well!

$\pi\pi \to \pi\pi$ and $\pi K \to \pi K$ scattering, $J = 0$

$$[t_0(s)]^{-1} \simeq \sum_{n=0}^{\infty} C_n\, \omega^n(s) + R(s, \tilde{s}_M) + \frac{s - \tilde{s}_M}{s_A - \tilde{s}_M} \frac{g_A}{s - s_A}$$

$$R(s, \tilde{s}_M) \equiv \frac{s - \tilde{s}_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - \tilde{s}_M} \frac{-\rho(s')}{s' - s}$$

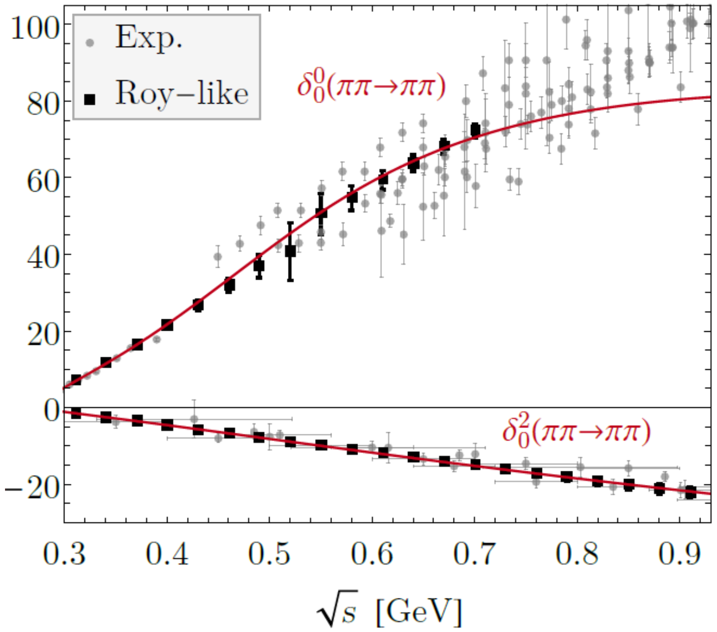
we perform the subtraction at the two-particle threshold, say $\tilde{s}_M = s_{th}$.

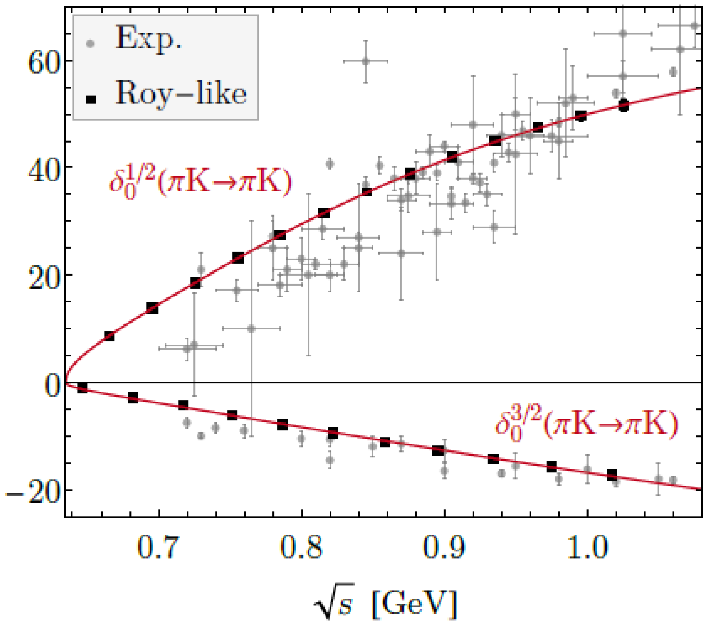
Adler zero was fixed by LO ChPT

$$s_A^{I=0} = \frac{m_\pi^2}{2}, \quad s_A^{I=2} = 2\, m_\pi^2$$

$$s_A^{I=3/2} = m_\pi^2 + m_K^2$$

$$s_A^{I=1/2} = \frac{1}{5}\left(m_\pi^2 + m_K^2 + 2\sqrt{4\,(m_K^2 - m_\pi^2)^2 + m_\pi^2\, m_K^2}\right)$$





