Gravitational form factor and D-term of hydrogen-like atoms in QED

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Excited QCD 2022

The talk is based on the recent work with Xiangdong-Ji:

Gravitational Tensor-Monopole Moment of Hydrogen Atom To Order O(α).
Arxiv: 2208.05029.

Outline

- Energy Momentum Tensor and mass distribution of bound-states.
- EMT form-factor for hydrogen-like atom: scale separation, IR sensitivity and NRQED.
- EMT form-factor in NRQED.
- Results and discussion.

Energy Momentum Tensor (EMT)

- Energy-momentum tenor (EMT) is not an unfamiliar object .

1. $T^{\mu\nu}(x)$ describes the distribution of mass and momentum flow in many systems. Such as:

- Classical continuum matter. σ^{ij} called stress tensor. Often decomposed further into pressure (trace) and shear-force (traceless).
- Classical electromagnetic dynamics. T^{0i} is the famous Poynting vector. T^{ij} is the famous Maxwell stress tensor.

EMT in QFT.

- In QFT, $T^{\mu\nu}(x)$ is more subtle.

1. Formally as Noether's currents of space-time translation symmetry.
2. $T^{00}(x)$: the energy density. $T^{0i}(x)$ the momentum density. $T^{ij}(x)$: momentum current density. $\partial_\mu T^{\mu\nu} = 0$.
3. Ambiguity exists. Even for the ``canonical" one, explicit construction differs in regularization schemes. *Meaning: DR vs Lattice vs Pauli-Vilas, the same renormalized $T^{\mu\nu}$ looks quite different in terms of bare operators.*

Canonical EMT in practical QFT

- The canonical EMT for gauge theory (QED or QCD) in DR.

1. $T^{\mu\nu} = -F^{\mu\rho}F_{\rho}^{\nu} + \frac{1}{4}g^{\mu\nu}F^2 + \bar{\psi}i\gamma^{(\mu}D^{\nu)}\psi$.
2. Gluon and quark parts are not RG invariant by themselves.
3. $T^{\mu\nu} = T^{\mu\nu}_{twist-2} + \frac{g^{\mu\nu}}{4-2\epsilon}\left(m\bar{\psi}\psi - \frac{2\epsilon}{4-2\epsilon}F^2\right)$ decomposed into traceless and trace-full parts.
4. $T^{\mu}_{\mu} = m\,\bar{\psi}\psi + \left(\frac{\beta}{2g}F^2 + m\gamma_m\bar{\psi}\psi\right)$. The trace-anomaly.
5. $T^{00}(x)$: mass decomposition. Ambiguity & scheme & scale.

2105.03974. Ji&Liu,&Schafer

Hadronic EMT form factor.

- The EMT form factor of spin-$\frac{1}{2}$ particle ($q = P' - P, \bar{P} = \frac{P+P'}{2}$)
- $\langle P'|T^{\mu\nu}|P\rangle =$

$$\bar{u}(P')\left(A(q)\gamma^{(\mu}\bar{P}^{\nu)} + B(q)\frac{\bar{P}^{(\mu}i\sigma^{\nu)\alpha}q_{\alpha}}{2M} + C(q)(q^{\mu}q^{\nu} - g^{\mu\nu}q^{2})\right)u(P)$$

1. The Mass, Spin form factors. $A(0) = B(0) = 1$. Mass and spin sum-rule.
2. What about $C(0)$? This is called the D-term and remains unknown.
3. Is $C(0)$ always negative due to stability?

Hadronic EMT form factor and GPDs

hep-ph/9609381, Ji
hep-ph/9807358, Ji
hep-ph/0307382, Diehl

- Where the EMT form factor can be probed ?

1. Twist-two parts. First moment of the Generalized parton distributions (GPD). $F(x,\xi,q) \sim H + E$.

2. $\int_{-1}^{1} x\, H_g(x,\xi,q) = A_g(q) + (2\xi)^2 C_g(q)$,
   $\int_{-1}^{1} x\, E_g(x,\xi,q) = A_g(q) - (2\xi)^2 C_g(q)$. And similar for quark.

3. Can be probed in DVCS, where $\xi = -\frac{q^+}{2\,\bar{P}^+}$ is called the skewness.

4. Near threshold J/ψ production: Clue to $C(0)$?

PhysRevD.103.096010, Guo, Ji, Liu
PhysRevD.106.086004,Mamo&Zahed

DVCS and vector meson production

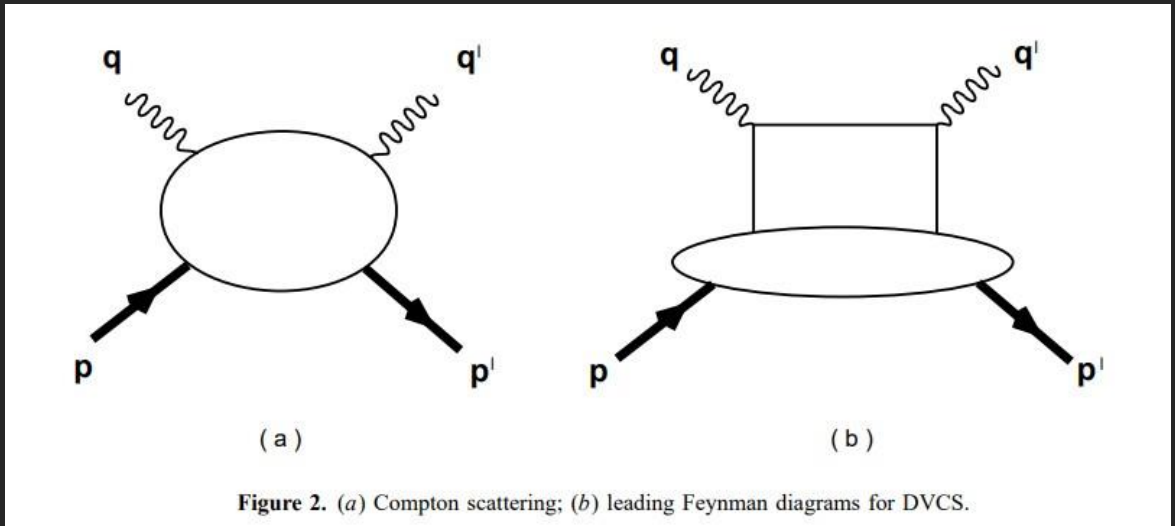


Figure 2. (*a*) Compton scattering; (*b*) leading Feynman diagrams for DVCS.

