International School on Theory & Analysis in Particle Physics

Basic Concepts: Relativistic notation, kinematics, and wave equations

Contents: Special Relativity, Lorentz transformations, four-vectors, natural units, collisions, Klein-Gordon equation, Dirac equation

Textbook: Introduction to Elementary Particles, Griffiths

Supplementary textbooks:

Modern Elementary Particle Physics, Kane
Quantum Field Theory, Ryder
Introducing Einstein's Relativity, d'Inverno

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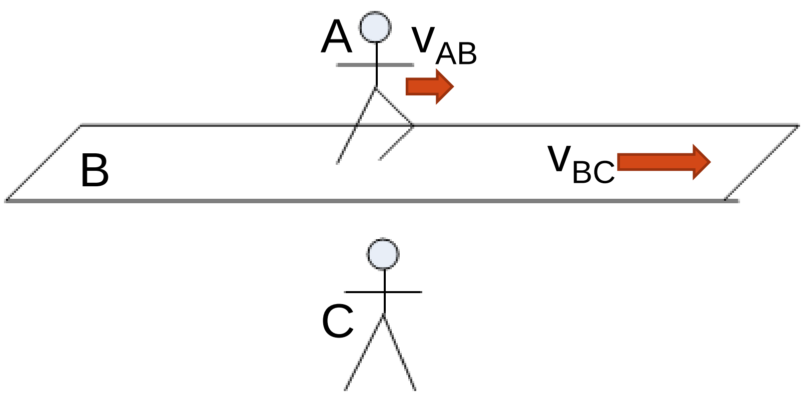
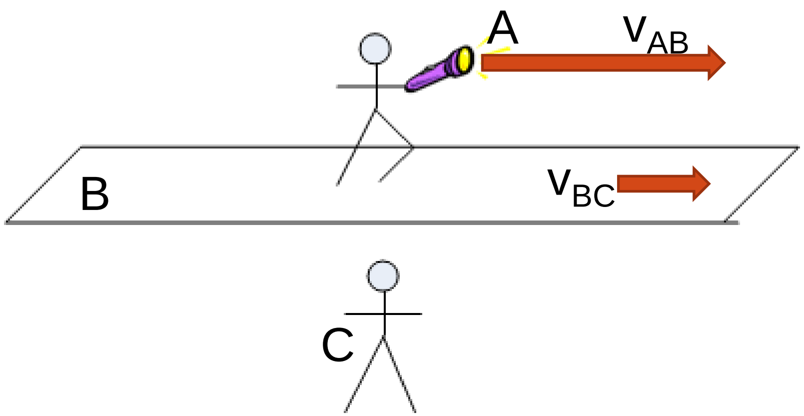
Special Relativity (Einstein 1905)

1. The principle of relativity
   (The same laws apply in all inertial reference frames)

2. The universal speed of light
   (The speed of light in vacuum (c) is the same in all inertial ref. frames)





$$v_{AC} = v_{AB} + v_{BC}$$

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)}$$

$$v_{AC} = v_{AB} = c$$

Consequences of Special Relativity

i. Relativity of simultaneity

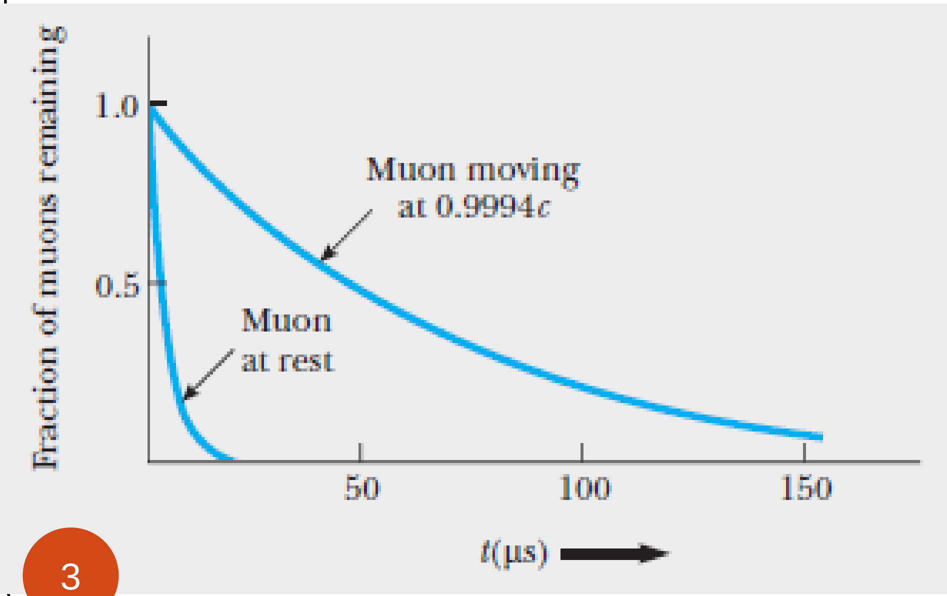
Two events that are simultaneous in S are in general not simultaneous in S´

