# Static quark operators based on Laplacian eigenmodes

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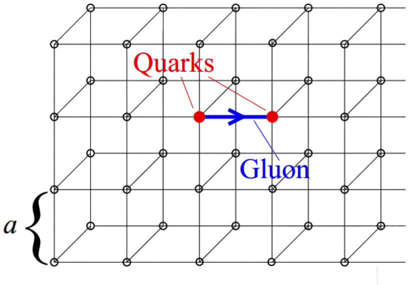
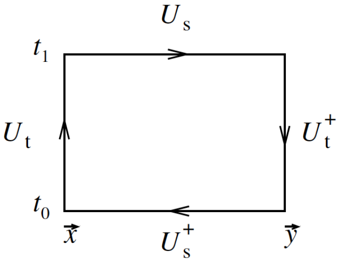
# Lattice QCD




- link variables $U_\mu(x) = \exp(i \int_x^{x+a\hat{\mu}} A_\mu dx^\mu)$
- Wilson line $U_s(\vec{x}, \vec{y}) = \exp(i \int_{\vec{x}}^{\vec{y}} A_\mu dx^\mu) = \prod U_\mu$
- path-ordered product of link variables, on-/off-axis
- plaquette, Wilson loop $W(R,T)$, static $Q\bar{Q}$ pair

# Motivation

- calculate the static potential energy with high resolution
  - matching the lattice QCD potential with the perturbative potential to determine $\Lambda_{\overline{MS}}$ in Fourier space, *e.g.*, Karbstein et al. (2014)
  - observation of string breaking in QCD, *e.g.*, Bali et al. (2008), Bulava et al. (2019)

$\Rightarrow$ we have to work with off-axis separated quarks

- the spatial part of the Wilson loop has to go over stair-like paths through the lattice $\rightarrow$ not unique, computationally expensive

$\Rightarrow$ alternative operator which ensures gauge covariance of the quark-anti-quark $Q(\vec{x})\bar{Q}(\vec{y})$ trial state

- required gauge transformation behavior:

$$U_s'(\vec{x},\vec{y}) = G(\vec{x})U_s(\vec{x},\vec{y})G^\dagger(\vec{y})$$

# Laplacian Eigenmodes

- Consider the 3D covariant lattice Laplace operator:

$$\begin{aligned}\Delta V \quad = \quad & \frac{1}{a^2}[U_x^\dagger(x-a,y,z)V(x-a,y,z) - 2V(\vec{x}) + U_x(\vec{x})V(x+a,y,z) \\ & + U_y^\dagger(x,y-a,z)V(x,y-a,z) - 2V(\vec{x}) + U_y(\vec{x})V(x,y+a,z) \\ & + U_z^\dagger(x,y,z-a)V(x,y,z-a) - 2V(\vec{x}) + U_z(\vec{x})V(x,y,z+a)]\end{aligned}$$

- transformation behavior: $\Delta' = G(\vec{x})\Delta G^\dagger(\vec{y})$
- consider $V(\vec{x})$ an eigenvector: $\Delta V(\vec{x}) = \lambda V(\vec{x})$

$$\begin{aligned}\Delta' V'(\vec{x}) &= \lambda V'(\vec{x}) \\ G(\vec{x})\Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda V'(\vec{x}) \\ \Delta G^\dagger(\vec{x})V'(\vec{x}) &= \lambda G^\dagger(\vec{x})V'(\vec{x})\end{aligned}$$

- $V(\vec{x})$ and $G^\dagger(\vec{x})V'(\vec{x})$ are members of the same eigen-space

