Anomalies and Generalized Symmetries in QFT

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The goal of this talk is to highlight some recent developments on the study of global symmetries. These ideas are universally applicable and are quickly becoming essential tools to improve our understanding of quantum field theory.

Global Symmetry

Continuous global symmetries $G^{(0)}$ in QFT are encoded by local currents operators $j^{(1)}$ obeying the conservation equation:

[Noether 1918]

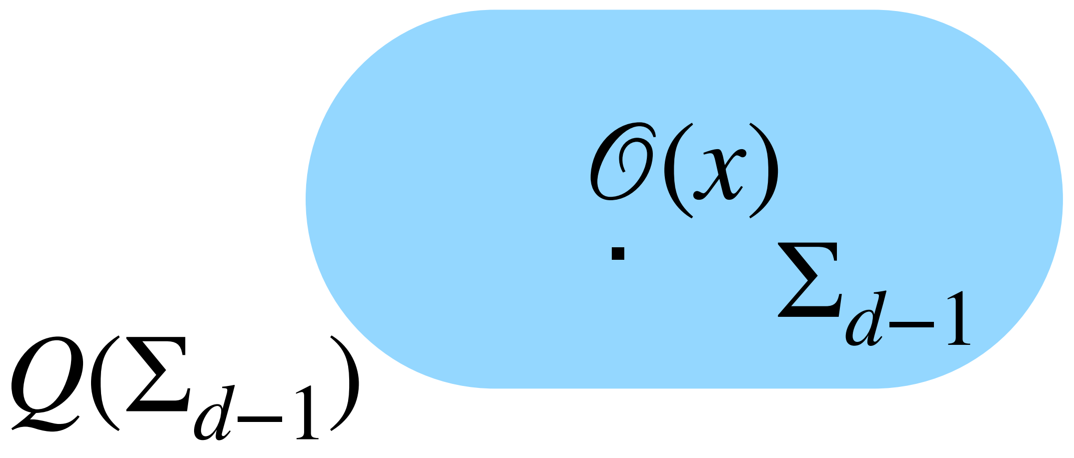
$$d * j^{(1)} = 0$$

Coupling to a symmetry means turning on a background source (gauge field) $A^{(1)}$

$$S[A^{(1)}, \cdots] \supset \int_{M_d} A^{(1)} \wedge *j^{(1)}$$

The global charge Q of a local operator $\mathcal{O}(x)$ is obtained by:

$$Q = \int_{\Sigma_{d-1}} * j^{(1)}$$



$$= q_{\mathcal{O}} \mathcal{O}(x)$$

where $q_{\mathcal{O}}$ is the $G^{(0)}$ charge of the local operator $\mathcal{O}(x)$

Topological Operators

[Gaiotto-Kapustin-Seiberg-Willet 2014]

The modern way to understand continuous or discrete global symmetries is to introduce extended operator (defects) of codimension-1 $U_g(\Sigma_{d-1})$ with $g \in G^{(0)}$.

Key properties:

- Topological: small deformations of $\Sigma \to \Sigma + \delta\Sigma$ do not modify $U_g(\Sigma)$
- Fusion Algebra: $U_g(\Sigma)U_h(\Sigma) = U_{gh}(\Sigma)$
- Linking: $\langle U_g(\Sigma)\mathcal{O}(x)\ldots\rangle = \langle g(\mathcal{O}(x))U_g(\Sigma)\ldots\rangle$

Example: $U(1)$ Global Symmetry

Defect Operator	$G^{(0)}$-symmetry $U_g(\Sigma)$	Example: $U(1)^{(0)}$-symmetry $U_\alpha(\Sigma_{d-1}) = \exp\left(i\alpha \int_{\Sigma_{d-1}} * j^{(1)}\right)$
Codimension	1	$* j^{(1)}$ is a (d-1)-form
Topological	Yes	$d * j^{(1)} = 0$
Fusion Rule	Group Law: $U_g U_h = U_{gh}$	U(1) Multiplication Law $U_\alpha U_\beta = U_{\alpha+\beta}$