In the Bjorken limit, the DVCS amplitudes factorizes into GPDs and Hard kernels.

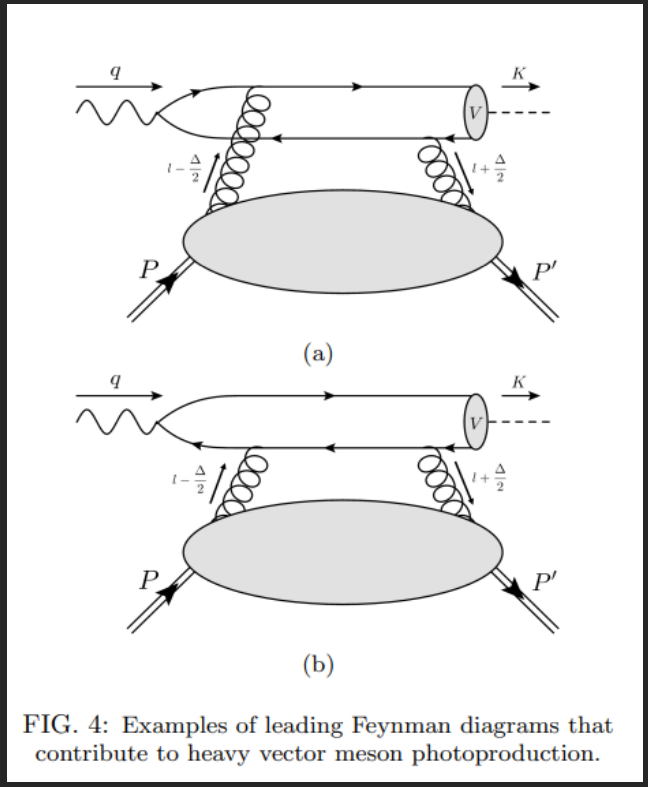
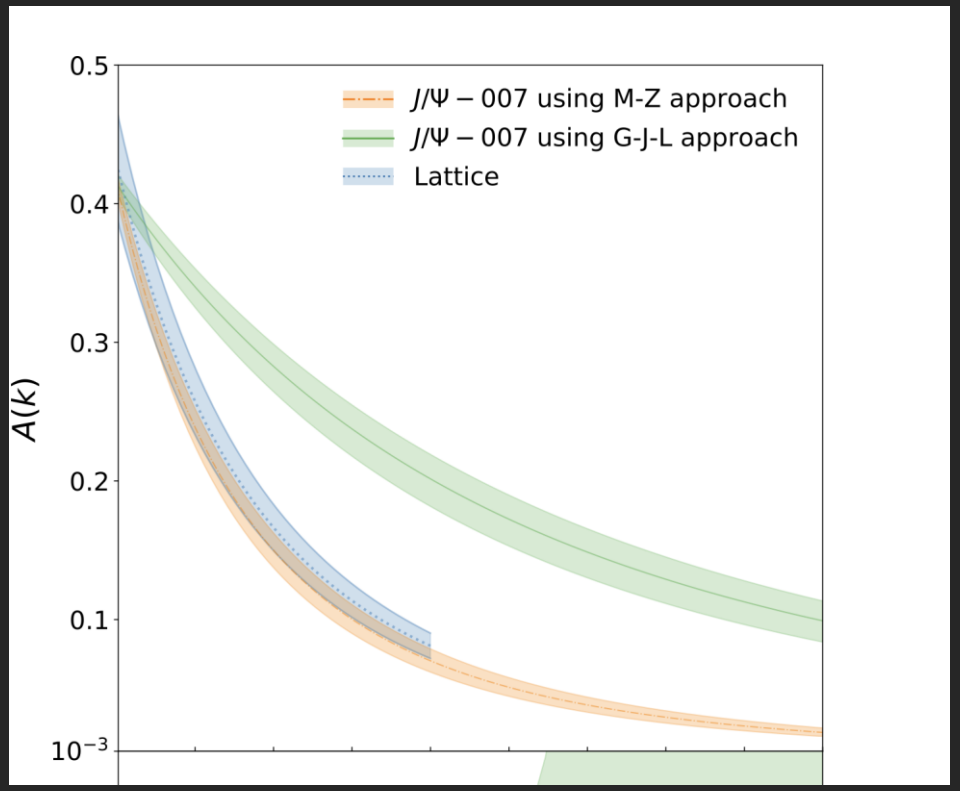


FIG. 4: Examples of leading Feynman diagrams that contribute to heavy vector meson photoproduction.

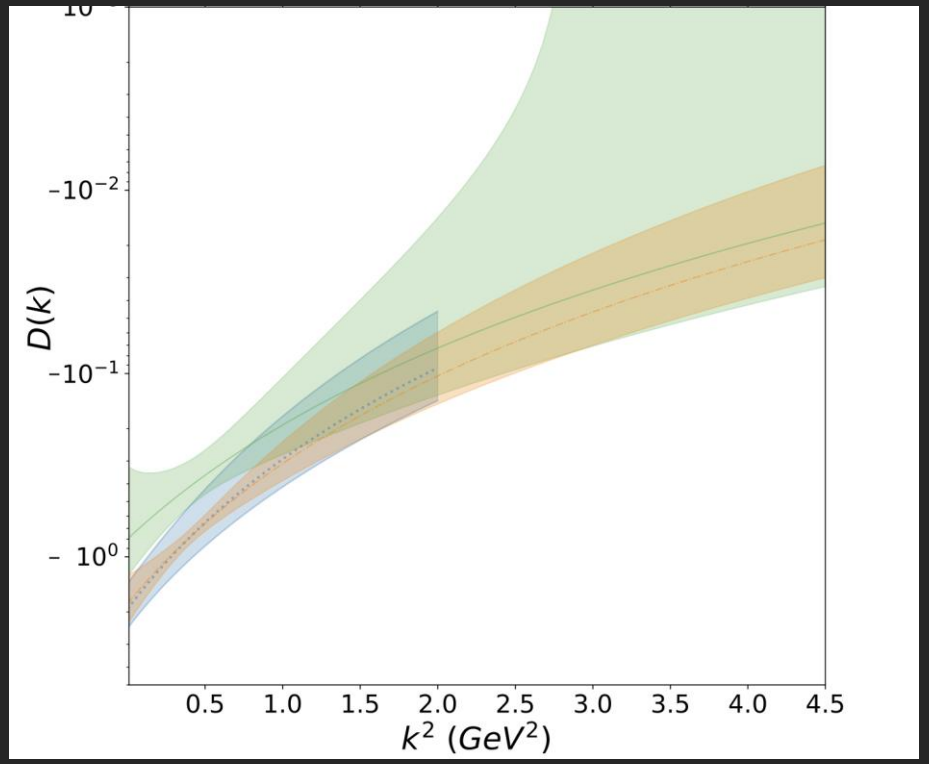
The vector meson production can also be used to probe GPDs. Near threshold may probe $\xi \sim 1$ region.