# Static $Q\bar{Q}$ pair, Wilson loop alternative...

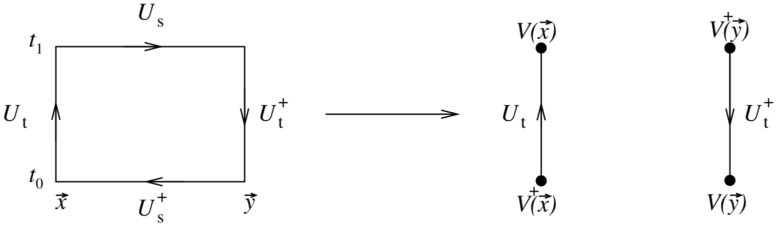
- idea taken from Neitzel et al. (2016) SU(2)
- SU(3): the eigenvalues are in general non-degenerate
- spatial Wilson line: $U'_s(\vec{x},\vec{y}) = G(\vec{x})U_s(\vec{x},\vec{y})G^\dagger(\vec{y})$

$$V'(\vec{x})V'^\dagger(\vec{y}) = G(\vec{x})V(\vec{x})V^\dagger(\vec{y})G^\dagger(\vec{y})$$

- Wilson loop of size $(R = |\vec{x}-\vec{y}|) \times (T = |t_1 - t_0|)$

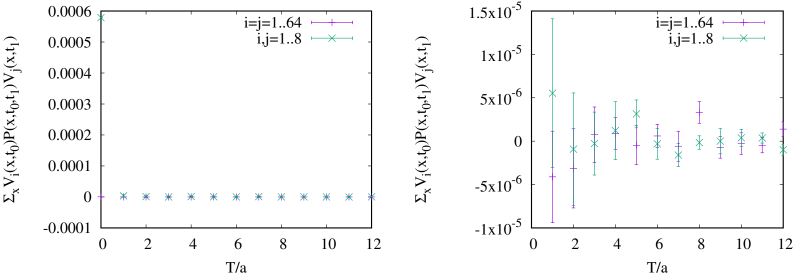
$$W(R,T) = \langle \mathrm{tr}\,[U_t(\vec{x};t_0,t_1)U_s(\vec{x},\vec{y};t_1)U_t^\dagger(\vec{y};t_0,t_1)U_s^\dagger(\vec{x},\vec{y};t_0)]\rangle$$

$$\rightarrow \langle \mathrm{tr}\,[U_t(\vec{x};t_0,t_1)V_j(\vec{x},t_1)V_j^\dagger(\vec{y},t_1)U_t^\dagger(\vec{y};t_0,t_1)V_i(\vec{y},t_0)V_i^\dagger(\vec{x},t_0)]\rangle$$



# Static quark line, SVD, GEVP

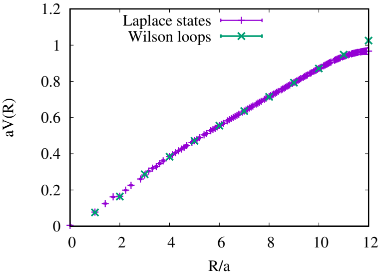
$$Q(T) = \sum_{i,j}^{N_v} \sum_{\vec{x},t_0} \langle V_i^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_1) V_j(\vec{x},t_1) \rangle$$

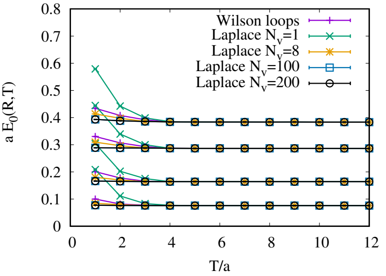



- SVD: $W = UDV^\dagger$; $U$, $V$ unitary, columns are orthonormal bases
- GEVP: $W(t)v_k = \rho_k W(t_0) v_k$, $\rho_k$ give effective energies
- $u_i$ or $v_k$ can be used to get 'optimal' profiles for energy states

# Simulations

- $24^3 \times 48$, $\beta = 5.3$, $N_f = 2$, $\kappa = 0.13270$, $a = 0.0658$ fm
- (on-axis) Wilson loops from 4646 measurements, 4D HYP
- Laplace states from 1161 measurements, 20 APE (0.5) for Lanczos, 1 4D HYP for temporal Wilson lines
- static potential $aV(R) = \lim_{T\to\infty} \log[W(R,T)/W(R,T+1)]$
- at $R = 12a$ the force between $Q\bar{Q}$ must vanish due to symmetry
- more eigenvectors gives earlier plateau and increase precision