Higher-Form Global Symmetries

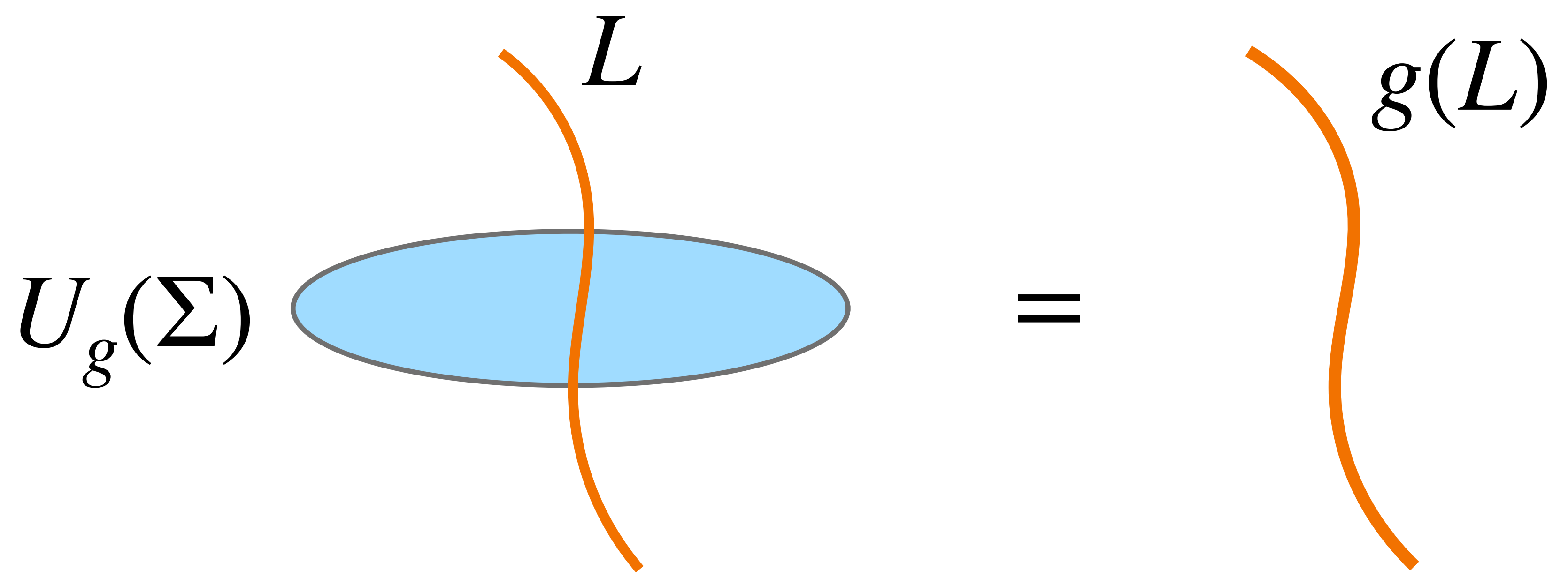
[Gaiotto-Kapustin-Seiberg-Willet 2014]

Let us consider extended operators of larger codimension:

	$G^{(p)}$-symmetry	Example: $U(1)^{(p)}$-symmetry
Defect Operator	$U_g(\Sigma_{d-(p+1)})$	$U_\alpha(\Sigma_{d-(p+1)}) = \exp\left(i\alpha\int_{\Sigma_{d-(p+1)}} * j^{(p+1)}\right)$
Codimension	> 1	$* j^{(p+1)}$ is a (d-p-1)-form
Topological	Yes	$d * j^{(p+1)} = 0$
Fusion Rule	Group Law: $U_g U_h = U_{gh}$	U(1) Multiplication Law: $U_\alpha U_\beta = U_{\alpha+\beta}$

Higher-Form Global Symmetries

Crucially, higher-form $G^{(p)}$ global symmetries act on operators of dimension p



For $p \geq 1$ a $G^{(p)}$ symmetry can be continuous or discrete but it is necessarily Abelian

We can still couple to a $G^{(p)}$-symmetry by turning on a background $(p+1)$-form gauge field $A^{(p+1)}$

Extended operators are ubiquitous in QFT, higher-form symmetries are inevitable.

