Google Summer of Code
Symplectic Integrators

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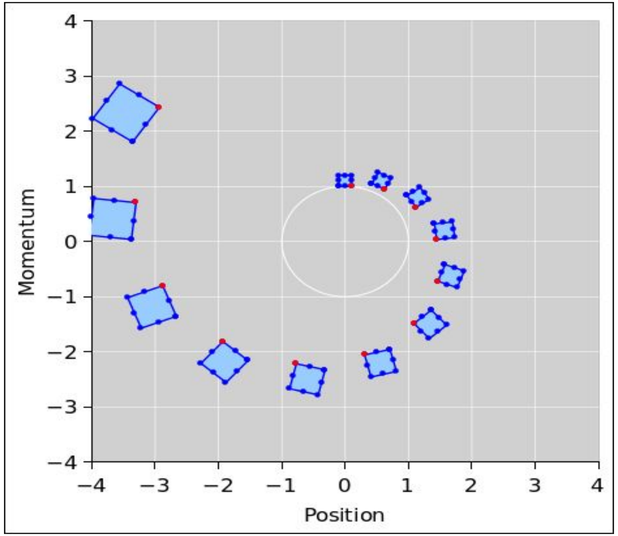
Motivation

- The Muon g-2 experimenters examine the precession of muons that are subjected to a magnetic field.
- To do this it measures the polarization of the muon to seek new physics over thousands of revolutions in its custom accelerator ring.
- To replicate this in Geant4, the toolkit must track muons for 5,000 turns with near-perfect energy conservation and no drift in particle momentum.

Symplectic Integrators

- While following the time evolution of a dynamical system, it is often necessary to use numerical integrations schemes.
- Symplectic integration methods can be superior to general purpose ordinary differential equation solvers.
- These method do not conserve the total energy and the computed energy value drifts from the true value.

- Symplectic methods belong to the larger class of geometric numerical integration algorithms.
- Symplectic methods are so named because, when applied to problems in Hamiltonian mechanics, the algorithms preserve the linear symplectic structure inherent in the phase space representation of the dynamics.
- For time independent Hamiltonian, these preserve the total energy and momentum.



The Boris Method

- We decided that a good starting point will be the boris algorithm.
- Due to explicitness, it is fast and conserves energy to 2nd order.
- It still has implicit properties that are typically need for obtaining accuracy.
- It conserves phase space volume, even though it’s not symplectic!
- The global bound on energy error typically associated with symplectic algorithms still holds for the Boris algorithm, making it an effective algorithm for the multi-scale dynamics of plasmas.

- Given the phase space coordinate (x_k, v_k) at the k-th time-step $t_k = k\Delta t$, the Boris algorithm solves for the phase space coordinate of the particle (x_{k+1}, v_{k+1}) at the (k + 1)-th time-step $t_{k+1} = (k + 1)\Delta t$ from the discretized equation of motion:

$$\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\Delta t} = \mathbf{v}_{k+1/2},$$

$$\frac{\mathbf{v}_{k+1/2} - \mathbf{v}_{k-1/2}}{\Delta t} = \frac{q}{m}\left(\mathbf{E}_k + \frac{\mathbf{v}_{k+1/2} + \mathbf{v}_{k-1/2}}{2} \times \mathbf{B}_k\right),$$

- An explicit form of the above equation can be obtained by separating the electric and magnetic field followed by a rotation.

