Chern-Simons on the Null Plane

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Abelian Chern-Simons Theory

The abelian Chern-Simons theory is describe by the following Lagrangian density:

$$\mathcal{L}_{CS} = \frac{\kappa}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha}, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian above is invariant by the following transformations,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \quad , \quad \mathcal{L}_{CS} \to \mathcal{L}_{CS} + \partial_\alpha\left(\frac{\kappa}{4e}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}\Omega\right). \tag{2}$$

The field equations are:

$$\frac{\kappa}{2}\varepsilon^{\rho\sigma\alpha}F_{\sigma\alpha} = 0, \tag{3}$$

The null plane time x^+ and longitudinal coordinate x^- are defined, respectively, as

$$x^+ \equiv \frac{x^0 + x^3}{\sqrt{2}} \qquad x^- \equiv \frac{x^0 - x^3}{\sqrt{2}}, \tag{4}$$

with the transverse coordinate kept unchanged. In the null-plane coordinates the metric is given by,

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5}$$

Explicitly, $x^+ = x_-$ and the derivatives with respect to x^+ and x^- are defined as: $\partial_+ \equiv \frac{\partial}{\partial x^+}$ and $\partial_- \equiv \frac{\partial}{\partial x^-}$, with $\partial^+ = \partial_-$. The Levi-Civita tensor has the following components,

$$\varepsilon^{+-1} = \varepsilon_{+-1} = 1. \tag{6}$$

Dirac Formulation

From (1) it is easy to write the first-order Lagrangian by introducing the momentum π^{μ} with respect to the fields A_{μ},

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \left(\partial_{+} A_{\mu}\right)} = \frac{\kappa}{2} \varepsilon^{+\rho\alpha} A_{\alpha}. \tag{7}$$

The theory has the following set of primary constraints:

$$\Omega_{1} \equiv \pi^{+} \approx 0 \qquad , \qquad \Omega_{2,a} \equiv \pi^{a} - \frac{\kappa}{2} \varepsilon_{ab} A_{b} \approx 0, \qquad a = -, 1, \tag{8}$$

where we have defined the following antisymmetric tensor,

$$\varepsilon_{-1} = -\varepsilon_{1-} = 1. \tag{9}$$

The canonical Hamiltonian density is defined by,

$$\mathcal{H}_C = \pi^{\mu}\partial_+ A_\mu - \mathcal{L} = -\kappa A_+ \varepsilon_{ab}\partial_a A_b. \tag{10}$$

The canonical Hamiltonian take the form,

$$H_C = -\kappa \int d^2x A_+ \varepsilon_{ab}\partial_a A_b, \tag{11}$$

where $d^2x \equiv dx^- dx^1$. Thus, the dynamics is determined by the primary Hamiltonian defined by,

$$H_P = H_C + \int d^2y \left[u^1(y)\,\Omega_1(y) + u^{2,a}(y)\,\Omega_{2,a}(y) \right]. \tag{12}$$

The fundamental Poisson brackets of the theory are,

$$\left\{ A_{\mu}\left(x\right), \pi^{\nu}\left(y\right) \right\} = \delta^{\nu}_{\mu}\delta^{2}\left(x-y\right), \tag{13}$$

The consistence conditions on the constraints are,

$$\dot{\Omega}_1\left(x\right) = \left\{ \Omega_1\left(x\right), H_P \right\} = \kappa\varepsilon_{ab}\partial_a A_b \approx 0,$$

Then, a secondary constraint arise defined by,

$$\Omega_4 \equiv \varepsilon_{ab}\partial_a A_b \approx 0. \tag{14}$$

The consistence of Ω_4 determine,

$$\dot{\Omega}_4(x) = \left\{ \Omega_4(x), H_P \right\} = \varepsilon_{ab}\partial_a u^{2,b} \approx 0, \qquad (15)$$

thus, no more constraints are generated from Ω_4.

The consistence condition of $\Omega_{2,a}$ determine,

$$\dot{\Omega}_{2,a}(x) = \left\{ \Omega_{2,a}(x), H_P \right\} = u^{2,a} - \varepsilon_{ab}\partial_b A_+ \approx 0. \qquad (16)$$

This relations is consistente with (18).