$G^{(p)}$- Symmetries in Gauge Theory

Example: Maxwell electrodynamics in $4d$ has a $U(1)^{(1)}_{\mathsf{e}} \times U(1)^{(1)}_{\mathsf{m}}$ global symmetry

The symmetry $U(1)^{(1)}_{\mathsf{e}}$ acts on Abelian Wilson lines of charge $q \in \mathbb{Z}$:

$$W_q = \exp\left(iq\int_\gamma a^{(1)}\right)$$

similarly $U(1)^{(1)}_{\mathsf{m}}$ acts on magnetic 't Hooft lines $\mathcal{T}_q(\gamma)$.

Important example: Pure $SU(N)$ Yang-Mills theory has a 1-form center symmetry $\mathbb{Z}_N$ acting on fundamental Wilson lines W_F.

$$U(\Sigma_2)W_F = e^{\frac{2\pi i}{N}}W_F$$

Dynamical Consequences

Higher-form symmetries can have ’t Hooft anomalies and can be spontaneously broken

$\Rightarrow$ Opportunity for new non-perturbative constraints on the RG flow

Consider $4d$ $SU(N)$ Yang-Mills theory, we have numerical, analytical, and experimental evidence that theory has unique vacuum that is gapped and color confining in the IR

There is an important topological term:

$$S \supset \frac{\theta}{8\pi^2}\int Tr(F \wedge F) \qquad \theta \in [0,2\pi)$$

At $\theta = 0$ and $\theta = \pi$ the theory is time-reversal T (equivalently CP) invariant.

Q: What is the IR fate for general θ ?

Dynamical Consequences

[Gaiotto-Kapustin-Komargodski-Seiberg 2017]

As we now know, for any value of θ, YM has a $\mathbb{Z}_N^{(1)}$ symmetry.

At $\theta = \pi$ there is a new ’t Hooft anomaly between T and $\mathbb{Z}_N^{(1)}$:

$\Rightarrow$ The IR cannot be both gapped and non-degenerate at $\theta = \pi$

- Gapless (CFT)
- Deconfined (TQFT)
- T spontaneously broken (2 degenerate vacua)

Most compelling scenario (Large-N, Lattice) is that T is spontaneously broken.

Idea: we can exploit generalized symmetries and their properties to establish new results about phase diagrams of general QFTs

Higher-Groups

[Kapustin-Thorngren 2013] [Cordova-Dumitrescu-Intriligator 2018] [Benini-Cordova-Hsin 2018]

Higher-group global symmetry: mixture of global symmetries with different p-degree.

<u>Example</u>: Two $U(1)$ Goldstones χ, ϕ coupled to a $U(1)$ gauge field $c^{(1)}$ in $3d$

$$S = -\frac{1}{2e^2}\int f_c^{(2)} \wedge *f_c^{(2)} - \frac{1}{2}\int d\chi \wedge *d\chi - \frac{1}{2}\int d\phi \wedge *d\phi + \frac{ik}{4\pi^2}\int \phi d\chi \wedge f_c^{(2)}$$

For any integer $k > 1$ this model has a large set of $G^{(0)}$ and $G^{(1)}$ global symmetries that are fused in a 2-group global symmetry:

[Aguilera-Damia-Argurio-LT to appear]

