Exotic tetraquarks with some heavy quarks in lattice QCD

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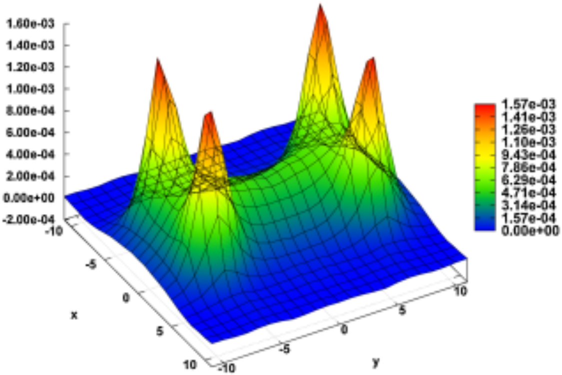
Abstract

We review [64] all the different direct and indirect approaches that lattice QCD has been employing to study multiquarks, focusing in the tetraquarks with heavy quarks.

In the new millennium, the interest in tetraquarks exploded with several experimental discoveries of tetraquark resonances with heavy quarks, starting with the Z_c and Z_b and contibuing with many more tetraquarks such as the the T_{cc}.

Lattice QCD predicted the T class of tetraquark, but so far it has not yet been able to comprehend the Z class of tetraquarkss.



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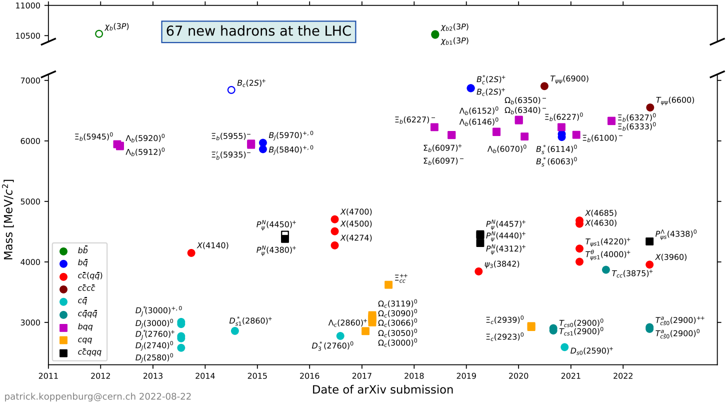

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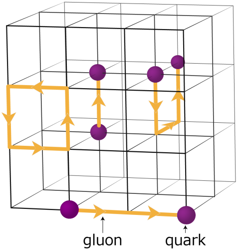
Figure: New hadrons discovered at the LHC, most of them at LHCb, including baryons, tetraquarks and pentaquarks, plotted as mass versus preprint submission date [49].

Tetraquarks were first proposed by Jaffe within the bag model in 1977 [46, 47]. In the early eighties, Richard [3, 10, 80, 36] and colleagues proposed tetraquarks with some heavy quarks would most likely form boundstates. After many decades, finally multiquark resonances have been confirmed in more than forty different experiments and lattice QCD computations, Fig. 1 shows the discoveries at LHC only [35, 49].



Concerning lattice QCD, we all know it is a discretization of QCD, needing

- a large ensemble of configurations,
- a large volume V,
- a small lattice spacing *a*
- and a physical pion mass m_π

to approach nature.

Notice it uses the Euclidean time evolution of correlations to compute the energies of observables. For instance, denoting $|O_1(t)\rangle$ the physical quantum state corresponding to the operator $O_1(t)$ at time, it can be decomposed in eigenvectors of the hamiltonian,

$$|O_1(0)\rangle = \sum_i c_{1\,i} |v_i\rangle \quad \Rightarrow \quad |O_1(t)\rangle = \sum_i c_{1\,i}\, e^{-\lambda_i t} \,, |v_i\rangle \tag{1}$$

and the matrix element between two operators at different times is

$$\langle O_2(t)|O_1(0)\rangle = \sum_i c_{2\,i}^{*} c_{1\,i}\, e^{-\lambda_i t} \,, \tag{2}$$

clearly for a time long enough the groundstate dominates this matrix element.