δ_0^0

- 2 fit parameters: χ^2/d.o.f. = 0.4, pole : $\sqrt{s_p}$ = 468(8) − i 239(4) MeV
- Roy pole: $\sqrt{s_p} = 457^{+14}_{-13} - \mathrm{i}\, 279^{+11}_{-7}$ MeV

δ_0^2

- 2 fit parameters, χ^2/d.o.f. = 0.0

$\delta_0^{1/2}$

- 2 fit parameters: χ^2/d.o.f. = 2.1, pole: $\sqrt{s_p}$ = 707 − i 246 MeV
- 3 fit parameters: χ^2/d.o.f. = 0.0, pole: $\sqrt{s_p}$ = 684 − i 312 MeV
- Roy pole: $\sqrt{s_p}$ = 648(7) − i 280(16) MeV

$\delta_0^{3/2}$

- 2 fit parameters, χ^2/d.o.f. = 0.5

$\pi\pi \to \pi\pi$ and $\pi K \to \pi K$ scattering on a lattice, J=0

$$[t_0(s)]^{-1} \simeq \sum_{n=0}^{\infty} C_n\, \omega^n(s) + R(s, \tilde{s}_M) + \frac{s - \tilde{s}_M}{s_A - \tilde{s}_M} \frac{g_A}{s - s_A}$$

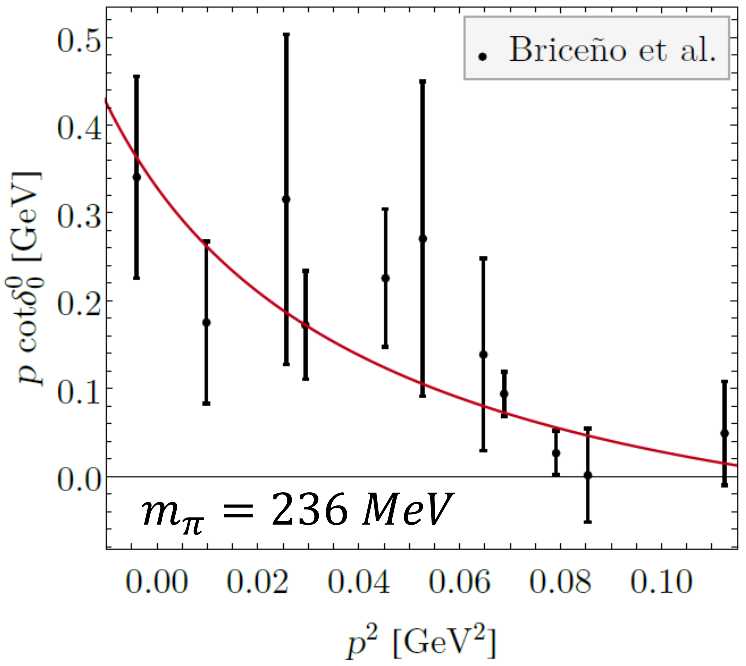
$$R(s, \tilde{s}_M) \equiv \frac{s - \tilde{s}_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - \tilde{s}_M} \frac{-\rho(s')}{s' - s}$$

we perform the subtraction at the two-particle threshold, say $\tilde{s}_M = s_{th}$.