Threshould J/Ψ production and EMT form factor

Experiment performed in J-lab. 2207.05212



Extraction of the proton A form factor based on various approaches vs Lattice.



Extraction of the proton $D = 4C$ form factor.

PhysRevD.103.096010, Guo, Ji, Liu
PhysRevD.106.086004,Mamo&Zahed

EMT form factor and Sign of $C(0)$: clue from QED?

- Despite progress, uncertainly still large for $C(0)$ and its physical meaning. Is $C(0)$ always negative due to stability? hep-ph/9902451, Polyakov&Weiss
- Insights from QED will be helpful.
- The C-form factor for single electron has been used as an example.

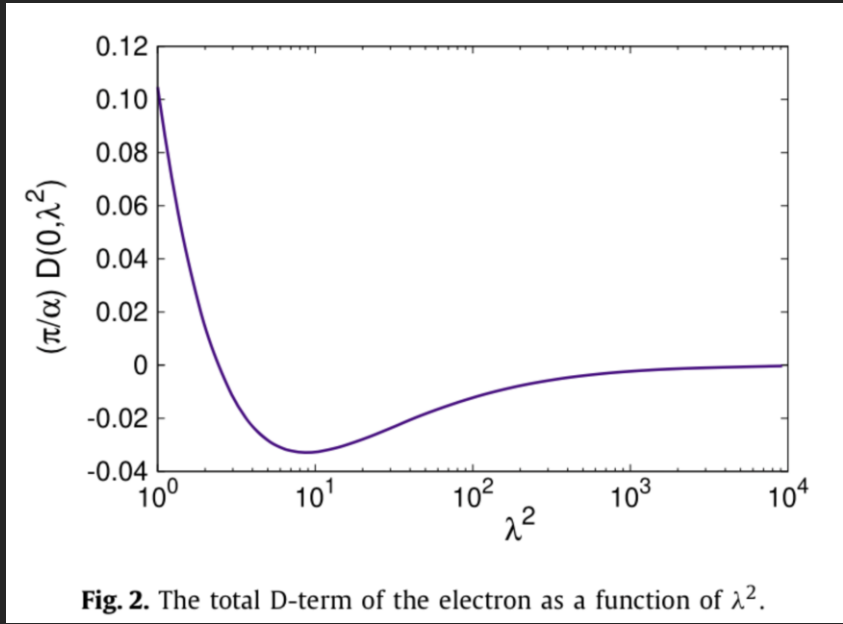


Fig. 2. The total D-term of the electron as a function of λ^2.

Phys.Lett.B 820 (2021) 136501, Metz&Pasquini&Rodini
Phys. Rev. D 15 (1977) 538, Milton,

The D-term of single electron is negative for small photon mass.

Figure from Metz&Pasquini&Rodini

However, single electron is not bound state.

But hydrogen atom is.

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Bound states in QED: the microscopic theory

- The most famous bound state in QFT: the hydrogen atom.
- Field theoretical approach: heavy proton / dressed-Dirac theory.

1. $\mathcal{L} = \bar{N} i D^0 N + \mathcal{L}_{QED}$. Bethe-Salpeter approach.
2. $\mathcal{L} = -e\bar{\psi}\gamma^\mu\psi\mathcal{A}_\mu + \mathcal{L}_{QED}$. $\mathcal{A}^\mu = \frac{e}{4\pi|\vec{r}|}\delta^{\mu,0}$: the static Coulomb field of the proton.
3. Can be shown to be equivalent.

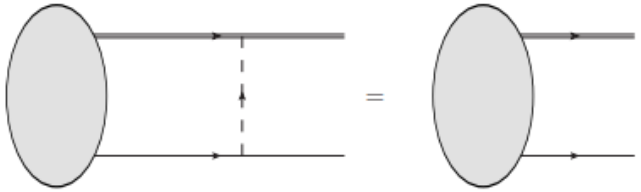


FIG. 1: The Bethe-Salpeter equation for the wave function Φ denoted by the oval blob. Double line represents propagator of proton field and single line represents the electron propagator. The dashed line represents the exchange of a Coulomb photon.

Bound states in QED: the microscopic theory

4. Renormalized using the same $Z_1 = Z_2$ and Z_3 as the free QED.

$$\delta\mathcal{L} = (Z_1 - 1)\bar{\psi}_R(i\gamma \cdot \partial - m)\psi_R - e(Z_1 - 1)\bar{\psi}_R\gamma^\mu\psi(A_{\mu,R} + \mathcal{A}_\mu)$$
$$-\delta_m m Z_1 \bar{\psi}_R\psi_R - \frac{Z_3 - 1}{4}F_R^{\mu\nu}F_{\mu\nu,R} - \frac{Z_3 - 1}{2}F_R^{\mu\nu}\mathcal{F}_{\mu\nu} \ ,$$

5. The free theory defined by the complete set of solutions of Dirac equation in the background field.

6. Radiative correction added perturbatively in terms of standard Feynman rules. Main difference: the Coulomb-Dirac propagator.

Weinberg, The Quantum theory of fields. Vol. 1

Bound states in QED: the microscopic theory

4. Example: self-energy correction to bound-state.

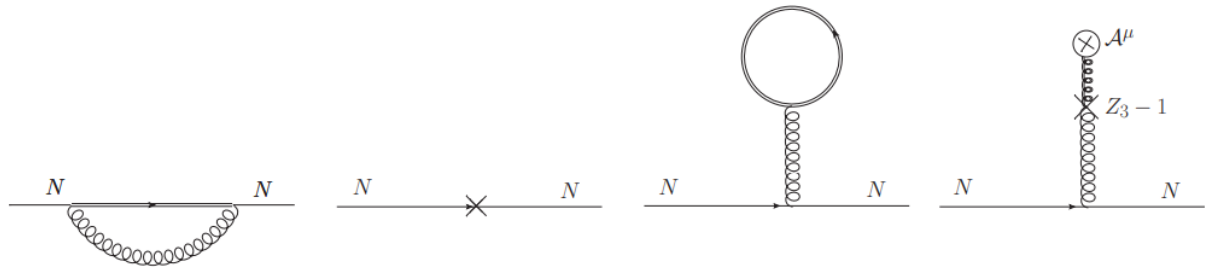


Figure 4: The Feynman diagram for the one-loop electron self energy. The double lines are dressed electron propagators and the crosses are counter terms. They contribute to the energy shift δE_N. Notice that the forth diagram corresponds to the counter term $\frac{Z_3-1}{2} F^{\mu\nu}\mathcal{F}_{\mu\nu}$ that couples the background and radiative photon fields.