The theory is characterized by the following set of constraints,

$$\begin{aligned} \Omega_1 &\equiv \pi^+ \approx 0, \\ \Omega_{2,a} &\equiv \pi^a - \frac{\kappa}{2}\varepsilon_{ab}A_b \approx 0, \\ \Omega_4 &\equiv \varepsilon_{ab}\partial_a A_b \approx 0. \end{aligned} \quad (17)$$

The theory has the following set of first class constraints.

$$\Theta_1 \equiv \pi^+ \approx 0 \qquad , \qquad \Theta_2 \equiv \partial_a \pi^a + \frac{\kappa}{2}\varepsilon_{ab}\partial_a A_b \approx 0. \quad (18)$$

and a set os second class constraints,

$$\Phi_a \equiv \pi^a - \frac{\kappa}{2}\varepsilon_{ab}A_b \approx 0, \qquad a = -, 1, \quad (19)$$

where,

$$\left\{ \Phi_a(x), \Phi_b(y) \right\} = -\kappa\varepsilon_{ab}\delta^2(x-y). \quad (20)$$

We are going to eliminate Φ_a using the following Dirac Brackets between two dynamical variables,

$$\left\{ A_m\left(x\right), B_n\left(y\right)\right\}_{D1} = \left\{ A_m\left(x\right), B_n\left(y\right)\right\} - \int d^2u d^2v \left\{ A_m\left(x\right), \Phi_a\left(u\right)\right\} C_{ab}^{-1}\left(u,v\right)\left\{ \Phi_b\left(v\right), B_n\left(y\right)\right\}, \quad (21)$$

where C_{ab}^{-1} is the inverse of the second class constraint matrix defined by,

$$C_{ab}\left(u,v\right) \equiv \left\{ \Phi_a\left(x\right), \Phi_b\left(y\right)\right\} = -\kappa\varepsilon_{ab}\delta^2\left(x-y\right), \quad (22)$$

where C_{ab}^{-1} is determined by,

$$\int d^2z C_{ac}\left(x,z\right) C_{cb}^{-1}\left(z,y\right) = \delta_{ab}\delta^2\left(x-y\right). \quad (23)$$

It is possible to show,

$$C_{ab}^{-1}\left(x,y\right)=\frac{1}{\kappa}\varepsilon_{ab}\delta^{2}\left(x-y\right), \tag{24}$$

Under this DB, the constraint Φ_a is a strong identity, thus, we can assume,

$$\pi^{a}=\frac{\kappa}{2}\varepsilon_{ab}A_{b}. \tag{25}$$

then A_b are the freedom degree. Thus, it is possible determine:

$$\left\{A_{a}\left(x\right),A_{b}\left(y\right)\right\}_{D1}=\frac{1}{\kappa}\varepsilon_{ab}\delta^{2}\left(x-y\right). \tag{26}$$

The first class constraints can be written,

$$\Theta_1 = \pi^+ \approx 0 \qquad , \qquad \Theta_2 = \kappa\varepsilon_{ab}\partial_a A_b. \tag{27}$$

To eliminate the first class constraints it is necessary to introduce two gauge condition,

$$\Theta_3 = A_+ \approx 0 \qquad , \qquad \Theta_4 = A_- \approx 0, \tag{28}$$

and define a matrix with the following elements,

$$D_{ab}\left(x,y\right) \equiv \left\{ \Theta_a\left(x\right), \Theta_b\left(y\right) \right\}_{D1}. \qquad a = 1,2,3,4 \tag{29}$$

Now, we can define the final set of DB of the theory for two dynamical variables,

$$\begin{aligned}\left\{ A_m(x), B_n(y) \right\}_D &= \left\{ A_m(x), B_n(y) \right\}_{D1} - \int d^2u d^2v \\ &\quad \left\{ A_m(x), \Theta_a(u) \right\}_{D1} D^{-1}_{ab}(u,v) \left\{ \Theta_b(v), B_n(y) \right\}_{D1}.\end{aligned} \quad (30)$$

where

$$D^{-1}(z,y) \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\partial^x_-} \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\partial^x_-} & 0 & 0 \end{pmatrix} \delta^2(x-y). \quad (31)$$

It is possible to show

$$\left\{ A_1(x), A_1(y) \right\}_D = \frac{1}{\kappa} \frac{\partial^x_1}{\partial^x_-} \delta^2(x-y). \quad (32)$$

Conclusions

- We have calculated the complete set of constraints of the theory.
- The constraints have been classified and it was shown that one of the first class constraints result of a combination of constraints.
- The inverse of the second class constraint matrices have been uniquely derived.
- The degrees of freedom have been determined and their corresponding DB calculated.