Emergence of hydrodynamics in expanding relativistic plasmas

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Based on work done in collaboration with Li Yan
(see arXiv: 2106.10508 and earlier papers)

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Emergence of hydrodynamics

- Hydrodynamics work amazingly well in describing the evolution of matter produced in heavy ion collisions

- Fluid behavior requires (some degree of) local equilibration How is this achieved? How does the system evolve from an initial, non equilibrium, collection of partons, to the nearly thermal state reached in the late stages of the collisions?

A very general problem in kinetic theory

Consider the (non-relativistic) Boltzmann equation

distribution function

$$\frac{\partial}{\partial t} f + \frac{\vec{p}}{m} \cdot \vec{\nabla}_r f - \vec{\nabla}_r U \cdot \vec{\nabla}_p f = C[f] \qquad f = f(\vec{r}, \vec{p}, t)$$

free motion force field collisions

Introduce density, velocity field, and pressure

$$n(\vec{r}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(\vec{r}, \vec{p}, t) \qquad n\vec{v}(\vec{r}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\vec{p}}{m} f(\vec{r}, \vec{p}, t) \qquad P_{ij}(\vec{r}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p_i \frac{p_j}{m} f(\vec{r}, \vec{p}, t)$$

When collisions dominate, the pressure becomes isotropic $P_{ij}(\vec{r}, t) = \delta_{ij} P(\vec{r}, t)$

and the kinetic equation reduces to hydrodynamics

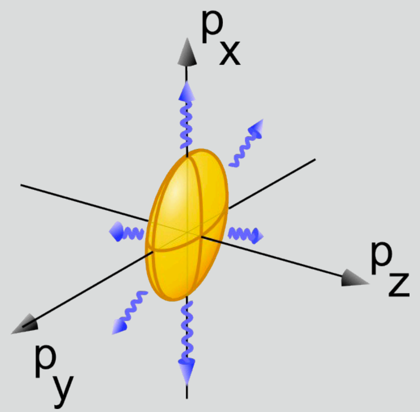
$$\partial_t n + \vec{\nabla}(n\vec{v}) = 0$$

$$m\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} U + \frac{1}{n} \vec{\nabla} P = 0$$

Conservation laws for particle number and momentum

Longitudinal expansion hinders isotropization

- The fast expansion of the matter along the collision axis tends to drive the momentum distribution to a very flat (oblate) distribution



- Translates into the existence of two different pressures

$$P_L = \int_{\boldsymbol{p}} \frac{p_z^2}{p_0} f \qquad P_T = \int_{\boldsymbol{p}} \frac{p_T^2}{p_0} f$$

(longitudinal) (transverse)

- Anisotropy $(\sim P_L - P_T)$ relaxes slowly, like a 'collective' variable associated to a conservation law

Simple kinetic equation (Bjorken flow)

- 1+1 dimensional expansion, in relaxation time approximation

$$\left[\partial_\tau - \frac{p_z}{\tau}\partial_{p_z}\right] f(\boldsymbol{p}/T) = -\frac{f(\boldsymbol{p},\tau) - f_{\rm eq}(p,\tau)}{\tau_R}$$

expansion collisions

- Describes the transition from the collisionless regime $(\tau \ll \tau_R)$ to the regime dominated by collisions, leading eventually to hydrodynamics $(\tau \gg \tau_R)$

Special moments of the momentum distribution

(JPB, Li Yan, 2017, 18, 19)

Special moments $p_z = p\cos\theta$

$$\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos\theta) f(\boldsymbol{p})$$

(Legendre polynomial)

$$P_0(z) = 1 \qquad P_2(z) = \frac{1}{2}(3z^2 - 1)$$

Why these moments ?

- There is too much information in the distribution function
- We want to focus on the angular degrees of freedom

The energy momentum tensor is described by first two moments

$$T^{\mu\nu} = \int_p f(\boldsymbol{p}) p^\mu p^\nu \qquad \mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

We are looking for an effective theory for these two moments

Coupled equations for the moments

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \geq 1)$$

(Free streaming) (collisions)

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left[a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right]$$

- The coefficients a_n, b_n, c_n are pure numbers ($a_0 = 4/3$, $c_0 = 2/3$)
- Interesting system of coupled linear equations
- Exact solution provides exact values for the energy density and pressures, but does not allow the complete reconstruction of the distribution function
- The competition between expansion and collisions is made obvious. Note the absence of collisional damping for the energy density.