5. In general, analytic calculation is very difficult.

Vacuum Polarization in a Strong Coulomb Field, Wichmann & Kroll. Phys.Rev.101,1956

Scale separation and NRQED

- However, when $\alpha \ll 1$, simplification occurs due to the emergent scale separation of the bound states:

1. The soft scale $k_s \sim \alpha m_e$. The size of the NR wave function.
2. The ultra-soft scale $k_{us} \sim \alpha^2 m_e$. The binding energy/kinematic energy.
3. The UV scale m_e. The electron mass.

- When expanded in the scale separations $\frac{\alpha m_e}{m_e}$ or $\frac{\alpha^2 m_e}{m_e}$, expression simplifies a lot. Power & log in renormalizable theory.
- Similar in spirit to twist expansion/OPE in high-energy limit.

Scale separation and NRQED

Caswell/Lepage, Phys. Lett. B 167, 437.
Labelle, Phys. Rev. D 58, 093013.
Pineda and Soto, Nucl. Phys. B Proc. Suppl. 64,428

- The NRQED is the modern way to organize this expansion . One needs

1. NRQED Lagrangian in terms of effective fields:

$$\mathcal{L} = \Psi^{\dagger}\left(iD^{0} + \frac{\vec{D}^{2}}{2m} - \frac{c_F e}{2m}\vec{\sigma}\cdot\vec{B}\right)\Psi - \frac{1}{4}F^{2} + \mathcal{O}\left(\frac{1}{m^{2}}\right)$$

2. The matching coefficients c_F and others matches the UV of IR to the IR of UV. “Splitting of Logarithms”.

3. For individual operators, matching through local counter-terms.

EMT for NRQED

- One can construct the "tree-level" EMT in NRQED:

$$T^{ij}_{\text{tree}} = T^{ij}_e + T^{ij}_\gamma + T^{ij}_{\gamma p} + T^{ij}_p$$

1. $T^{ij}_e = -\frac{1}{4m}\Psi^\dagger D^i D^j \Psi - \frac{1}{4m} D^i D^j \Psi^\dagger \Psi + \frac{1}{8m}\left(D^{(i}\Psi\right)^\dagger D^{j)}\Psi.$
2. T^{ij}_γ is the standard photon EMT.
3. $T^{ij}_{\gamma p} = \delta^{ij}\nabla V_p \cdot \nabla V_e - \partial^i V_p \partial^j V_e - \partial^j V_p \partial^i V_e$ is the interference between the Coulomb fields of electron and proton.
4. $\nabla^2 V_e = e\Psi^\dagger\Psi$ is the Coulomb field of the electron. $V_p = \frac{e}{4\pi r}$.

EMT for NRQED

5. One can show that the above T^{ij}_{tree} is conserved in symmetric bound-states using EOM.
6. However, to match to the T^{ij} in QED, local counter-terms are required.

- In this work we calculate to order $\frac{\alpha}{m}$. At this order, one needs only one counter-term

$$T^{ij}_{NRQED} = T^{ij}_{tree} + d_0\left(\partial^i\partial^j - \delta^{ij}\partial^2\right)\Psi^\dagger\Psi + \mathcal{O}\left(\frac{\alpha}{m^2}\right).$$

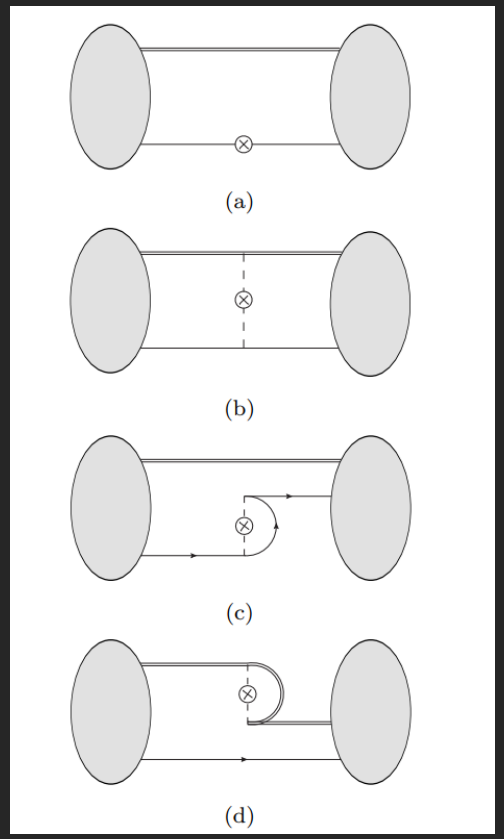
The overall-strategy of the calculation.

1. To obtain the matching coefficient d_0. One calculates using Free-NRQED and match to QED.

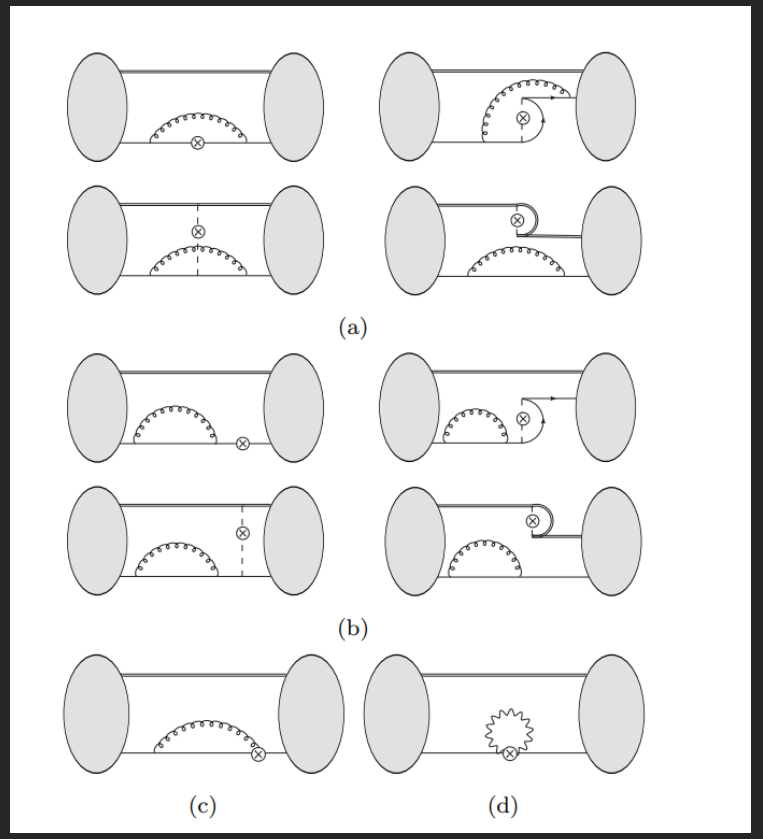
   Reason: Free NRQED and dressed NRQED shares the same UV.