We will review the different lattice QCD results on tetraquarks with some heavy quarks, obtained with dynamical quarks for the *u*, *d* , *s*, *c* quarks, non-relativistic quarks for *b* quarks and static quarks as an approximation of *b* quarks.

Static potentials and colour field densities

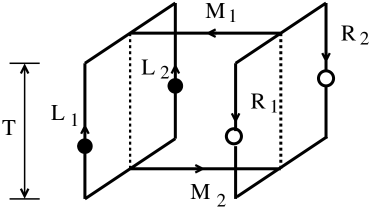


Figure: The tetraquark (4Q) Wilson loop for the calculation of the 4Q potential V_{4Q}, utilized in Ref. [62].

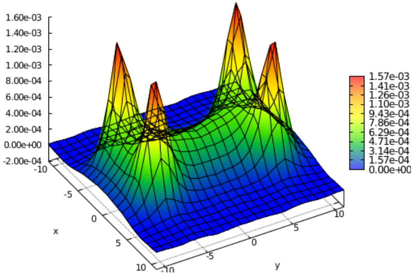


Figure: Lagrangian density 3D plot for a tetraquark, from Ref. [26], showing a clear tetraquark double-Y flux tube.

The first computations for tetraquarks in lattice QCD used static quarks only, less expensive than dynamical quarks, and can be computed in pure gauge QCD [63, 62] and Alexandrou et al. [5].

The authors find that the potential V_{4Q} is described by the OGE Coulomb plus a four-body potential V_{c4Q}, with a double-Y extension of the three-body linear potential,

$$V_{4Q} = -\alpha_{4Q}\{(\frac{1}{r_{12}}+\frac{1}{r_{34}})+\frac{1}{2}(\frac{1}{r_{13}}+\frac{1}{r_{14}}+\frac{1}{r_{23}}+\frac{1}{r_{24}})\}+\sigma_{4Q}L_{\min}+C_{4Q} \qquad (3)$$

The colour electric and magnetic square field densities were studied with four static quarks, by Cardoso et al. [26, 27]. Fig. 3 shows a double-y with two Steiner junctions.

String flip-flop potentials with static quarks

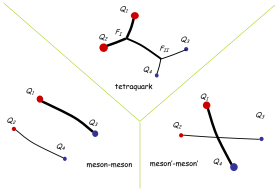


Figure: Triple string flip-flop potential [11].

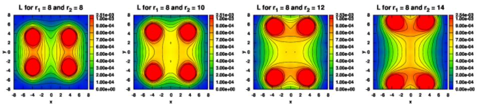


Figure: Evidence of flip-flop of the flux tube in the colour field densities for the $QQ\bar{Q}\bar{Q}$ system [25].

Depending on the geometry of our system, we can have three different groundstate colour singlets two different $1\,1$ and one $\bar{3}\,3$. Moreover, the orthogonal colour singlet to the $\bar{3}\,3$ is the $6\,\bar{6}$ and the orthogonal colour singlet to the $1\,1$ is the $8\,8$.

In lattice QCD, the dependence of the potential on the positions of four quark systems has been studied. The string flip-flop was already observed in the first study of tetraquark potentials [62]. Then Cardoso et al [25] confirmed this flip-flop studying the flip-flop of the colour field densities, see Fig. 5. Recently the computation, not only of the groundstate potential, but also of the first excited potential was performed [12]. Indeed, the flip-flop was observed both in the groundstate and in the excited potential. The mixing of both potentials in the transition region was also observed.

However, the spin-dependence of the flip-flop potential has not yet been computed, this leads to more bound state tetraquarks of the type T_{bb} than the ones obtained with lattice QCD [11].