Implementation

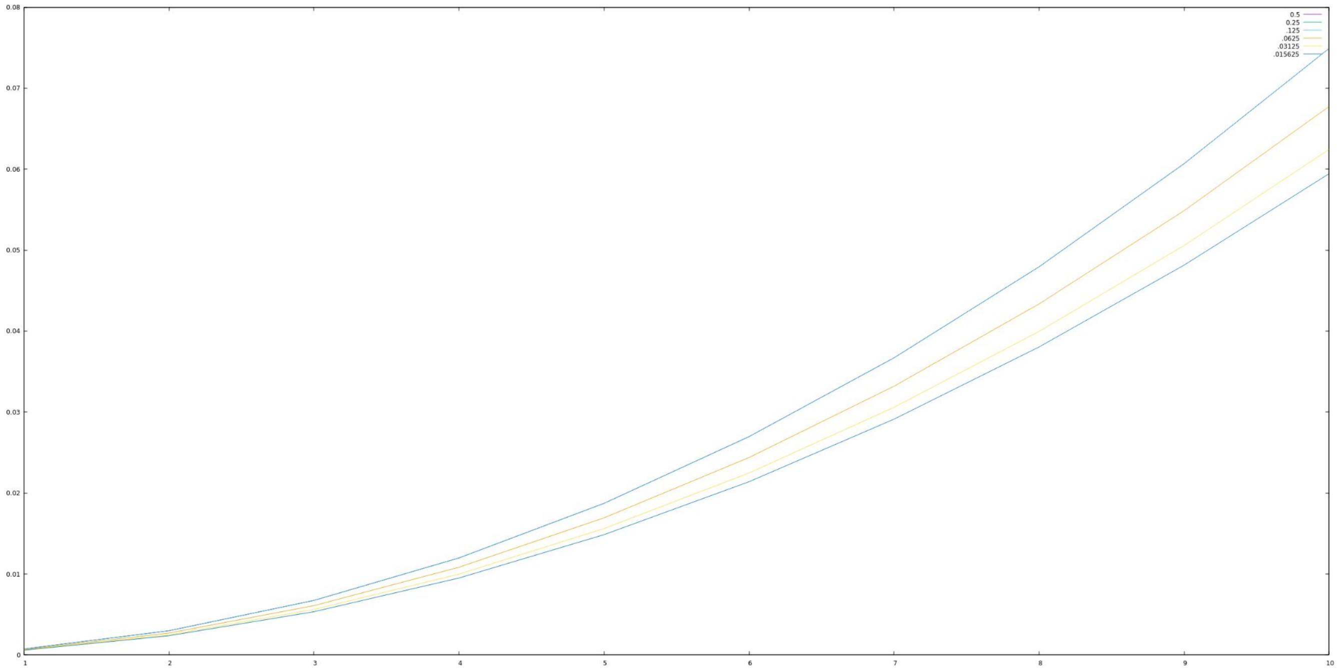
- The implementation of the boris method is split into the driver class `G4BorisDriver`, and the stepper class `G4BorisScheme`.
- The stepper class handles the actual running algorithm.
- If we notice the equation, the position and velocity updates are off by a time step. To tackle this issue, the position update is split into two parts.