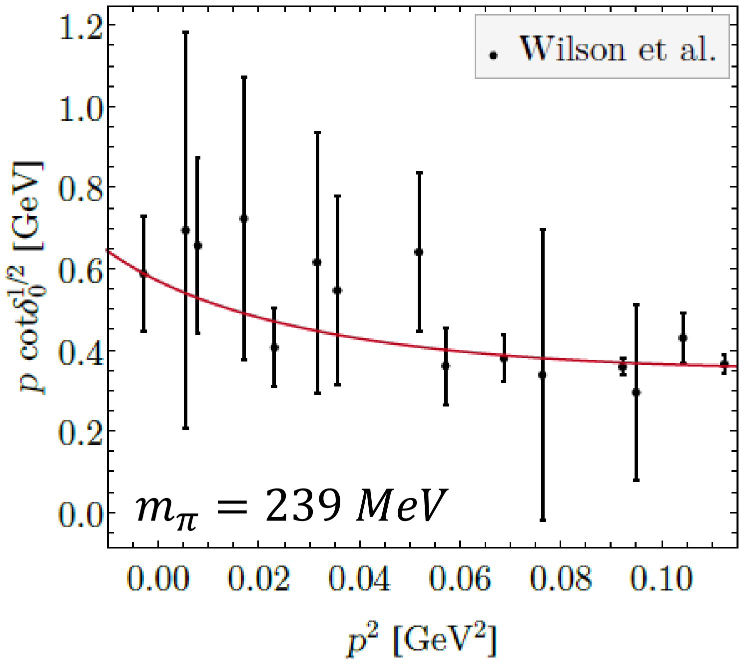
Adler zero was fixed by LO ChPT

$$s_A^{I=0} = \frac{m_\pi^2}{2},$$

$$s_A^{I=1/2} = \frac{1}{5}\left(m_\pi^2 + m_K^2 + 2\sqrt{4\,(m_K^2 - m_\pi^2)^2 + m_\pi^2\, m_K^2}\right)$$



- 2 fit parameters
- χ^2/d.o.f. = 0.8
- pole: $\sqrt{s_p} = 554 - i\,170$ MeV



- 2 fit parameters
- χ^2/d.o.f. = 0.4
- pole: $\sqrt{s_p} = 764 - i\,278$ MeV

DIA and chiral extrapolation

Let us consider the case of S-wave of $\pi\pi$ scattering

Only two input parameters are required:

1. Adler zero s_A
2. slope parameter g_A at the Adler zero

Fit parameters:

$$g_A = 0.56(9), \quad C_0 = 1.38(26),$$

The NLO ChPT parameters:

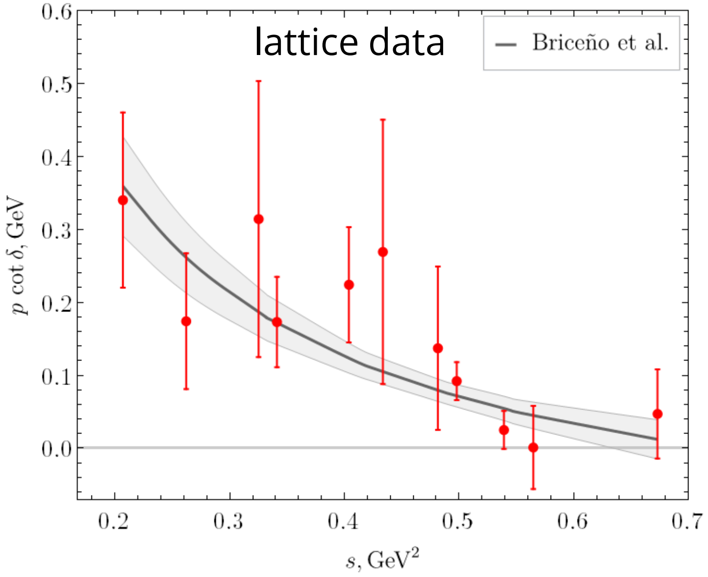
$$g_{A,\mathrm{NLO}} = \left(\frac{\mathrm{d}t_0^{\mathrm{NLO}}(s)}{\mathrm{d}s}\bigg|_{s=s_A} \right)^{-1} = 0.45(3),$$