Potentials with static heavy quarks for the T_{QQ} family

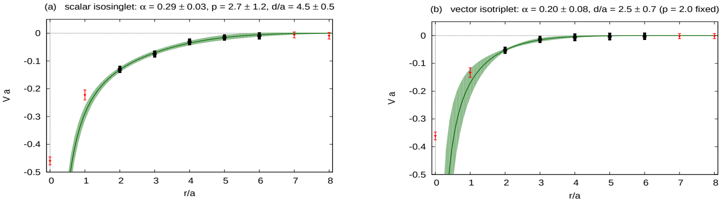


Figure: The lattice QCD potentials fitted in Ref. [78, 79, 18], left scalar-isoscalar and right vector-isovector.

The first lattice computations for T_{bb} and related states, proposed by Richard many years ago, utilized static potentials computed with static heavy quarks and dynamical light quarks, coupled to two heavy-light mesons, first with quenched light quarks [60, 59, 32, 31] and then for dynamical light quarks [78, 79, 8, 24, 18]. In the operators,

$$(\mathcal{C}\Gamma)_{AB}\Big(\bar{Q}_C(\mathbf{r}_1)\psi_A^{(1)}(\mathbf{r}_1)\Big)\Big(\bar{Q}_C(\mathbf{r}_2)\psi_B^{(2)}(\mathbf{r}_2)\Big), \tag{4}$$

$\bar{Q}$ denotes a static quark operator, ψ a light antiquark operator, A, B and C are spin indices and $\mathcal{C} = \gamma_0\gamma_2$ is the charge conjugation matrix [78, 79]. More recently heavy quark spin effects were partly included [17], the extrapolation to the chiral limit was performed [18, 14, 15, 17, 13, 16] and the light quarks were extended from the u, d flavours to the s and c flavours [14]. Using an educated ansatz for the potential, the dynamics provided to the heavy quarks in the Schrödinger equation with the Born-Oppenheimer approximation [20] led to the prediction of boundstates.

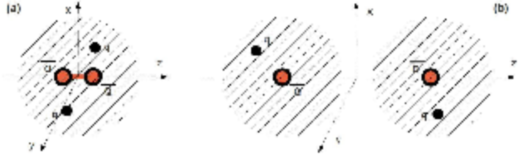


Figure: The picture of perturbative one-gluon exchange at short distances and of meson wavefunction screening at large distances, shown in Ref. [18].

channel	α	d/a	p	χ^2/dof
scalar isosinglet	0.293(33)	4.51(54)	2.74(1.20)	0.35
vector isotriplet	0.201(77)	2.48(69)	2.0 (fixed)	0.06

Table: χ^2 minimizing fit results of the ansatz (5) to the lattice static antiquark-antiquark potential; fitting range $2 \leq r/a \leq 6$; lattice spacing $a \approx 0.079\,\mathrm{fm}$

We expect a Coulomb potential at short distances, and the two heavy antiquarks are in the groundstate if they are in a triplet colour state 3, s-wave and thus spin 1. At large distances we expect a screening typical of the static-light wavefunction, sketched in Fig. 7. Our best ansatz is,

$$V(r) = -\frac{\alpha}{r} \exp\left(-\left(\frac{r}{d}\right)^p\right), \tag{5}$$

A binding energy of -90 ± 43 MeV is predicted. Other tetraquarks such as $ls\bar{b}\bar{b}$, $lc\bar{b}\bar{b}$, $sc\bar{b}\bar{b}$, $ll\bar{c}\bar{b}$, $ll\bar{c}\bar{c}$ do not bind in this approach, but a p-wave resonance with the same flavour was also predicted. Partly including the spin of the heavy antiquarks reduces binding slightly to -60 ± 45 MeV, in the binding channel of a BB^*. Importantly, the quantum numbers predicted for the T_{bb} are $I(J^P) = 0(1^+)$. At small distance r, the heavy quarks have spin 1, and the light quarks are in a scalar-isoscalar state.

Diquark-antidiquark and meson-meson in a bounstate

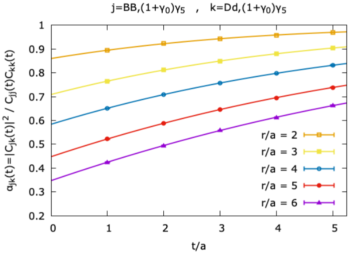


Figure: The squared overlap α_{jk} for for several fixed r distances as a function of t for ensemble B40.24 of Ref. [16]. The normalized trial states are $j = BB, (1+\gamma_0)\gamma_5$, $k = Dd, (1+\gamma_0)\gamma_5$.