Algorithm 3 BorisBuneman$(\mathbf{X}, \mathbf{V}, t, h)$

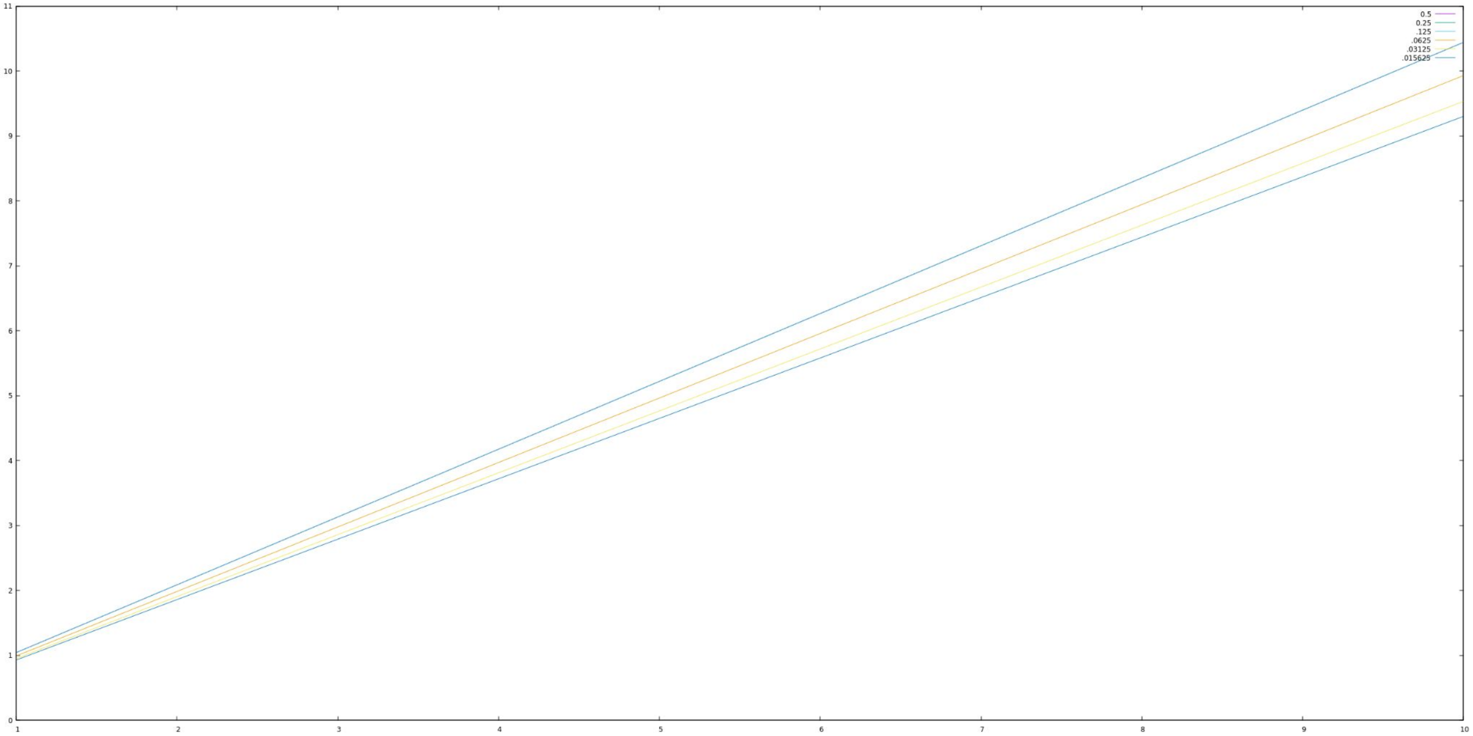
1: $(\mathbf{X}, t) \leftarrow \text{BBPush}(\mathbf{X}, \mathbf{V}, t, h/2)$
2: $\mathbf{E} \leftarrow \text{QueryExternalEField}(\mathbf{X}, t) + \text{CalculateSpaceChargeE}(\mathbf{X}, \mathbf{V})$
3: $\mathbf{B} \leftarrow \text{QueryExternalBField}(\mathbf{X}, t) + \text{CalculateSpaceChargeB}(\mathbf{X}, \mathbf{V})$
4: $\mathbf{V} \leftarrow \text{BBKick}(\mathbf{V}, \mathbf{E}, \mathbf{B}, h)$
5: $(\mathbf{X}, t) \leftarrow \text{BBPush}(\mathbf{X}, \mathbf{V}, t, h/2)$
6: **return** $(\mathbf{X}, \mathbf{V}, t)$