$$G_A^{(0)} \times_k G_B^{(1)}$$

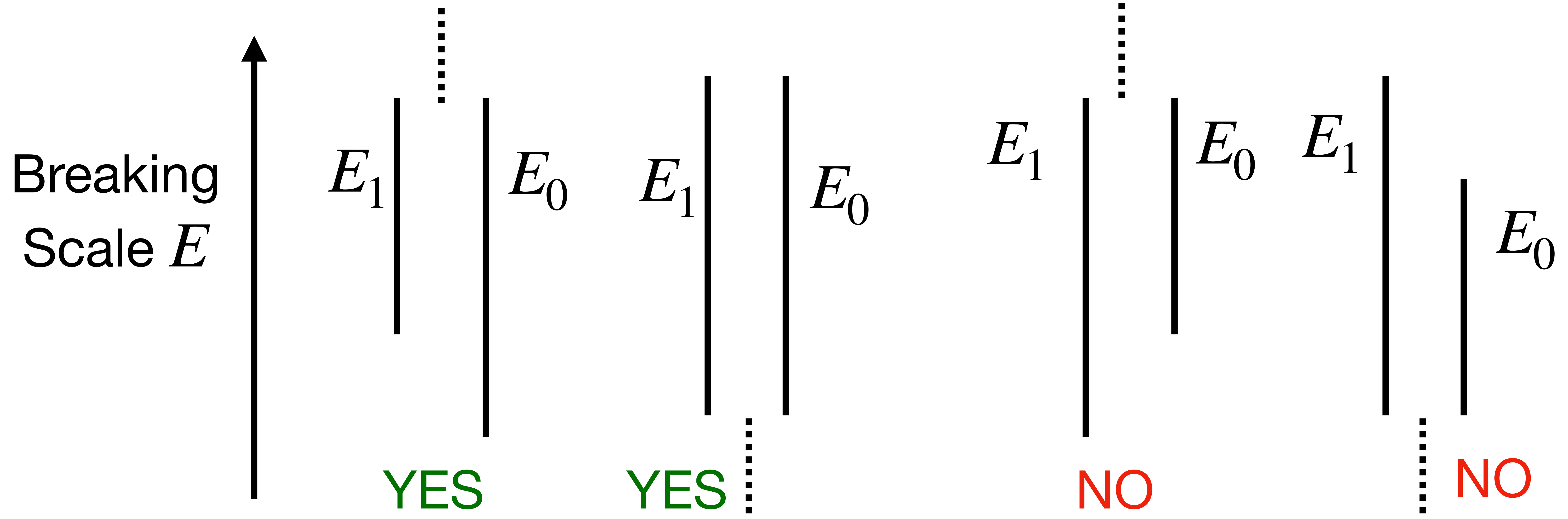
$$A^{(1)} \to A^{(1)} + d\lambda_A^{(0)} \qquad\qquad B^{(2)} \to B^{(2)} + d\Lambda_B^{(1)} + \frac{k}{2\pi}\lambda_A^{(0)} F_A^{(2)}$$

Background fields have mixed transformation rules!

Constraints from Higher-Groups

Higher groups can be found in a wide variety of examples in different dimensions. What are their dynamical implications?

2-groups have a peculiar symmetry breaking pattern: $G^{(0)} \times_k G^{(1)} \to G^{(1)}$, $G^{(0)} \times_k G^{(1)} \to \varnothing$



<u>Example</u>: Trying to evade these inequalities leads to a loss of control in simple axion EFTs in $4d$. The energy scale at which axion strings can decay is always larger than the mass scale of charged particles.

[Brennan-Cordova 2020]

Non-Invertible Symmetries

Do all global symmetries form a group? Surprisingly NO!

Symmetries should be encoded by topological extended operators, these objects need not obey a group multiplication law.

Defect Operator	$G^{(0)}$ $U_g(\Sigma_{d-1})$	$G^{(p)}$ $U_g(\Sigma_{d-(p+1)})$	Non-invertible symmetry $\mathscr{D}(\Sigma_{d-(p+1)})$
Codimension	1	>1	≥ 1
Topological	Yes	Yes	Yes
Fusion Rule	g x h = gh	g x h = gh	$g \times h = \sum_l N_{gh}^l \, l$

Non-Invertible Symmetries

[Frohlich-Fuchs-Runkel-Schweigert 2006]
[Chang-Lin-Shao-Wang-Yin 2018]

An extended operator $\mathscr{D}$ is non-invertible whenever it is not possible to find an inverse operator such $\mathscr{D}^{-1}$ that $\mathscr{D} \times \mathscr{D}^{-1} = 1$

It has been advocated that the non-invertible topological operators $\mathscr{D}$ should be viewed as generalizations of ordinary global symmetries. There is a rapidly growing list of theories in $d \geq 2$ exhibiting these symmetries.