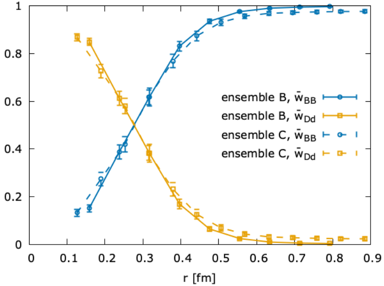


Figure: Comparing the *real* tetraquark Dd versus the *molecular* BB in Ref. [16]. Fitted normalized absolute squares of the coefficients $\bar{w}_{BB}$ and $\bar{w}_{Dd}$ as functions of the distance r.

Continuing Ref. [78, 79, 14, 15], Ref. [16] compared two frequently discussed competing structures for the exotic T_{bb}, a $\bar{b}\bar{b}ud$ system with quantum numbers $I(J^P) = 0(1^+)$a tetraquark: the *real* tetraquark Dd versus the *molecular* BB, where the two $\bar{b}$ quarks are assumed to be infinitely heavy at frozen distance r.

By minimizing effective energies and by solving a generalized eigenvalue problem (GEVP) [19], they determined the importance of the meson-meson and the diquark-antidiquark creation operators with respect to the ground state. The estimated meson-meson to diquark-antidiquark ratio of this tetraquark is around 60% to 40%.

Potentials with static heavy quarks $Q\bar{Q}$ for Z_b family

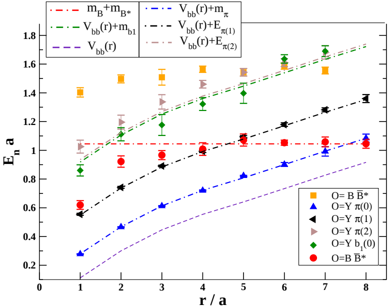


Figure: Eigen-energies of $\bar{b}b\bar{d}u$ system, computed in Ref. [71], for various separations r between static quarks b and $\bar{b}$ are shown by points. The eigenstate dominated by $B\bar{B}^*$ (red circles) has energy significantly below $m_B + m_{B^*}$ and shows sizable attraction. Lattice spacing is $a \simeq 0.124$ fm.

The approach of computing, in a first step, potentials in lattice QCD using static quarks for the heavy quarks, has also been applied to the case where the heavy pair is a $Q\bar{Q}$. In the Z_b family of tetraquarks, with flavour $u\bar{d}b\bar{b}$, we have two possible flavours for the meson-meson pairs, either the pair $u\bar{b} = B^{+(*)}$ and $\bar{d}b = \bar{B}^{0(*)}$ or the pair $u\bar{d}$ and $b\bar{b}$, say corresponding to a meson π^+ and a quarkonium meson such as an Υ. Potentials possibly leading to a boundstate in the $B^{+(*)} - \bar{B}^{0(*)}$ channel was found in Refs. [68, 69, 72, 71, 75, 4]. Nevertheless a difficulty resides in identifying the different open coupled channels, because the different channels are not orthogonal.

HAL QCD non-static potentials

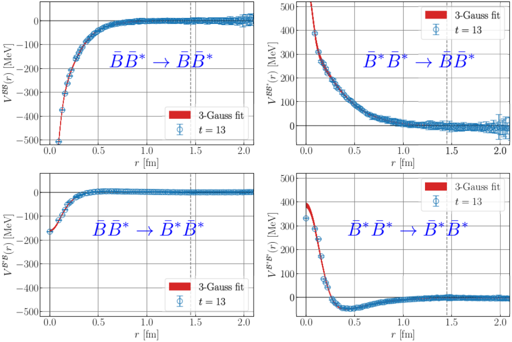


Figure: Coupled channel potentials obtained in Ref [6] with the HAL QCD method using dynamical light quarks and non-relativistic heavy quarks.