2. Combine d_0 with bound-state calculation to get full result.
3. DR with $D = 3 - 2\epsilon$ is adopted consistently. Coulomb gauge is chosen.
4. Power-counting in α eliminates all the fermionic diagrams.

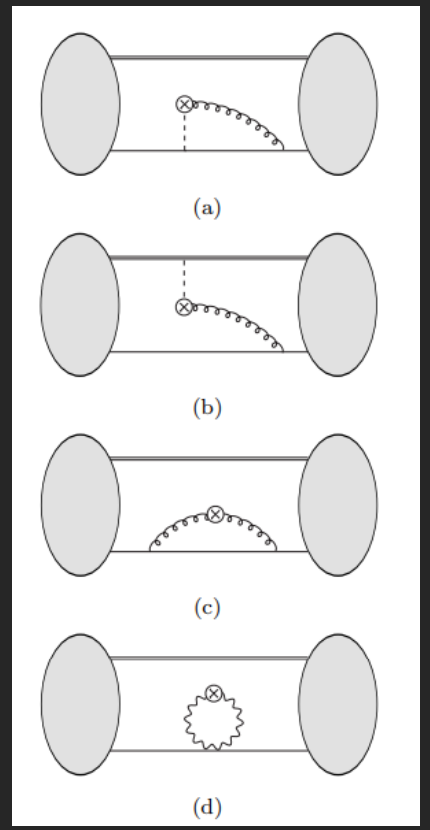
The overall-strategy of the calculation.



The leading order diagrams.



The one-loop fermionic diagrams.



The photonic diagrams.

The overall-strategy of the calculation.

- An example of power-counting in α

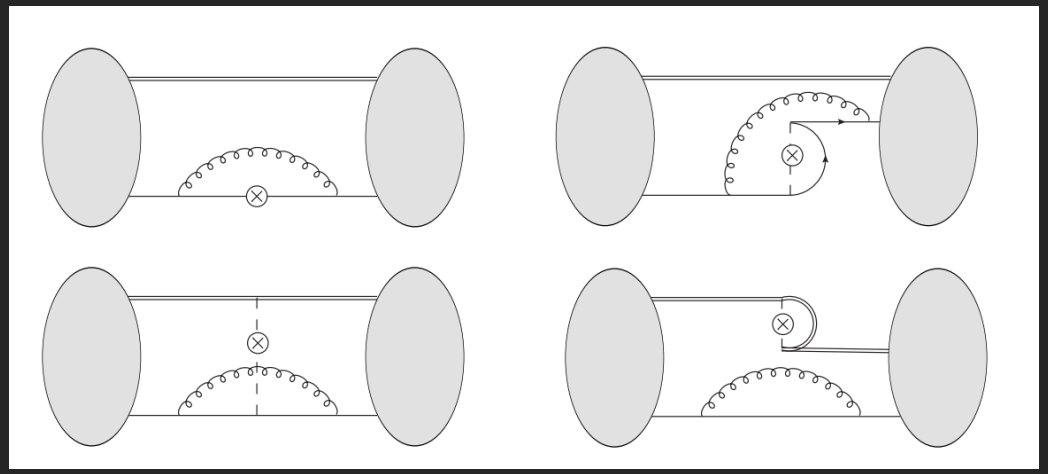


Fig. (6a) in 2208.05029

$$\langle T_0^{ij}\rangle_{6a}(\vec{q}) = e^2\mu^{2\epsilon}\sum_{M,M'}\int\frac{d^D\vec{k}}{(2\pi)^D}$$
$$\frac{(1-\frac{1}{D})\vec{v}_{0M'}(\vec{k})\cdot\vec{v}_{M0}(\vec{k})}{(E_0-|\vec{k}|-E_M)(E_0-|\vec{k}|-E_{M'})2|\vec{k}|}\langle M'|T_0^{ij}(\vec{q})|M\rangle\ , \tag{64}$$

where the $T_0^{ij}(\vec{q})$ is defined in Eq. (62), including the proton part when $M = M'$. It is easy to see that for $D = 3$, when $|\vec{k}| = \mathcal{O}(\alpha m_e)$ or $|\vec{k}| = \mathcal{O}(\alpha^2 m_e)$, the two energy denominators and one phase-space measure $2|\vec{k}|$ for the photon contributes to $(\alpha m_e)^{-3}$ or $(\alpha m_e)^{-6}$, which is always canceled by the integration measure $\int d^3\vec{k} = (\alpha m_e)^3$ or $(\alpha m_e)^6$, respectively. The two form-factors for the velocity operators will contributes to α^2, while the matrix element $\langle M'|T_0^{ij}(q)|M\rangle$ as shown above will contributes to $\mathcal{O}(\alpha^2)$ at order $\mathcal{O}(q^0)$ and $\mathcal{O}(1)$ to $\mathcal{O}(q^2)$. Therefore, together with the overall e^2, Fig. 6a will contributes at order $\mathcal{O}(\alpha^3)$ to the coefficients of q^2, and to $\mathcal{O}(\alpha^5)$ for coefficients of q^0, therefore not relevant for our calculation. Similar argument can be used to show that diagrams in Fig. 6b and Fig. 6c (T_{e1}^{ij}) will be irrelevant to NLO as well. The last diagram Fig. 6d,

The overall-strategy of the calculation.

- DR for bound states.

1. All quantities are in D dimensions, including NR wave functions.
2. However, the sum-rule $\sum_M \frac{2\vec{v}_{NM}\cdot\vec{v}_{MN}}{D(E_M-E_N)} = \frac{1}{m}$ will guarantee that the coefficients of the $\frac{1}{\epsilon}$ poles are D-independent constants.
3. Similar to the simplified calculation to the Lamb' shift.