Effective theory obtained by 'eliminating' moments $\mathcal{L}_{n>1}$

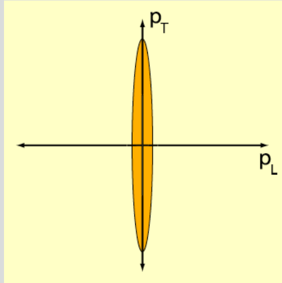
Two-moment truncation

(effective theory)

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix}a_0 & c_0\\ b_1 & a_1+\frac{\tau}{\tau_R}\end{pmatrix}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix}$$

- Contains second order viscous hydrodynamics à la Israel-Stewart
- Views hydrodynamics as a coupled mode problem
- Amenable to analytic solution, bringing insight into the notions of attractors, general features of the gradient expansion, and its resummations in terms of trans-series, etc. [not discussed here]
- Captures most important features of more sophisticated approaches, and can be made quantitatively accurate with a simple renormalization of a second order transport coefficient (a1) [see later].

Collisionless regime



In the absence of collisions,

$$f(p_T, p_z, \tau) = f_0(p_T, p_z\tau/\tau_0)$$

All moments decay with the same power law at late time

$$g_n(\tau) = \tau\partial_\tau \ln \mathcal{L}_n, \qquad g_n(\tau \to \infty) \to -1$$

Many moments are needed to accurately describe the late time distribution.

However a reasonably accurate description of the first two moments is obtained from the truncation [can be improved - see later]

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix}a_0 & c_0\\ b_1 & a_1\end{pmatrix}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix}$$

Two eigenmodes $\lambda_0 = 0.929366$ $\lambda_1 = 2.21349$ [exact values: $\lambda_0 = 1, \quad \lambda_1 = 2$]

$$\mathcal{L}_i = \alpha_i\left(\frac{\tau_0}{\tau}\right)^{\lambda_1} + \beta_i\left(\frac{\tau_0}{\tau}\right)^{\lambda_2}$$

NB: one mode is less damped and (trivially) plays the role of "attractor"

Free streaming fixed point

One can transform the coupled linear equations into a single non linear differential equation for the quantity $g_0(\tau) = \frac{\tau}{\tau_R}\frac{\partial \mathcal{L}_0}{\partial \tau}$ $\left(\frac{P_L - P_T}{\varepsilon} = -\frac{1}{c_0}(a_0 + g_0)\right)$

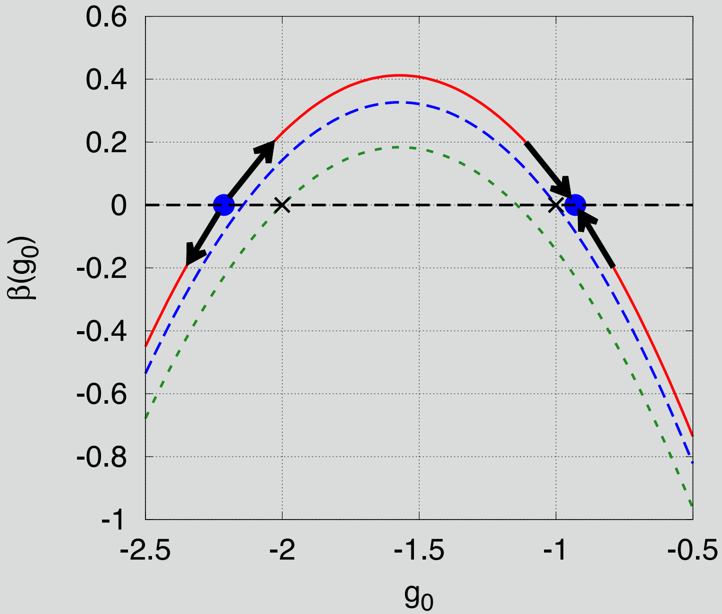
$$\tau \frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + (a_0 + a_1)\, g_0 + a_1 a_0 - c_0 b_1 - \left[c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0}\right] = 0$$

Write this as

$$\tau \frac{dg_0}{d\tau} = \beta(g_0) \qquad \beta(g_0) = -g_0^2 - (a_0 + a_1)g_0 - a_0 a_1 + c_0 b_1$$

$g_0^* = -\lambda_1 = -2.21$

(unstable)



$g_0^* = -\lambda_0 = -0.929$

(stable)

exact fixed point (-1) can be recovered by adjusting L2 to its exact value, known near a fixed point, e.g. at the stable fixed point,

$\mathcal{L}_2/\mathcal{L}_0 = 3/8$

- **This fixed point structure is only moderately affected by higher moments**
- **This structure is approximately captured by Israel-Stewart hydrodynamics**