The HAL QCD method computes potentials with dynamical quarks [44, 7, 45], extracting them from the wavefunction / Nambu-Bethe-Salpeter amplitude,

$$V(\boldsymbol{r}) = \frac{\Delta\phi(\boldsymbol{r})}{2m\phi(\boldsymbol{r})} + E \ . \tag{6}$$

from operators of two hadrons, each one composed of quarks.

Very recently, in Ref. [6] for the T_{bb} study, a 2×2 time dependent coupled channel was studied with pairs of B and B^* mesons. The potentials are shown in Fig. 11, for the 2×2 coupled channel system of a BB^* pair and a B^*B^* pair . They are in general comparable to the potentials obtained with static quarks in the same 2×2 coupled channel system of Ref. [17], where they are denominated V_j and V_5, with evidence for OGEP attraction in the BB^*. Moreover, observing in detail the HAL QCD potentials, there is a new evidence for some OPEP attraction in the B^*B^* pair.

Search for tetraquark resonances high in the spectrum

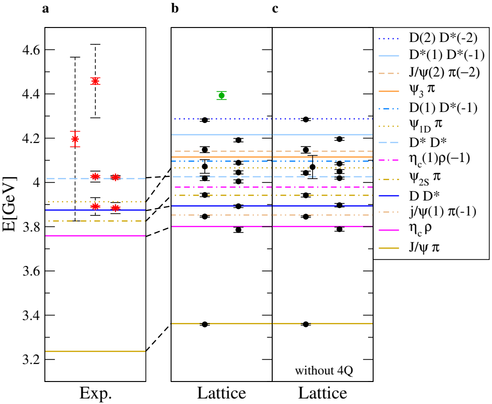


Figure: The spectrum for quantum numbers $I^G(J^{PC}) = 1^+(1^{+-})$ from Ref. [73]. (a) Position of the experimental Z_c^+ candidates [21]. (b,c) The discrete energy spectrum from our lattice simulation: (b) shows energies based on complete 22 × 22 matrix of interpolators, (c) is based on the 18 × 18 correlator matrix without diquark-antidiquark interpolating fields $\mathcal{O}^{4q}_{1-4}$.

After Z_c was discovered in three different experiments, full dynamical lattice QCD calculations were performed [74, 73, 37, 28]. The authors used the technique of comparing the spectrum just with several meson-meson operators, and the spectrum after adding diquark-antidiquark operators. Representative examples are

$$\mathcal{O}_1^{\psi(0)\pi(0)} = \bar{c}\gamma_i c(0)\ \bar{d}\gamma_5 u(0) \qquad (7)$$

for meson meson operators, and for the diquark-antidiquark operators,

$$\mathcal{O}_1^{4q} \propto \epsilon_{abc}\epsilon_{ab'c'}\,(\bar{c}_b C\gamma_5 \bar{d}_c\ c_{b'}\gamma_i C u_{c'} - \bar{c}_b C\gamma_i \bar{d}_c\ c_{b'}\gamma_5 C u_{c'})\ .$$

While this same technique succeeded in identifying the $X(3872)$ state in the spectrum [66], no evidence for the Z_c was found in the spectrum, see for instance Fig. 12.

Boundstate search with non-relativistic bottom quarks

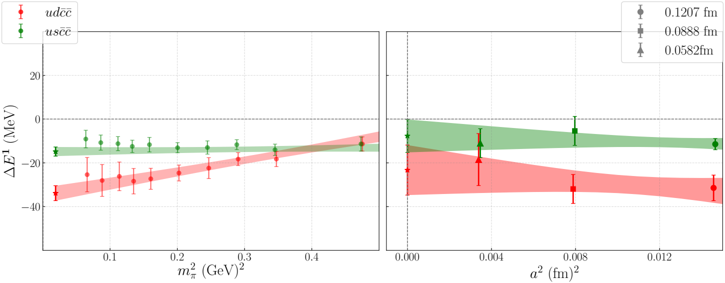


Figure: Results of Ref. [48] for the $ud\bar{c}\bar{c}$ and $us\bar{c}\bar{c}$ doubly charm tetraquark states, colour coded. Left panel: chiral extrapolation for several pion masses at $a = 0.1207$ fm for each of the states. Right panel: Continuum extrapolation at the chiral extrapolation to the physical pion mass.