Caswell/Lepage, Phys. Lett. B 167, 437.
Labelle, Phys. Rev. D 58, 093013.
Pineda and Soto, Nucl. Phys. B Proc. Suppl. 64,428

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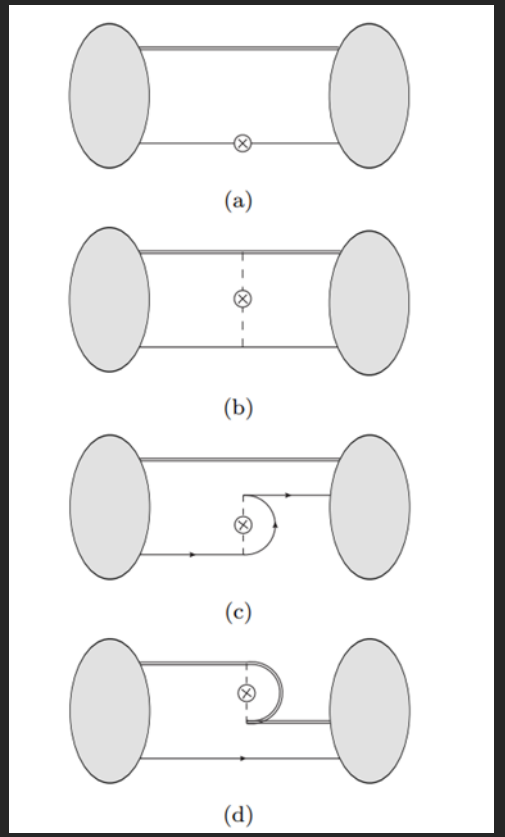
The Leading order results

- $\langle 0|T^{ij}(q)|0\rangle_H = (q^i q^j - \delta^{ij} q^2)\frac{C_H(q)}{m_e}$

$$\frac{C_{\rm H}(q)}{m_e} = \frac{1}{2m_e(\frac{q^2}{\alpha^2 m_e^2}+4)} - \frac{\alpha}{4|q|}\left(\frac{\pi}{2} - \text{Arctan}\frac{q}{2\alpha m_e}\right) + \frac{\alpha\pi}{|q|}\frac{1}{(\frac{q^2}{\alpha^2 m_e^2}+4)^2} + \frac{\alpha\pi}{|q|}\frac{1}{(\frac{q^2}{\alpha^2 M^2}+4)^2}.$$

- $\frac{C_H(0)}{m_e} = \frac{1}{4m_e}$ is positive.



The leading order diagrams.

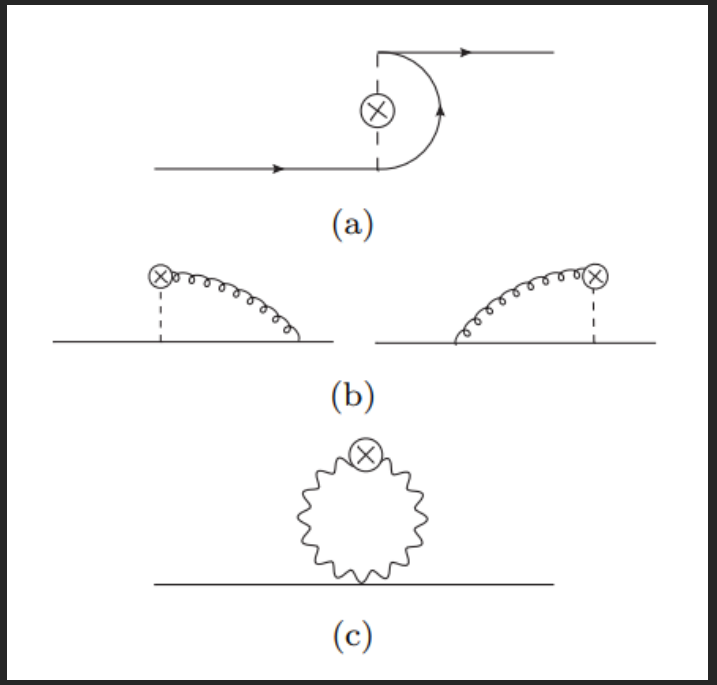
The matching coefficients d_0

$$T^{ij}_{NRQED} = T^{ij}_{tree} + d_0(\partial^i\partial^j - \delta^{ij}\partial^2)\Psi^\dagger\Psi$$

- The condition for matching

$$\left\langle \vec{Q}\middle|T^{ij}_{NRQED}\middle|-\vec{Q}\right\rangle = \left\langle \vec{Q}\middle|T^{ij}_{QED}\middle|-\vec{Q}\right\rangle$$

- Only photonic diagrams are relevant.
- Coulomb-self, mixed and tadpole diagram.
- Tadpole diagram originates from backward moving diagram in QED.

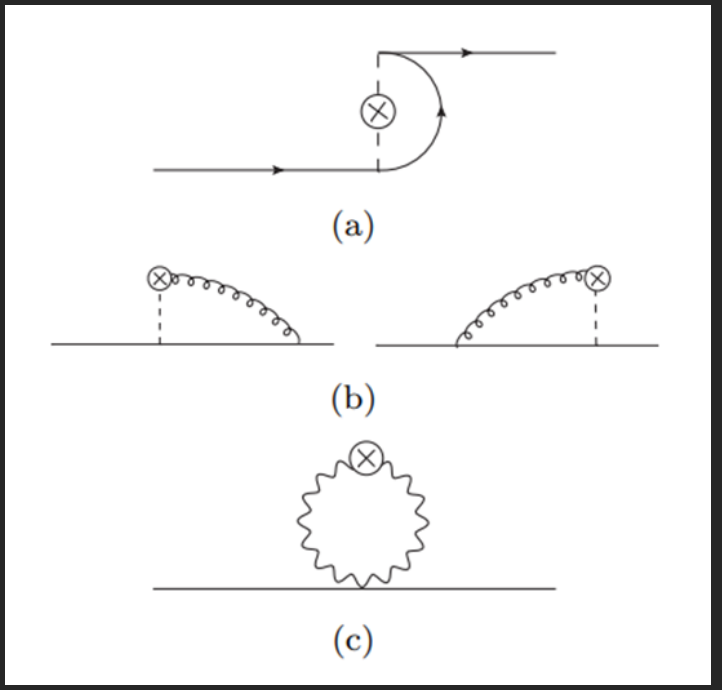


The relevant diagrams in free NRQED for T^{ij}_{tree}.