ii. Lorentz contraction

Moving objects are shortened by a factor $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

iii. Time dilation

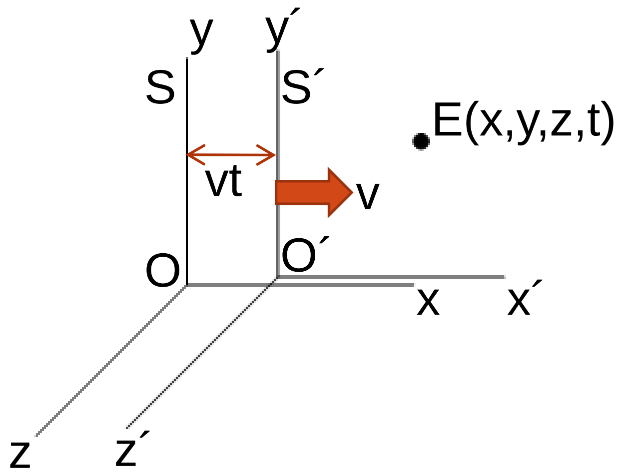
Moving clocks run slow by the same factor γ



μ at rest: $\tau_0 = 2 \times 10^{-6}$ μs

μ moving at v = 0.9994c:

$\tau = \gamma\,\tau_0 = 58 \times 10^{-6}$ μs



Galilean Transformations

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Lorentz Transformations

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

HMW 1: Verify relativity of simultaneity, time dilation, length contraction, and velocity addition rule, using Lorentz transformations.

Position-time four vector: x^μ, μ=0,1,2,3

$x^0=ct$, $x^1=x$, $x^2=y$, $x^3=z$

Lorentz Transformations

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

, where $\beta \equiv \frac{v}{c}$

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \qquad (\mu = 0,1,2,3)$$

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

Lorentz Transformations: $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

HMW 2: Rapidity $\phi = \tanh^{-1} \beta$

a) Show that $\Lambda = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) Show that $\phi_{AC} = \phi_{AB} + \phi_{BC}$

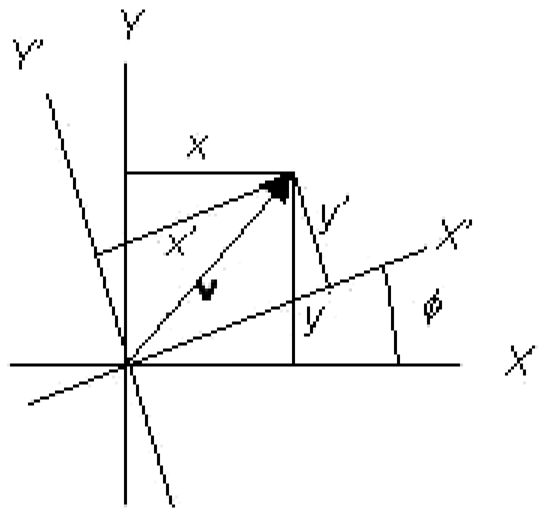
Lorentz Transformation: $\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rotation:

$$x' = x \cdot \cos\phi + y \cdot \sin\phi$$
$$y' = y \cdot \cos\phi - x \cdot \sin\phi$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$r^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

r^2 is invariant under R

$$I \equiv c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

I is invariant under Λ: same in any inertial system

HMW 3: Verify this.

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 = (x'^1)^2 + (x'^2)^2 + (x'^3)^2$$

$$I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$

$I = g_{\mu\nu} x^\mu x^\nu$ where $g_{\mu\nu}$ are the components of the Metric: $\mathrm{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$$x^\mu = (x^0, x^1, x^2, x^3)$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3)$$

→ $I = x^\mu x_\mu$

Given *any two* four-vectors, the scalar product

$a^\mu b_\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$ is invariant.

$$a^\mu = (a^0, a^1, a^2, a^3)$$

$$a_\mu = (a^0, -a^1, -a^2, -a^3)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (\partial_0, \nabla)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = (\partial^0, -\nabla)$$

$$\partial_\mu a^\mu = \partial^\mu a_\mu = \frac{\partial a^0}{\partial t} + \nabla \cdot \vec{a}$$

$$\Box = \partial_\mu \partial^\mu = g_{\mu\nu} \partial^\nu \partial^\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \Delta$$

"Natural" Units for high energy physics: $\hbar = 1, \quad c = 1$

$$\hbar = 6.6 \times 10^{-25} \mathrm{GeV \cdot sec} = 1, \quad c = 3.0 \times 10^{8} \mathrm{m/sec} = 1$$

$$[\mathrm{Energy}] = [\mathrm{Mass}] = [\mathrm{Length}]^{-1} = [\mathrm{Time}]^{-1}$$

HMW 4: a) Show that 200 MeV fm $\cong$ 1, where 1 fm = 10^{-15} m.

b) Show that mass of a proton (1.7×10^{-27} kg) is $\cong$ 1 GeV.