With no static quarks, the T_{bb} fully exotic tetraquark has been studied by Francis et al. [34], Leskovec et al. [51] and Junnarkar et al. [48], using the NRQCD lattice action [76, 50, 58, 9, 23, 30, 52, 53] to calculate bottom quark propagators. All the lattice QCD computations agree that there is a bound T_{bb} tetraquark, and as well in T_{bbs}. There is still some tension in the other quantum numbers of this family of tetraquarks; in general Ref. [48] tends to find binding in T_{cc}, T_{ccs} and T_{bbc} where the other groups don't. The later group employed the chiral extrapolation to the physical m_π, and the continuum limit to vanishing a , only the thermodynamic limit with large volumes remain to be checked. This is illustrated, as in Fig. 13 for the tetraquarks T_{cc} and T_{ccs}. Also notice Hughes et al found a null evidence for a full bottom tetraquark boundstate in Ref. [43].

Scattering study with the Lüscher method for phase shifts

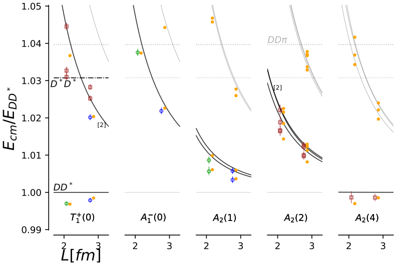


Figure: The center-of-momentum energy $E_{cm} = (E^2 - \vec{P}^2)^{1/2}$ of the $cc\bar{u}\bar{d}$ system [67] in various lattices. The lattice energies are shown by large circles and squares. The free DD^* energies are shown by lines.

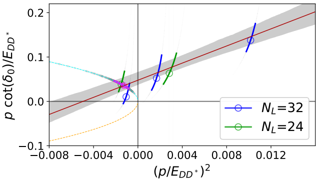


Figure: $p\cot\delta_{l=0}^{(J=1)}$ for DD^* scattering (red line) and $ip = +|p|$ (cyan line) versus p^2. The virtual bound state occurs [67] at the momenta indicated by the magenta octagon, where two curves intersect.

Lüscher et al and others developed a technique to compute phase shifts from the momenta and energy of open channels [56, 57, 56, 55, 54, 22, 65]. For instance in a single channel with two particles, using the effective range approximation [29, 61],

$$k^{2l+1}\cot\delta_l(k) = \frac{-1}{a_l} + \frac{r_l}{2}k^2 + o(k^4) \tag{8}$$

with scattering length a_l and effective range r_l, a pole is equivalent to $\cot\delta_l + i = 0$. The scattering of a pair of pairs of B and B^* mesons was also studied [6, 77], and the T_{bb} boundstate was obtained. However, very recently the T_{cc} was studied in the channel DD^* by Padmanath and Prelovsek [67]. However only a virtual boundstate pole was found, with $\text{Im}(k) < 0$, whereas for a boundstate we should have $\text{Im}(k) > 0$.