The matching coefficients d_0

- T^{ii}(trace) and $Q^i T^{ij} \mathrm{Q}^{\mathrm{j}}$ (longitudinal) calculated separately.

1. The Coulomb-self diagram is conserved separately.
2. Longitudinal parts cancel between the mixed and tadpole diagrams.
3. T^{ii} of the mixed diagram contains the logarithm.



The relevant diagrams in free NRQED.

The matching coefficients d_0

- The total results reads

$$\langle T^{ij}_{\rm tree}\rangle(2\vec{Q}) = (Q^iQ^j - \delta^{ij}Q^2)\tilde{C}(Q^2)\ , \qquad (97)$$

where

$$\tilde{C}(Q^2) = \frac{\alpha\pi}{8|Q|} + \frac{e^2}{6m_e\pi^2}\left(-\frac{1}{\epsilon} + \ln\frac{Q^2}{\mu^2} + \gamma_E - \ln\pi - \frac{7}{6}\right)\ . \qquad (98)$$

- Same in IR with QED, differs in UV

$$\tilde{C}_{\rm QED}(Q^2) = \frac{\alpha\pi}{8|Q|} + \frac{e^2}{6m_e\pi^2}\ln\frac{4Q^2}{m_e^2} - \frac{11e^2}{72m_e\pi^2} \qquad (99)$$

The matching coefficients d_0

- As a result, one obtains the d_0

$$d_0 = -\frac{\alpha}{6\pi m_e}\left(\frac{1}{\epsilon} + \ln\frac{4\mu^2}{m_e^2} + \ln\pi - \gamma_E + \frac{1}{4}\right). \quad (101)$$

- It contains UV logarithms $\ln\frac{\mu^2}{m_e^2}$, but no IR sensitivity.

The NLO contribution for bound-state.

Figure (b) vanishes.

- For the bound-state, the pure-radiative diagram is activated in the ultra-soft region.
- Dipoles expansion performed in ultra-soft region as usual.
- Tadpole diagram remains the same as Free-NRQED.
- Cancellation of longitudinal parts between (a), (c) and (d).

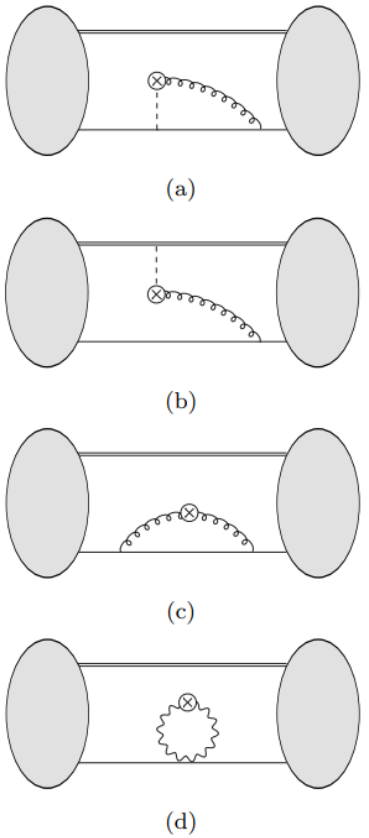


FIG. 7: The order-$\mathcal{O}(\alpha)$ contributions to $T_\gamma^{ij} + T_{\gamma p}^{ij}$ for a bound state. Dashed lines represent Coulomb photons and crossed circles denote the operator insertions.

The NLO contribution: checking conservation.

- For Figure (a)

$$Q^i \langle T^{ij}_{\gamma\parallel\perp} \rangle_{7\mathrm{a}} Q^j = \frac{Q^4 e^2}{\pi^2} \sum_M \frac{2\vec{v}_{M0}\cdot\vec{v}_{0M}}{D(E_M - E_0)} \left(-\frac{1}{15\epsilon} + \frac{1}{15}\ln\frac{(E_M-E_0)^2}{\mu^2} + \frac{-15\ln\pi + 15\gamma_E - 1}{225} \right) .$$

- For Figure (c)

$$Q^i \langle T^{ij}_{\gamma\perp} \rangle_{7\mathrm{c}} Q^j = \frac{Q^4 e^2}{\pi^2} \sum_M \frac{2\vec{v}_{0M}\cdot\vec{v}_{M0}}{D(E_M - E_0)} \left(\frac{1}{15}\ln\frac{Q^2}{(E_M-E_0)^2} + \frac{15\ln 4 - 46}{225} \right) .$$

Cancellation works!

- For Figure (d)

$$Q^i \langle T^{ij}_{\gamma\perp} \rangle_{7\mathrm{d}} Q^j = \frac{e^2}{15 m_e \pi^2 \epsilon} Q^4 + \frac{e^2}{15 m_e \pi^2} Q^4 \left(-\ln\frac{Q^2}{\mu^2} + c_2' \right) ,$$

$$c_2' = \gamma_E - 3 + \ln 4 + \ln\pi + 2\psi\left(\frac{7}{2}\right) \equiv \frac{47}{15} - \gamma_E + \ln\pi - \ln 4 .$$

The NLO contribution: The total results

- As a result, from T^{ii} one has

$$\langle T^{ij}_{\rm tree}\rangle(2\vec{Q}) = (Q^iQ^j - \delta^{ij}Q^2)\tilde{C}_{\rm s}(Q^2)\ ,$$

$$\tilde{C}_{\rm s}(Q^2=0) = \frac{e^2}{6\pi^2}\sum_M \frac{2\vec{v}_{M0}\cdot\vec{v}_{0M}}{D(E_M-E_0)}\left(-\frac{1}{\epsilon} + \ln\frac{(E_M-E_0)^2}{\mu^2} + \gamma_E - \ln\pi - \frac{1}{2}\right)\ .$$

- To match to QED, simply adds $-4d_0$. This leads to our result.

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The NLO contribution: The total results

- The D-term for hydrogen-atom

1. $\langle 0|T^{ij}(q)|0\rangle_H = (q^i q^j - \delta^{ij} q^2)\frac{C_H(q)}{m_e}$

2. $\frac{C_H(0)}{m_e} = \tau_H = \frac{1}{4m_e} + \frac{\alpha}{6\pi}\sum_{M\neq 0}\frac{2\vec{v}_{0M}\cdot\vec{v}_{M0}}{D(E_M - E_0)}\left(\ln\frac{4(E_M-E_0)^2}{m_e^2} - \frac{1}{4}\right)$

- The logarithm couples the UV and IR. Similar to the Bethe logarithm.
- Both continuum and discrete spectrum contributes.