Some Results from the Method

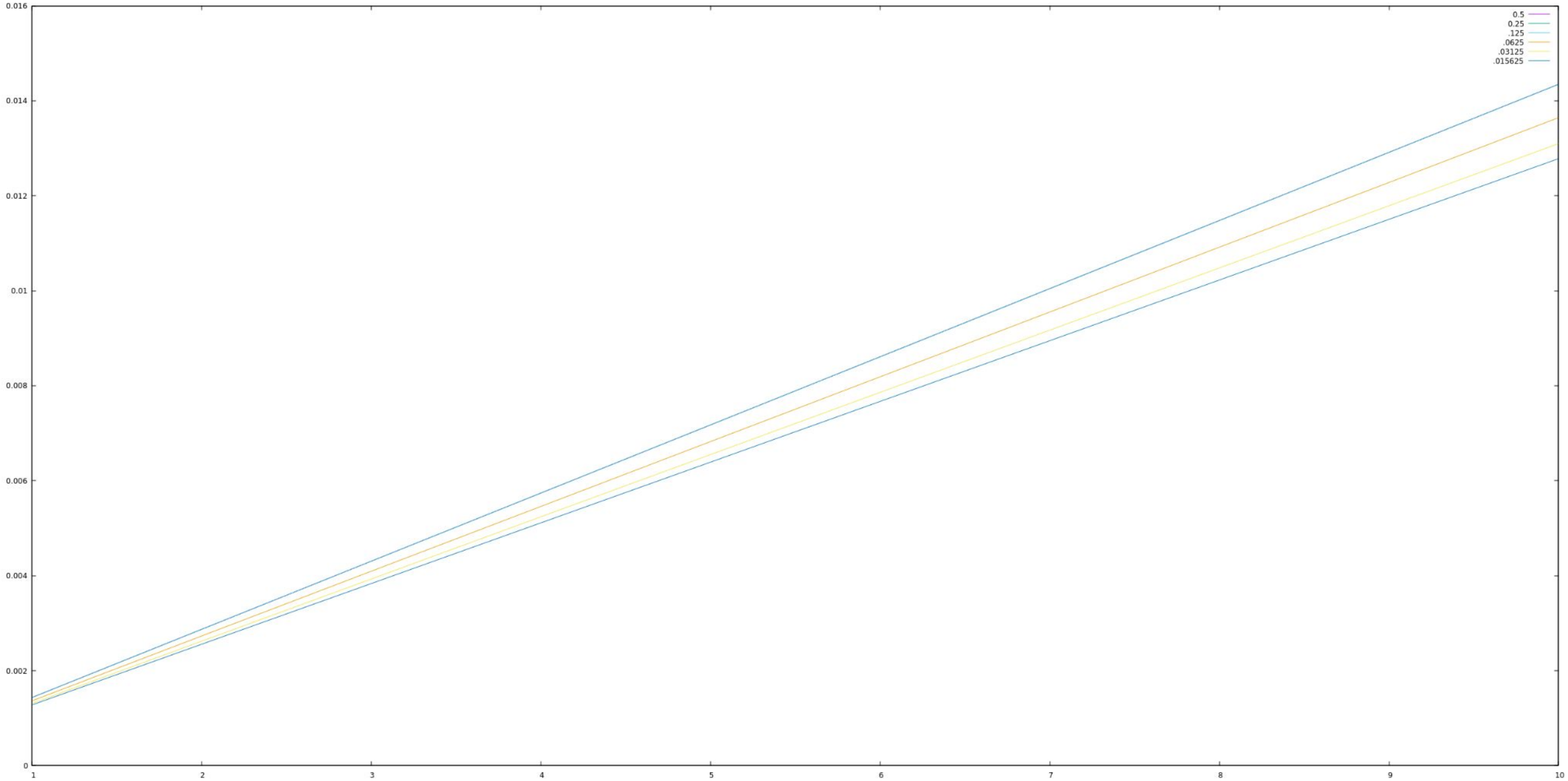
- Deviation in x direction after every turn for different values of step length parameter.