Outlook for tetraquarks with lattice QCD

state	qnumber	δmass	width	decay mode	significance	experiment
Tcc(3874)	$ud\bar{c}\bar{c}$?	-360 ± 44 KeV	48 ± 16 KeV	virtual $D^0 D^{*+}$	$15.5 \pm 6.5\sigma$	LHCb [1, 2]
	$ud\bar{c}\bar{c}$ 1^+	-23 ± 11 MeV	0	-	-	dynamical lattice QCD [48]
		vBS ~ -9 MeV	0	-	-	dynamical lattice QCD [67]
Tccs	$us\bar{c}\bar{c}$ 1^+	-8 ± 8 MeV	0	-	-	dynamical lattice QCD [48]
Tbc	$us\bar{b}\bar{c}$ 1^+, 0^+	$\sim -40 \pm 50$ MeV	0	-	-	heavy quark lattice QCD [77]
Tbb	$ud\bar{b}\bar{b}$ 1^+	-90 ± 43 MeV	0	-	-	static lattice QCD [18, 14, 15, 13, 16]
		-59 ± 38 MeV	0	-	-	2×2 static lattice QCD [17]
		-189 ± 13 MeV	0	-	-	heavy quark lattice QCD [33]
		~ -113 MeV	0	-	-	heavy quark lattice QCD [34, 42]
		-143 ± 34 MeV	0	-	-	heavy quark lattice QCD [48]
		-128 ± 34 MeV	0	-	-	heavy quark lattice QCD [51]
		~ -120 MeV	0	-	-	heavy quark lattice QCD [41]
		-154.8 ± 37.2 MeV	0	-	-	scattering lattice QCD [6]
		-83.0 ± 30.2 MeV	0	-	-	scattering lattice QCD [6]
	$ud\bar{b}\bar{b}$ 0^+	-50.0 ± 5.1 MeV	0	-	-	static lattice QCD [24]
		-5 ± 18 MeV	0	-	-	heavy quark lattice QCD [48]
Tbbs	$us\bar{b}\bar{b}$, $bs\bar{b}\bar{b}$ 1^+	-98 ± 10 MeV	0	-	-	heavy quark lattice QCD [33]
		~ -36 MeV	0	-	-	heavy quark lattice QCD [34, 42]
		-87 ± 32 MeV	0	-	-	heavy quark lattice QCD [48]
		~ -80 MeV	0	-	-	heavy quark lattice QCD [70]
		-86 ± 32 MeV	0	-	-	heavy quark lattice QCD [77]
Tbbc	$uc\bar{b}\bar{b}$ 1^+	-6 ± 11 MeV	0	-	-	heavy quark lattice QCD [48]
Tbbcs	$sc\bar{b}\bar{b}$ 1^+	-8 ± 3 MeV	0	-	-	heavy quark lattice QCD [48]

Table: The tetraquark boundstates, or very narrow resonances [64]. Experimentally only the T_{cc} has been observed, thus we include the lattice QCD predictions as well, where we detail the approach to address the heavy quarks (static, heavy quark effective theory, or dynamical) [48, 67, 77, 18, 14, 15, 13, 16, 17, 33, 34, 42, 51, 41, 6, 24, 70, 77].

The case where lattice has most success is in the boundstates of the T family, such as the T_{bb} which hass not yet been observed but is presently used as a benchmark for different lattice QCD studies.

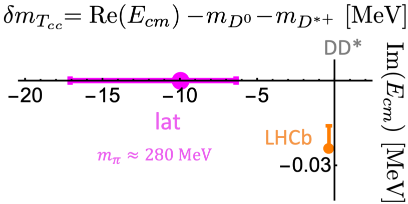


Figure: The pole in the scattering amplitude related to T_{cc} in the complex energy plane: the lattice result [67] (magenta) and the LHCb result (orange).

The T_{cc} recently found at LHCb is an excellent case study for future lattice QCD efforts. The extension of the Lüscher technique to several channels and to three particle resonances is under development by Hansen et al, [38, 39, 40], this should enable the study of the T_{cc}. Another new development is the Master field approach, where a very large lattice is used. The statistical average over several configurations is no longer necessary. In this case the One Pion Exchange Potential should be able to use its full extent, and this is important for deuteron-like molecules. T_{cc} is expected to have both a Dd-like attraction at short distances and a π exchange at large distances.

However, for the $Z_b,\ Z_c \ldots$ family, high in the spectrum, with many decay coupled channels, other new technical advances are still necessary to study them with lattice QCD.
We expect Tetraquarks will become a priority for the lattice QCD community, with increased precision, new techniques, and computations to directly address theoretical questions on the properties of tetraquarks.

MERCI !!!

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