Including collisions

Two-moment truncation

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau}\left(a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right)$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau}\left(b_1 \mathcal{L}_0 + a_1 \mathcal{L}_1\right) - \frac{1}{\tau_R}\mathcal{L}_1$$

Can be transformed into a non linear equation, similar to that resulting form Israel-Stewart formulation of second order viscous hydrodynamics and much studied in [Heller, Spalinski , 2015]

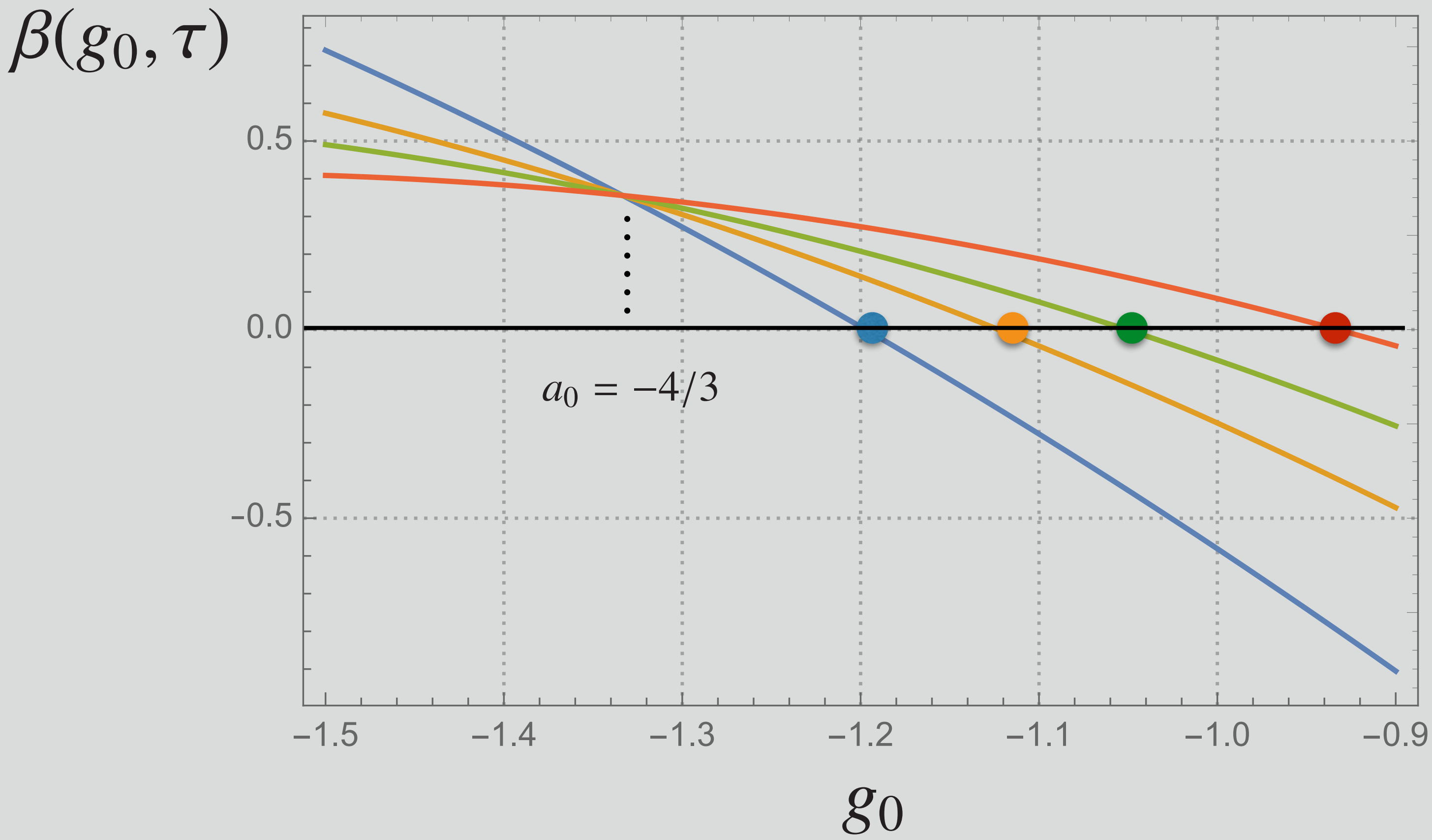
$$\tau \frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + \left(a_0 + a_1 + \frac{\tau}{\tau_R}\right) g_0 + a_1 a_0 - c_0 b_1 + \frac{a_0 \tau}{\tau_R} = 0$$

$\tau \ll \tau_R$ one recovers the two free streaming fixed points

$\tau \gg \tau_R$ $\quad g_0 + a_0 = 0, \quad g_0 = -4/3$ hydro fixed point

Attractor

Under the effect of collisions, the stable free streaming fixed point evolves "slowly" into the hydrodynamic fixed point



"Pseudo fixed points" move "slowly" towards -4/3 as time increases

The "attractor" is the solution $g_0(\tau)$ that joins the (stable) free streaming fixed point at early time to the hydrodynamic fixed point at late time.