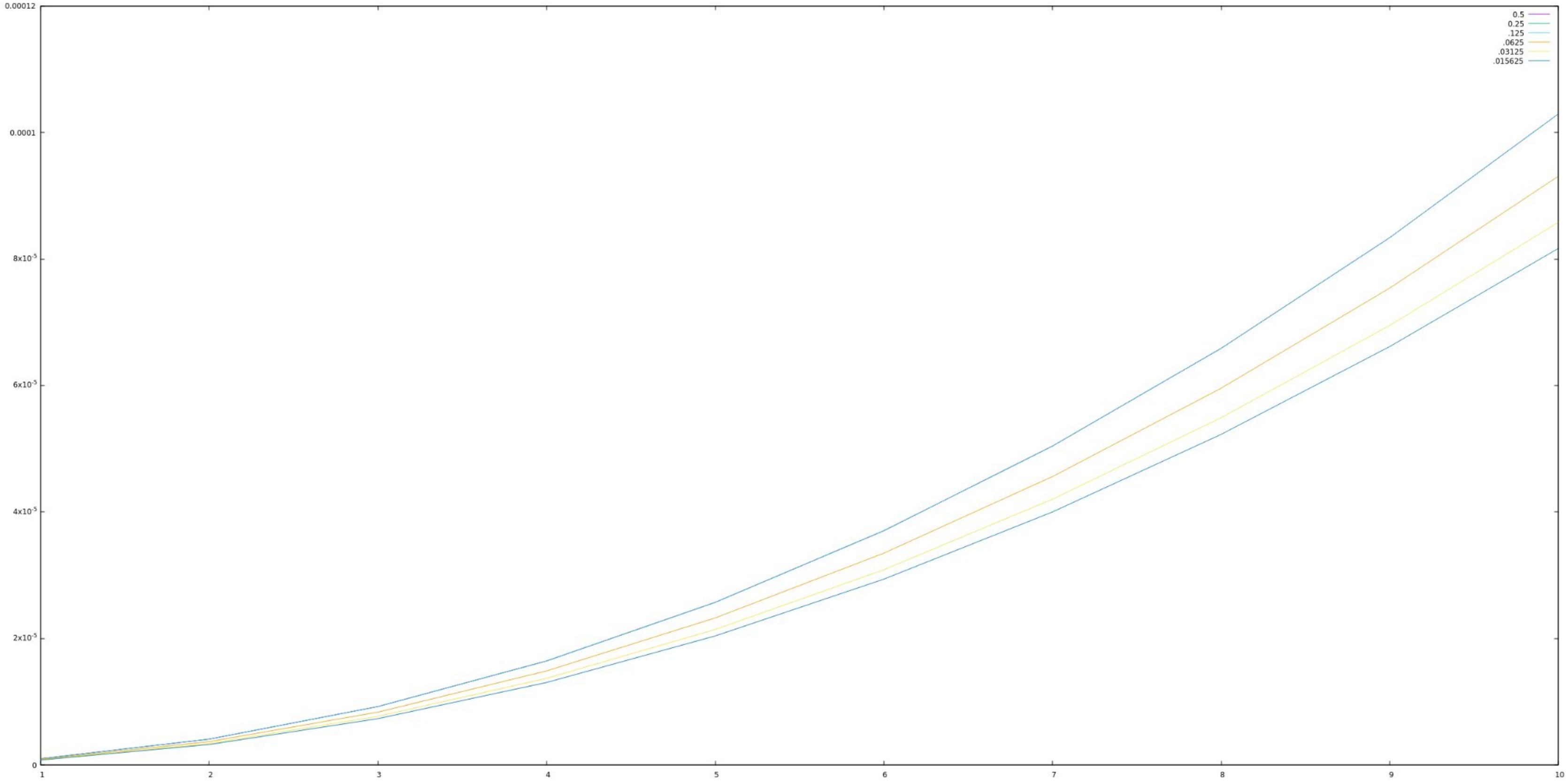
- Deviation in y-direction



- Deviation in X-direction



- Deviation in Y-direction



The Boris SDC

- A straightforward extension of the Boris method is the Boris - Spectral Deferred Corrections (SDC) method.
- Boris-SDC is fundamentally a collocation method solved via spectral deferred corrections, using the same trick as the Boris algorithm to avoid an implicit velocity dependence.
- Essentially, the particular SDC formulation chosen for the Lorentz system results in a node-to-node update step for velocity and position with the same kind of (seemingly) implicit dependence on velocity found in the Boris Algorithm.

Implementation

- I am following a similar structure to the Boris method.
- The method is more expensive, but gives tighter bounds on energy error.
- Currently Work in Progress.