# Gaussian profiles

$$W_{kl}(R,T) = \sum_{i,j}^{N_v} N_{kl}(\lambda_i,\lambda_j) \sum_{\vec{x},t_0} \left\langle V_i^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_1) V_j(\vec{x},t_1) V_j^\dagger(\vec{y},t_1) U_t^\dagger(\vec{y};t_0,t_1) V_i(\vec{y},t_0) \right\rangle$$

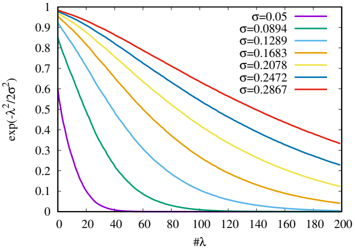
Gaussian profile functions:

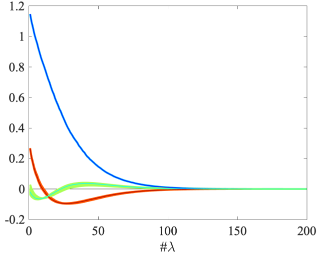
$N_{kl}(\lambda_i,\lambda_j) = \exp(-\lambda_i^2/2\sigma_k^2)\exp(-\lambda_j^2/2\sigma_l^2)$ using 7 different $\sigma$ values

$\Rightarrow$ prune $W_{kl}$ using 3 most significant singular vectors $u_i$ at $t_0$

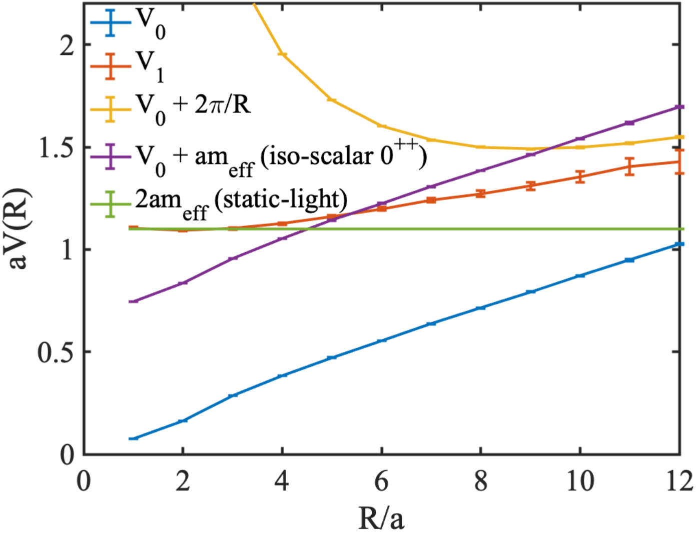
$\ldots$improves stability and keeps useful operators

$\Rightarrow$ pruned ("optimal") profiles $u_{i,j}\exp(-\lambda^2/\sigma_j^2)$



# Improved static potential

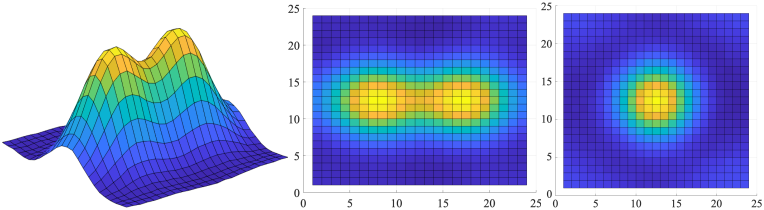


# Flux tube profiles

- $\square(\vec{x},\vec{y}) = V(\vec{x})V^\dagger(\vec{y})$ is idempotent, *i.e.*, $\square^2 = \square$, or
  $\sum_{\vec{z}} \square(\vec{x},\vec{z})\square(\vec{z},\vec{y}) = \square(\vec{x},\vec{y}) \quad \left( \sum_{\vec{z}} V^\dagger(\vec{z})V(\vec{z}) = \mathbb{1} \right)$
- place a 'test-charge' in the quark-antiquark pair source operator $V(\vec{x})V^\dagger(\vec{x}+R)$, which is not given by a trivial plaquette, but another eigenvector pair $V^\dagger(\vec{z})V(\vec{z})$

$$\psi_{ij}(\vec{z},R) = \sum_{\vec{x},t} ||V_i(\vec{x},t)V_i^\dagger(\vec{z},t)V_j(\vec{z},t)V_j^\dagger(\vec{x}+R,t)||_2$$

where instead of summing over $\vec{z}$, which would give the identity, the free coordinate scans the 3D time-slice at $t$.