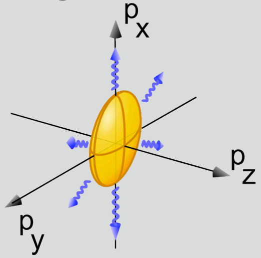
The transition from free streaming to hydrodynamics

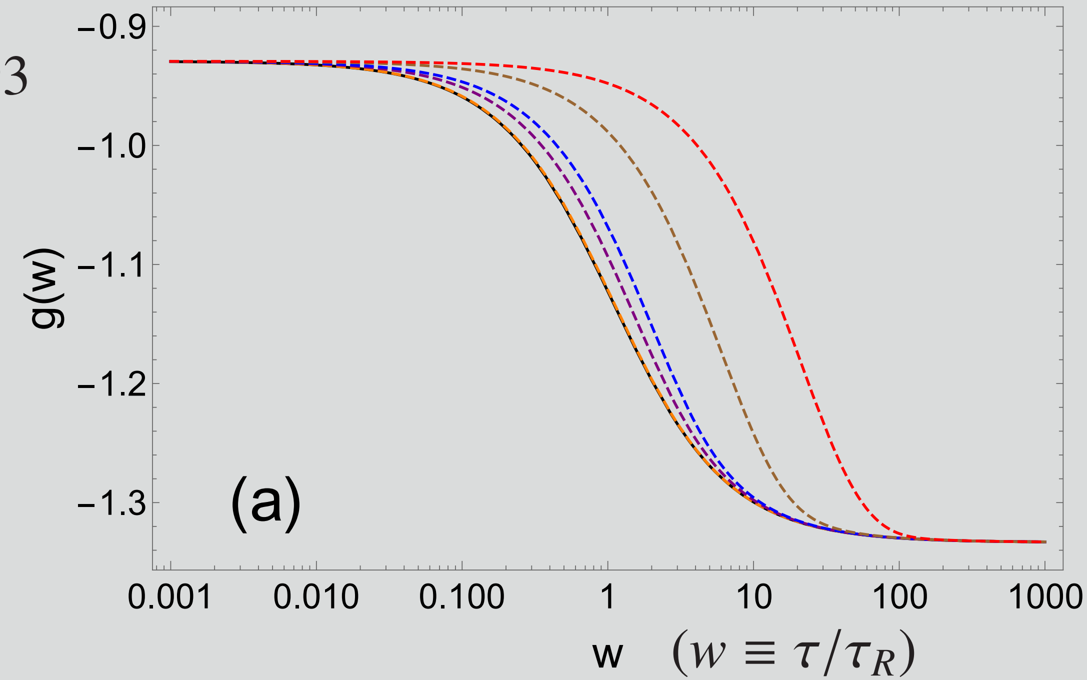
(Attractor solution)

Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

Free streaming fixed point

$g_0 \approx -0.93$

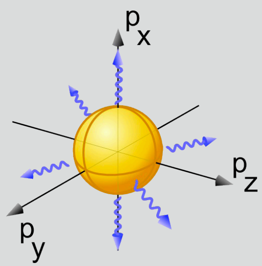




Hydro fixed point

$$g_0 = -\frac{4}{3}$$

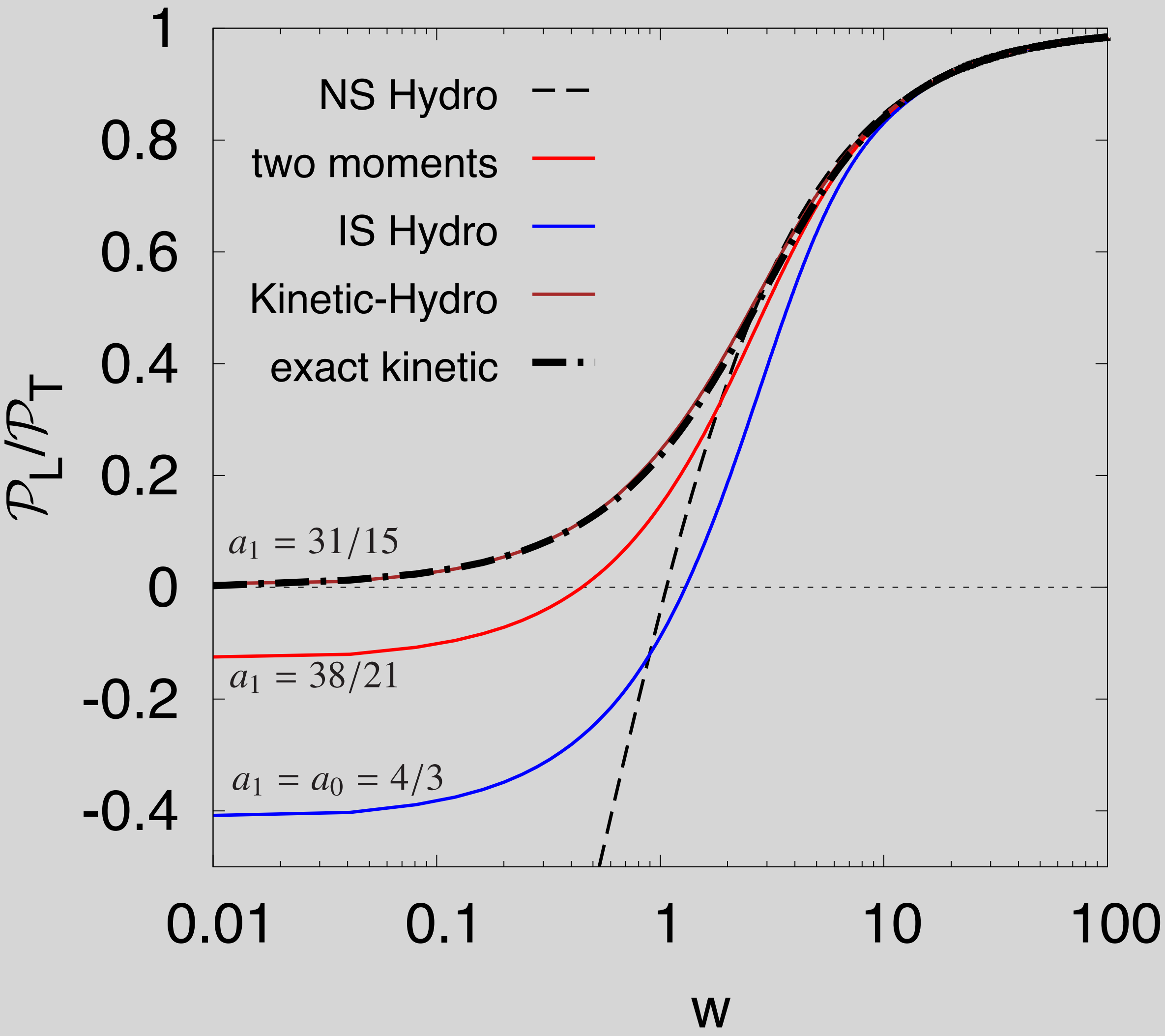
(Universal!)



The transition region occurs when the collision rate is comparable to the expansion rate $(\tau \sim \tau_R)$

Renormalizing a1 cures unphysical features of two-moment truncation (and other Israel-Stewart calculations)

$$\tau \frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + \left(a_0 + a_1 + \frac{\tau}{\tau_R}\right) g_0 + a_1 a_0 - c_0 b_1 + \frac{a_0 \tau}{\tau_R} = 0$$



Conclusions

The solution of a simple kinetic equation for Bjorken flow was analysed in terms of special moments of the distribution function.

The simplest two moment-truncation yields an 'effective' theory that captures the main qualitative features of the dynamics, in particular the transition from the collisionless regime to hydrodynamics. It encompasses all versions of second order (Israel-Stewart) hydrodynamics

The collisionless regime is characterized by two fixed points, one stable, the other unstable. The effect of the collisions is to move "slowly" the stable free streaming fixed point into the (universal) hydrodynamic fixed point.

Conclusions

The "attractor" emerges as the solution that joins the collisionless fixed point at t=0 to the hydrodynamic fixed point at large time. The vicinities of the two fixed points are easy to control (known ratios of moments in free streaming, Navier-Stokes in hydrodynamics).

Hydrodynamic behavior emerges where it is supposed to do so, i.e. when the collision rate is comparable to the expansion rate.

By 'improving' the transition region between the fixed points (i.e., adjusting the free streaming fixed point), one does not 'improve' hydrodynamics!

The present analysis extends with 'minor' modifications to the non-conformal case (2208.02750)