The NLO contribution: The total results

- More explicitly, one has

$$\tau_H = \frac{1}{4m_e} + \frac{\alpha}{6\pi m_e}\left(\ln \alpha^2 + \tau_d + \tau_c - \frac{1}{4}\right)$$

1. The discrete spectrum part

$$\tau_d = \sum_{n=2}^{\infty} \frac{2^8 n^5}{3(n^2-1)^4}\left(1-\frac{2}{n+1}\right)^{2n} \ln\left(1-\frac{1}{n^2}\right)^2 = -0.264\ .$$

2. The continuum spectrum part

$$\tau_c = \frac{1}{3}\int_0^{\infty} dE \frac{2^8 \ln(2E+1)^2}{(2E+1)^4} \frac{e^{-\frac{4}{\sqrt{2E}}\mathrm{Arccot}\left(\frac{1}{\sqrt{2E}}\right)}}{1-e^{-\frac{2\pi}{\sqrt{2E}}}} = 0.458\ .$$

The NLO contribution: The total results

- Therefore, one has

$$\frac{\tau_H}{\tau_0} - 1 = \frac{4\alpha}{3\pi}(\ln\alpha - 0.028) = -1.54 \times 10^{-2}.$$

- The NLO contribution is small and negative.

Discussion and Summary

- The positivity of LO results is due to the long-range force in QED.
- Heuristically, one can guess it from result in free-QED

$$\tilde{C}_{\mathrm{QED}}(Q^2) = \frac{\alpha\pi}{8|Q|} + \frac{e^2}{6m_e\pi^2}\ln\frac{4Q^2}{m_e^2} - \frac{11e^2}{72m_e\pi^2} \qquad (99)$$

- At small Q, the natural IR cutoff now is αm_e, therefore LO positive while NLO negative.
- Positivity holds for small photon mass.

Discussion and Summary

- To summarize, the EMT form factor in QFT are important quantities to understand the mass-structure of bound-states.
- We use the hydrogen-like atom in QED as a non-trivial example to show that the D-term is not necessarily negative.
- The example is multi-scale in nature and demonstrates the basic principles of EFT.

Ambiguity in T^{00}-based mass sum-rules

2105.03974. Ji, Liu, Schafer

- Consider the $O(N)$ non-linear sigma model in large N. One has

1. The classical energy : $H_c = \frac{1}{2}\int dx\left((\partial_t \pi^a)^2 + (\partial_x \pi^a)^2\right)$.
2. The trace-anomaly : $H_a = \frac{\beta(g_0)}{2g_0}\int dx\left(\partial_\mu \pi^a\right)^2$.
3. The true $H = \int dx\, T^{00}(x)$ looks different across schemes.

The contribution of H_c, H_a in different UV regularization schemes.

scheme	H_c	$H_a \equiv H_S$	H
$k^2 \leq \Lambda_{\rm UV}^2$	$\frac{m}{2}$	$\frac{m}{2}$	m
lattice	$\frac{m}{2}$	$\frac{m}{2}$	m
$\lvert k_1\rvert \leq \Lambda_{\rm UV}$	m	$\frac{m}{2}$	m
DR	m	$\frac{m}{2}$	m
$\frac{k_4^2}{\lambda^2} + k_1^2 \leq \Lambda_{\rm UV}^2$	$\frac{\lambda m}{1+\lambda}$	$\frac{m}{2}$	m

The NLO contribution for bound-state.

- We use the

1. Schwinger's α parameter for relativistic propagators
2. The λ parameter for NR propagators

   to disingularize all the Feynman integrals.

- Relations for Coulomb-gauge projections are used to simplify the integrand.
- Mellin transform trick is used to obtain the small q asymptotics.

The NLO contribution for bound-state.

- Typical parameter-space representation looks like

$$F_1(q) = -\frac{D}{4}\left(\frac{D}{2}+1\right)\left(1-\frac{1}{D}\right)\int_0^1 4x^2\bar{x}dx\int_0^\infty d\lambda\int_0^1 dt_1\int_0^1 dt_2\int_0^\infty \rho^{-\frac{D}{2}+1}d\rho e^{-\frac{\lambda^2}{4}-\lambda\sqrt{\rho x t_1}-4\rho x(1-x)t_2 q}\,, \tag{E15}$$

$$F_2(q) = \frac{D}{2}\left(1-\frac{1}{D}\right)\int_0^1 x(1+2x^2-2x)dx\int_0^\infty d\lambda\int_0^1 dt_1\int_0^\infty \rho^{-\frac{D}{2}+1}d\rho e^{-\frac{\lambda^2}{4}-\lambda\sqrt{\rho x t_1}-4\rho x(1-x)q}\,, \tag{E16}$$

$$F_3(q) = -\frac{1}{D}\frac{D}{2}(1-\epsilon)\int_0^1 4x(1-x)dx\int_0^\infty d\lambda\int_0^1 dt_1\int_0^\infty \rho^{-\frac{D}{2}+1}d\rho e^{-\frac{\lambda^2}{4}-\lambda\sqrt{\rho}-4\rho x(1-x)t_1 q}\,, \tag{E17}$$

$$F_4(q) = (1-\epsilon)\int_0^1 (2x-1)^2dx\int_0^\infty d\lambda\int_0^\infty \rho^{-\frac{D}{2}+1}d\rho e^{-\frac{\lambda^2}{4}-\lambda\sqrt{\rho}-4\rho x(1-x)q}\,, \tag{E18}$$

$$F_5(q) = -\frac{D}{2}\left(1-\frac{1}{D}\right)\int_0^1 x(2x-1)dx\int_0^\infty d\lambda\int_0^1 dt_1\int_0^\infty \rho^{-\frac{D}{2}+1}d\rho e^{-\frac{\lambda^2}{4}-\lambda\sqrt{\rho(xt_1+1-x)}-4\rho x(1-x)q}\,. \tag{E19}$$

Clearly, all the integrals are absolutely convergent for $D = 3$ and $q \neq 0$, thus one can set $D = 3$ and use the Mellin transform technique as before to obtain the small-q asymptotics. Direct calculation leads to

$$\sum_{i=1}^{5} F_i(s) = -\frac{8}{15s^2} + \frac{8(15\ln(4)-46)}{225s} + \mathcal{O}(1)\,, \tag{E20}$$