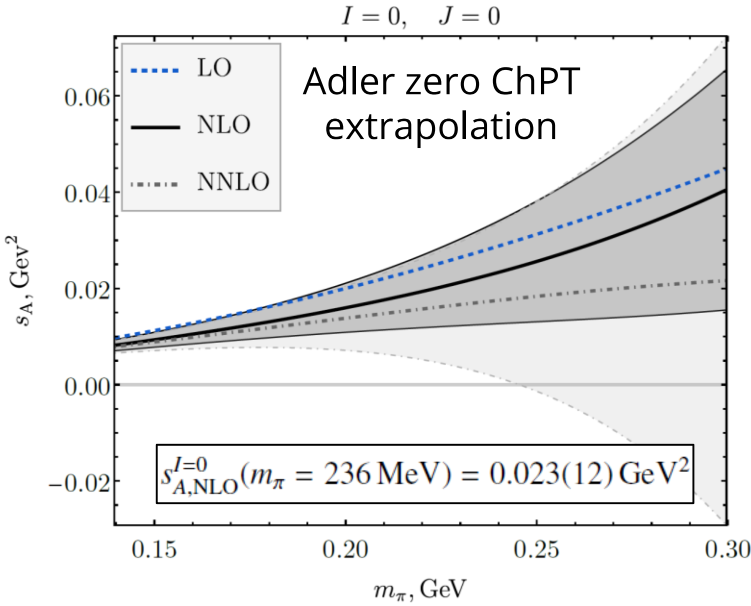
$$C_{0,\mathrm{NLO}} = \left[t_0^{\mathrm{NLO}}(s_{th})\right]^{-1} = 1.46(6).$$

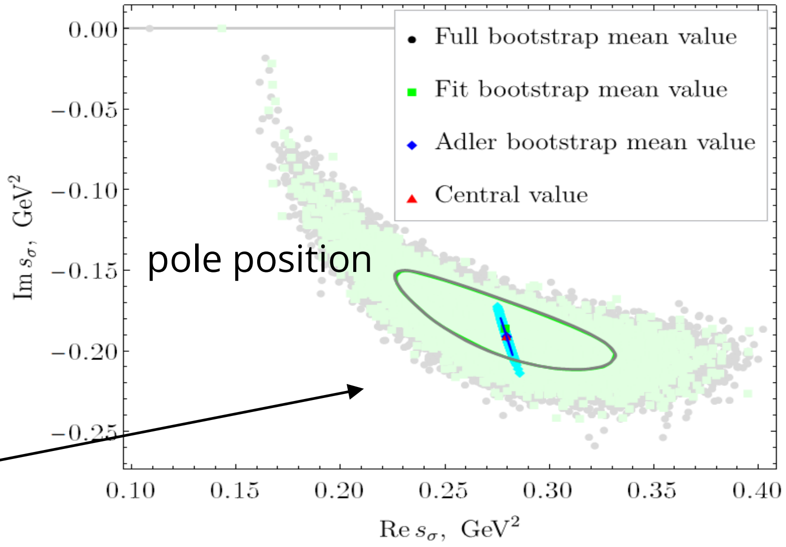
✓ The fitted parameters are consistent with ChPT extrapolation!

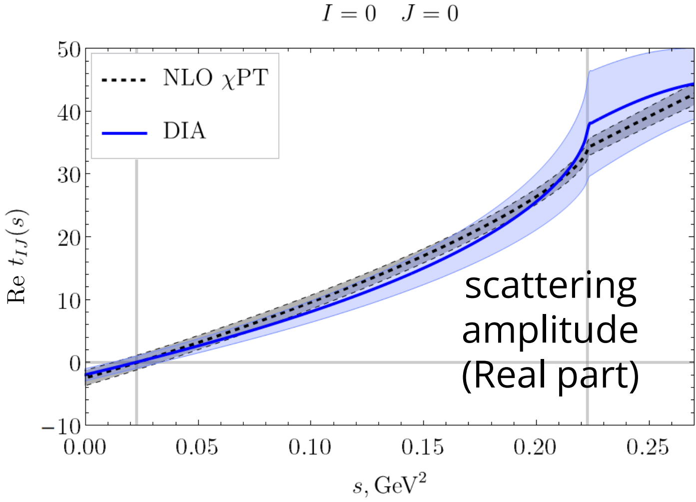
Pole position:

$$\sqrt{s_p} = 553^{+47}_{-54} - i\,168^{+19}_{-15}\ \mathrm{MeV}$$









DIA for higher spins, equal-mass case

The behavior of the amplitudes near the threshold:

$$t_J(s) \sim p(s)^{2J} \overset{m_1=m_2}{\sim} (s - s_{th})^J$$

Applying $J+1$ subtractions, one can write the dispersion relation for the quantity

$$\frac{p^{2J}}{t_J(s)}$$

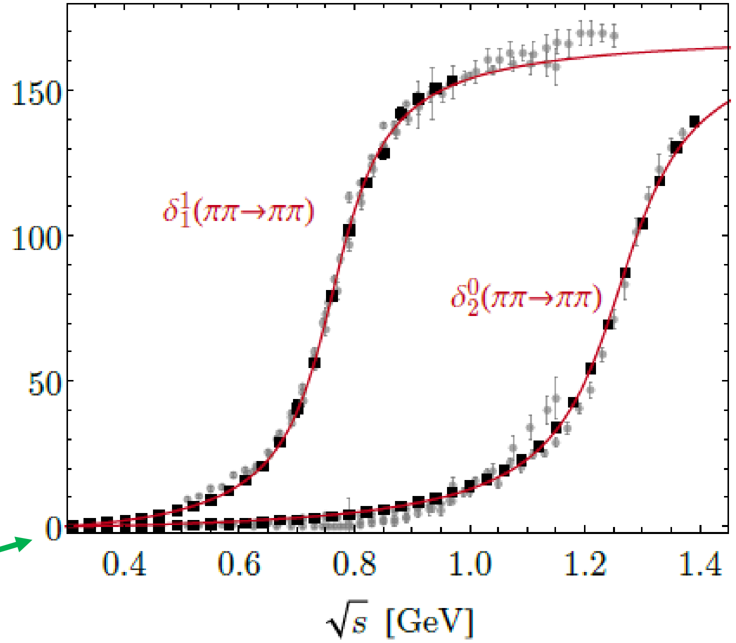
arriving to

$$J\ =\ 1: \quad [t_1(s)]^{-1} \simeq \frac{a}{s-s_{th}} + \sum_{n=0}^{\infty} C_n\,\omega^n(s) + R(s,s_{th})\,,$$

$$J\ =2: \quad [t_2(s)]^{-1} \simeq \frac{a}{(s-s_{th})^2} + \frac{b}{s-s_{th}} + \sum_{n=0}^{\infty} C_n\,\omega^n(s) + R(s,s_{th}).$$

… and so on …

where $$R(s,\tilde{s}_M) \equiv \frac{s-\tilde{s}_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s'-\tilde{s}_M} \frac{-\rho(s')}{s'-s}$$



$\boldsymbol{\delta_1^1}$

- 2 fit parameters: χ^2/d.o.f. = 3.0, pole: $\sqrt{s_p}$ = 758 – i 73 MeV
- 3 fit parameters: χ^2/d.o.f. = 0.9, pole: $\sqrt{s_p}$ = 762 – i 71 MeV
- Roy pole: $\sqrt{s_p} = 763.7^{+1.7}_{-1.5} - \mathrm{i}\,73.2^{+1.0}_{-1.1}$ MeV

$\boldsymbol{\delta_2^0}$

- 2 fit parameters, , χ^2/d.o.f. = 1.1 pole: $\sqrt{s_p}$ = 1261 – i 94 MeV
- Roy pole: $\sqrt{s_p} = 1267.3^{+0.9}_{-0.9} - \mathrm{i}\,87(9)$ MeV