Example (Again): Two $U(1)$ Goldstones χ, ϕ coupled to a $U(1)$ gauge field $c^{(1)}$ in $3d$

$$S = -\frac{1}{2e^2}\int f_c^{(2)} \wedge * f_c^{(2)} - \frac{1}{2}\int d\chi \wedge * d\chi - \frac{1}{2}\int d\phi \wedge * d\phi + i\frac{1}{4\pi^2}\int \phi d\chi \wedge f_c^{(2)}$$

For $k = 1$, there is no 2-group global symmetry. The $U(1)^{(0)}$ shift symmetries of χ (and ϕ) are completely broken:

$$d * j^{(1)} = \frac{1}{4\pi^2} d\phi \wedge f_c^{(2)}$$

Non-Invertible Symmetries

In this case we can define a non-invertible operator $\mathscr{D}$ as follows.

[Aguilera-Damia-Argurio-LT to appear]

First we introduce a new current $\hat{j}^{(1)}$ such that:

$$\hat{j}^{(1)} = j^{(1)} - \frac{1}{4\pi^2}\phi f_c^{(2)} \,, \qquad d * \hat{j}^{(1)} = 0 \,.$$

In terms of $\hat{j}^{(1)}$ we can define a new topological operator:

$$\hat{U}_\alpha(\Sigma_2) = \exp\left(i\alpha \int * j^{(1)} - \frac{1}{4\pi^2}\phi f_c^{(2)} \right)$$

For a generic value of $\alpha \in [0,2\pi)$ the operator $\hat{U}_\alpha$ is not gauge-invariant

However, for $\alpha = 2\pi/N$, there is a gauge-invariant and topological operator:

$$\mathscr{D}_{1/N}(\Sigma_2) = \exp\left[i \int_{\Sigma_2} \left(\frac{2\pi}{N} * j^{(1)} + \frac{1}{2\pi}\rho f_c^{(2)} + \frac{N}{2\pi}\rho db^{(1)} + \frac{1}{2\pi}\phi db^{(1)} \right) \right]$$

Non-Invertible Symmetries

However, for $\alpha = 2\pi/N$, there is a gauge-invariant and topological operator:

$$\mathscr{D}_{1/N}(\Sigma_2) = \exp\left[i\int_{\Sigma_2} \left(\frac{2\pi}{N} * j^{(1)} + \frac{1}{2\pi}\rho f_c^{(2)} + \frac{N}{2\pi}\rho db^{(1)} + \frac{1}{2\pi}\phi db^{(1)} \right) \right]$$

The dynamical fields ρ and the gauge field $b^{(1)}$ only live on Σ_2 and not in the full spacetime M_3 . We thus obtained $\mathscr{D}_{1/N}$ by dressing the naive operator $\hat{U}_{2\pi/N}$ with a 2d theory living on Σ_2 coupled to ϕ and $f_c^{(2)}$.

It is also possible to show that:

$$\mathscr{D}_{1/N} \times \mathscr{D}_{1/N}^{-1} \neq 1$$

The symmetry $U(1)^{(0)}$ is broken but gives rise to infinitely many non-invertible operators $\mathscr{D}$ labeled by rational angles $\alpha = 2\pi/N$.

Non-Invertible Symmetries

[Choi-Lam-Shao 2022]
[Cordova-Ohmori 2022]

Our construction inspired by the following observation:

Consider the QCD Lagrangian for the massless u and d quarks below the electroweak scale where $SU(2) \times U(1)$ is higgsed to $U(1)_{EM}$ and above the pion scale.

The axial current suffers from an ABJ anomaly:

$$j_\mu = \frac{1}{2}\bar{u}\gamma_5\gamma_\mu u - \frac{1}{2}\bar{d}\gamma_5\gamma_\mu d\ , \qquad d*j = \frac{1}{8\pi^2} f \wedge f$$

We already know that the classical $U(1)_A^{(0)}$ symmetry is broken to $\mathbb{Z}_2^{(0)}$ quantum mechanically.

A more surprising fact is that, as before, for more general rational angles $\alpha = \pi/N$ there are infinitely many non-invertible operators.

Conclusions

- We discussed three generalizations of global symmetries, higher-form symmetries, higher-group symmetries, and non-invertible symmetries.
- These tools open new possibilities to obtain nonperturbative results in quantum field theory.
- A short term goal is to widen our understanding of all the dynamical consequences that are implied by these symmetries.

Thanks!