(1 eV = 1.6×10^{-19} J)

Relativistic Energy and Momentum

$E = \gamma m, \quad \vec{p} = \gamma m \vec{v}$ where $\gamma = \dfrac{1}{\sqrt{1 - v^2}}$

for $v \ll 1$, $\gamma = (1 - v^2)^{-1/2} = 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 + \cdots$

$$E = \gamma m = m + \frac{1}{2} m v^2 + \cdots$$

Relativistic Kinetic Energy: $T = E - m = (\gamma - 1) m$

Energy-momentum four vector: $p^\mu = (E, p_x, p_y, p_z)$

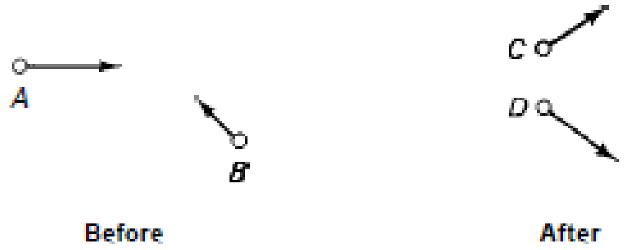
invariant: $p^\mu p_\mu = E^2 - \vec{p} \cdot \vec{p} = \gamma^2 m^2 (1 - v^2) = m^2$ ⇨ $E^2 = |\vec{p}|^2 + m^2$

Massless particle: $p^\mu p_\mu = 0, \; E = |\vec{p}|, \quad v = 1$

(for a massless particle, $E = h\nu$)

Relativistic Collisions

Total energy and momentum is conserved.



$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$$

If collision is elastic, T and therefore m is also conserved ($E = T + m$)

Conserved: Same before and after the collision.

Invariant: Same in any inertial reference frame.

$p_A^\mu + p_B^\mu$ conserved: Yes, invariant: No

$p_A^\mu p_{A\mu}$ conserved: No, invariant: Yes

$(p_A^\mu + p_B^\mu)(p_{A\mu} + p_{B\mu})$ conserved: Yes, invariant: Yes

Ex. A pion at rest decays into muon and a neutrino $(\pi^- \to \mu^- + \bar{\nu}_\mu)$

On the average, how far would the muon travel in vacuum before decaying?

$$d = v\tau = \gamma v \tau_0$$

Energy-momentum conservation: $p_\pi = p_\mu + p_\nu$

Solve for μ and square both sides: $p_\mu = p_\pi - p_\nu$

$$m_\mu^2 = m_\pi^2 - 2p_\pi \cdot p_\nu = m_\pi^2 - 2m_\pi E_\nu$$

$$p_\pi = (m_\pi, 0{,}0{,}0)$$

$E_\nu = |\vec{p}_\nu| = |\vec{p}_\mu|$ since ν is massless and total **p** = 0

$$|\vec{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = \gamma v m_\mu \Rightarrow d = \gamma v \tau_0 = \frac{m_\pi^2 - m_\mu^2}{2m_\pi m_\mu}\tau_0 = 186\ \mathrm{m}$$

$$(m_\pi = 139.6\ \mathrm{MeV}, \quad m_\mu = 105.7\ \mathrm{MeV}, \quad \tau_0 = 2.2\ \mu\mathrm{s})$$

Ex. A proton strikes another proton at rest, creating a proton-antiproton pair.

What is the threshold energy for this reaction? $(p + p \rightarrow p + p + p + \bar{p})$

LAB Frame

before

after

CM Frame

before

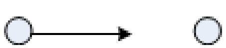
after

before, LAB: $p^{\mu}_{\text{total}} = (E + m, |\vec{p}|, 0,0)$

after, CM: $p'^{\mu}_{\text{total}} = (4m, 0,0,0)$

$p^{\mu}_{\text{total}} p_{\mu\text{total}}$ is both conserved and invariant: $(E + m)^2 - |\vec{p}|^2 = 16m^2$

$$E = 7m$$

Fixed target:  threshold T = 6m

Colliding beams:  threshold T = m

HMW 5: In a two-body scattering event, $A + B \rightarrow C + D$, it is convenient to introduce the ***Mandelstam variables***

$$s \equiv (p_A + p_B)^2/c^2$$
$$t \equiv (p_A - p_C)^2/c^2$$
$$u = (p_A - p_D)^2/c^2$$

The *theoretical* virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system. *Experimentally*, though, the more accessible parameters are energies and scattering angles.

a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$

b) Show that $E_{\mathrm{TOT}}^{\mathrm{CM}} = \sqrt{s}$

Single particle wave equations:

Schrödinger equation for a free particle

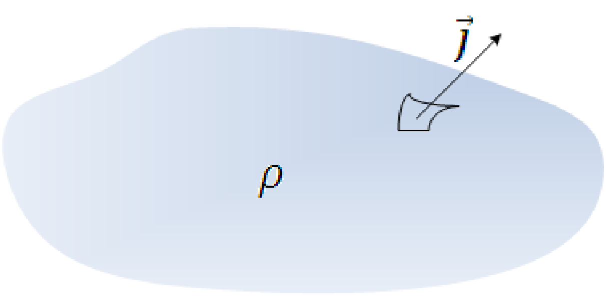
non-relativistic energy-momentum relation: $E = \frac{|\vec{p}|^2}{2m}$

$$\vec{p} \to -i\hbar\nabla, \;\; E \to i\hbar\frac{\partial}{\partial t} \;\Rightarrow\; i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi$$

probability density: $\rho = \phi^*\phi$

probability current: $\vec{j} = -\frac{i\hbar}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*)$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$



$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V + \int_V (\nabla\cdot\vec{j})\mathrm{d}V = 0$$

$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V + \int_S \vec{j}\cdot\mathrm{d}\vec{A} = 0$$

Klein-Gordon equation

relativistic energy-momentum relation: $p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$

$$\vec{p} \to -i\hbar\nabla, \;\; E \to i\hbar\frac{\partial}{\partial t} \;\Rightarrow\; \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{m^2c^2}{\hbar^2}\phi = 0$$

$$p^\mu \to i\partial^\mu \;\Rightarrow\; \boxed{(\partial^\mu\partial_\mu + m^2)\phi = 0} \text{ or } \boxed{(\Box + m^2)\phi = 0}$$

probability density and current: $j^\mu = (\rho, \vec{j}) = \frac{i\hbar}{2m}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = \partial_\mu j^\mu = 0$$

$$\rho = \frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$$: negative probability?

Klein-Gordon equation not a single particle wave equation

it is the equation for a *quantum field* whose quanta are scalar (spin 0) particles:

$|\phi|^2 \propto$ number of particles

Dirac Equation:

- first order
- wave eq. for spin ½ particles

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

ψ: Dirac spinor

$$[-(\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu) - i(\gamma^\mu \partial_\mu) m]\psi = 0,$$

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0.$$

Since $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, $\gamma^\mu \gamma^\nu \Rightarrow \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \equiv \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$

$$\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu \psi + m^2 \psi = 0.$$

$p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$, $p^\mu \to i\partial^\mu$, $(\partial^\mu \partial_\mu + m^2)\psi = 0$

$$(g^{\mu\nu} \partial_\mu \partial_\nu + m^2)\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad (\nu \neq \mu).$$

Dirac Matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad (\nu \neq \mu).$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\gamma^{0\dagger} = \gamma^0, \; \gamma^{i\dagger} = -\gamma^i$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Probability current j^μ for Dirac equation:

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0 \;\Longrightarrow\; \psi^\dagger(-\mathrm{i}\gamma^0\overleftarrow{\partial}_0 + \mathrm{i}\gamma^i\overleftarrow{\partial}_i - m) = 0.$$

multiply by γ^0

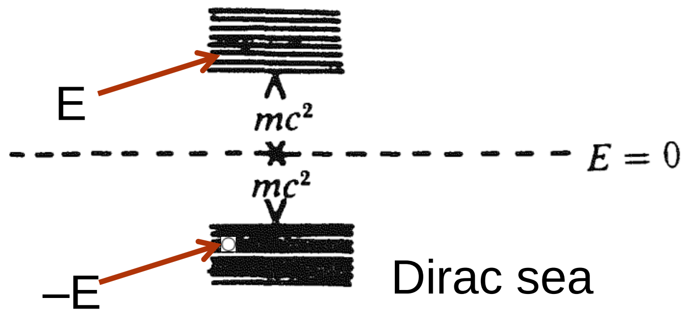
$\bar{\psi}(\mathrm{i}\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$ where $\bar{\psi} = \psi^\dagger\gamma^0$ (adjoint spinor)

$$j^\mu = \bar{\psi}\gamma^\mu\psi \;\Longrightarrow\; \partial_\mu j^\mu = 0$$

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \;\Longrightarrow$$ no negative probabilities..

$$p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2$$

$$E = \mp\sqrt{|\vec{p}|^2 + m^2}$$ ➡ Negative energies?



electron+hole→2E

LH hole with –E in Dirac sea corresponds to a RH **antiparticle** (positron) with E

modern (QFT) interpretation:

no Dirac sea required,

negative energy problem solved upon “second” quantization: [,] → { , }

Dirac’s prediction of antiparticles remains valid

(not as holes but as CPT conjugates)