DIA for higher spins, nonequal-mass case

It is possible to write down the general formula for spin J,
adopting J subtractions in the dispersion relation :

$$\boxed{[t_J(s)]^{-1} = \frac{1}{p^{2J}(s)}\left(\sum_{i=0}^{J-1} a_i (s-s_M)^i + (s-s_M)^J \sum_{n=0}^{\infty} C_n \omega^n(s) + (s-s_M)^J R_{p\,J}(s,s_M)\right)}$$

where $$R_{p\,J}(s,s_M) \equiv \frac{s-s_M}{\pi}\int_R \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s}\frac{p^{2J}(s')}{(s'-s_M)^J}$$

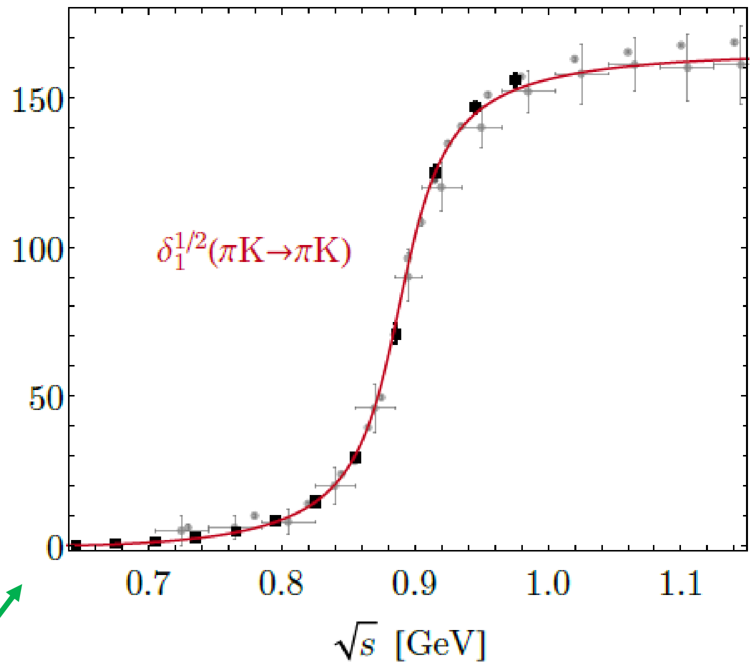
Particular case

$$J = 1: \quad [t_1(s)]^{-1} = \frac{a_0}{p^2(s)} + \frac{s-s_M}{p^2(s)}\sum_{n=0}^{\infty} C_n\,\omega^n(s) + \frac{s-s_M}{p^2(s)} R_{p\,1}(s,s_M)$$

Alternative form (applying two subtractions at the different points):

$$J = 1: \quad [t_1(s)]^{-1} = \frac{a+b\,s}{p^2(s)} + \sum_{n=0}^{\infty} C_n\,\omega^n(s) + R(s,0)$$

with an additional constraint $\sum_{n=0}^{\infty} C_n\,\omega^n(s=0) = 0$



$\boldsymbol{\delta_1^{1/2}}$

- 2 fit parameters: χ^2/d.o.f. = 0.7,
  pole: $\sqrt{s_p} = 889 - i\,27$ MeV
- Roy pole: $\sqrt{s_p} = 890(2) - i\,25.6(1.2)$ MeV

Conclusions

- Imroved parametrization for inverse scattering amplitudes for spinless particles was derived
- Derivation from the general principles – unitarity, maximal analiticity
- The test on the well-studied cases for $\pi\pi \to \pi\pi$ and $\pi K \to \pi K$ scattering was performed

- Ongoing collaboration with Lattice group (Mohler et al.)
- Determination of LEC (up to NNLO) using S and P wave lattice data

Thank you for attention!