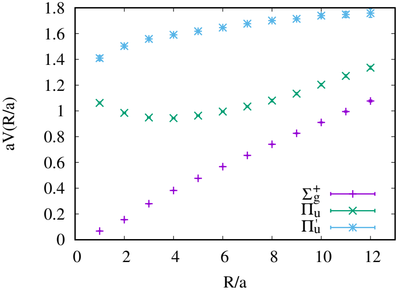
# Static-hybrid states, e.g., $\Pi_u$

gluonic excitations are realized via covariant derivatives

$$\nabla_{\vec{k}} V(\vec{x}) = \frac{1}{2}[U_k(\vec{x})V(\vec{x}+\hat{k}) - U_k^\dagger(\vec{x}-\hat{k})V(\vec{x}-\hat{k})]$$

*e.g.*, we could use the operator for $\Pi_u(R,T) = \sum_{\vec{x},t_0,\vec{k}\perp\vec{y}-\vec{x}}$

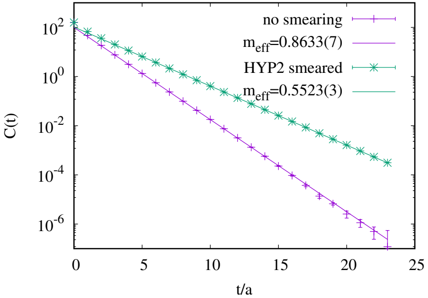
$$\Big\langle \mathrm{tr}\,\Big[U_t(\vec{x};t_0,t_1)\{\nabla_{\vec{k}} V_j(\vec{x},t_1)V_j^\dagger(\vec{y},t_1) + V_j(\vec{x},t_1)[\nabla_{\vec{k}} V_j]^\dagger(\vec{y},t_1)\}$$
$$U_t^\dagger(\vec{y};t_0,t_1)\{\nabla_{\vec{k}} V_i(\vec{y},t_0)V_i^\dagger(\vec{x},t_0) + V_i(\vec{y},t_0)[\nabla_{\vec{k}} V_i]^\dagger(\vec{x},t_0)\}\Big]\Big\rangle,$$



# Static-light (charm) meson

$$C(t) = -\sum_{t_0,i,j} \left\langle \mathrm{tr}_d\{[V_i^\dagger D^{-1} V_j](t_0+t,t_0)P_+\} \right.$$
$$\left. \sum_{\vec{x}} V_j^\dagger(\vec{x},t_0) U_t(\vec{x};t_0,t_0+t) V_i(\vec{x},t_0+t) \right\rangle$$

- with light (charm) perambulators $V^\dagger D^{-1} V$
- HYP smearing of temporal links in $U_t$ removes free energy



# Conclusions & Outlook

- ✓ alternative operator for a static quark-anti-quark pair based on Laplacian eigenmodes, replacing traditional Wilson loops
- ✓ improved version (several eigenvectors weighted with Gaussian profiles) gives earlier plateau and better signal
- ✓ much higher resolution of the potential energy as off-axis distances basically come "for free"
- ✓ measurements of flux tube profiles via $VV^\dagger$ 'test-charge'
- ✓ implementation of static-light (charm) correlator using "perambulators" $V(t_1)D^{-1}V(t_2)$ from distillation framework
- 🔧 putting together building blocks for observation of string breaking in QCD (mixing matrix of static and light quark propagators)
- 🔧 also working on hybrid static potentials (hybrid meson masses), instead of "gluonic handles" (excitations) use derivatives of $V$

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