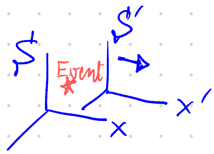
# SR Recap

▷ Lorentz Transformation & Spacetime




$\boxed{\text{L.T.}}$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

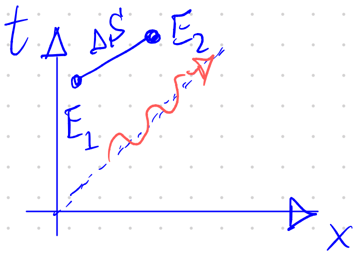
from Einstein's Postulates

- Laws of physics take the same form in all inertial frame
- c must be the same in all inertial frames

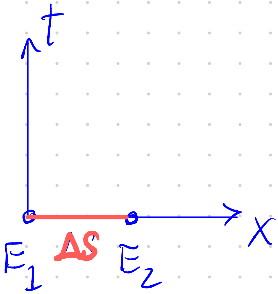
inertial frame

- no friction, no external foce (Galileo & Einstein)
  - ⇒ a body in uniform motion
- a body moving freely in gravity (Einstein)

Consider the events $E_1$ and $E_2$




Ex 1




$S: (\cancelto{0}{\Delta t}, \Delta x)$, $S': (\Delta t', \Delta x')$

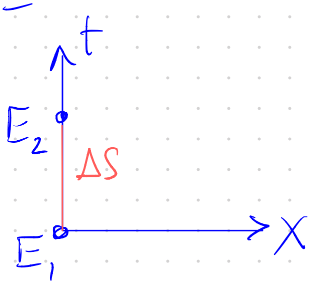
$$\Delta t' = \gamma\left(\cancelto{0}{\Delta t} - \frac{v}{c^2}\Delta x\right)$$

$$= -\frac{\gamma v \Delta x}{c^2} \rightarrow$$

| | Simultaneity |
|---|---|
| $S$ | ✓ |
| $S'$ | ✗ |

$$\Delta x' = \gamma \Delta x$$

Ex 2




$S: (\Delta t, \cancelto{0}{\Delta x})$, $S': (\Delta t', \Delta x')$

$$\Delta t' = \gamma \Delta t \rightarrow \text{time dilation}$$

$$\Delta x' = -\gamma v \Delta t$$

The message of L.T.

- ▷ space ⇒ not absolute (Newtonian, also)
- ▷ time ⇒ not absolute

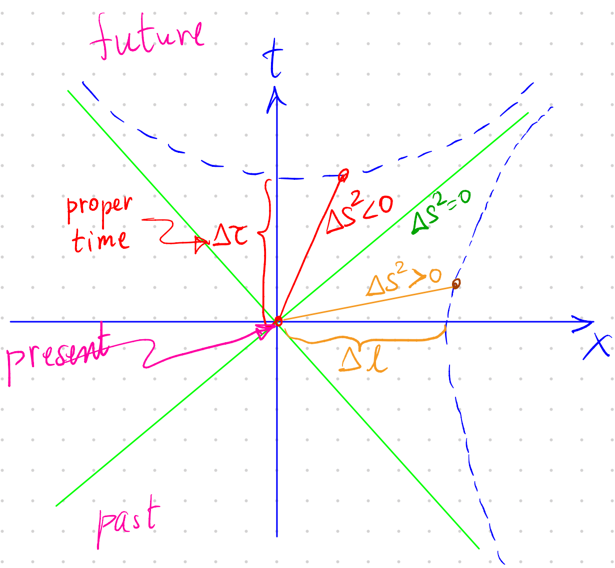
But
- ▷ spacetime can be considered to be absolute ! → Lorentz invariant

$$\boxed{(\Delta s)^2 = (\Delta s')^2}$$

Spacetime interval

$$ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2$$

convention: mostly minus $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$




$$\Delta s^2 \begin{cases} < 0 \ ; & \text{timelike} \\ = 0 \ ; & \text{lightlike or null} \\ > 0 \ ; & \text{spacelike} \end{cases}$$

Note for lightlike separation ($\Delta s^2 = 0$)

No L.T. can make two null-separated events either
- occur simultaneously
- at the same spatial position

▷ Metric tensor

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \quad ; \quad \alpha, \beta = 0, 1, 2, 3$$

$$\{\eta_{\alpha\beta}\} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$ds^2 = \eta_{00}\, dx^0 dx^0 + \eta_{ii}\, dx^i dx^i \quad ; \quad i = 1, 2, 3$$

$$= -(c\,dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^3$$

What is metric tensor

⇒ inner product (map one of two vectors to dual vector space)

$$A \cdot B = \eta_{\alpha\beta} A^\alpha B^\beta$$

$$A \cdot A = \eta_{\alpha\beta} A^\alpha A^\beta$$

⇒ raising and lowering indices

$$A_\alpha = \eta_{\alpha\beta} A^\beta \qquad \text{(lowing index)}$$

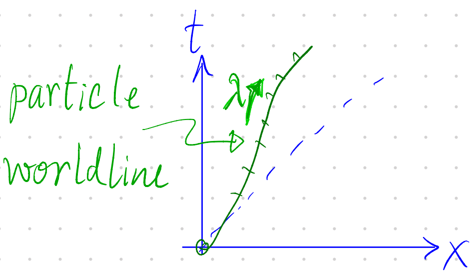
$$A^\alpha = \eta^{\alpha\beta} A_\beta \qquad \text{(raising index)}$$

$$A_0 = -A^0 \qquad\qquad \eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha{}_\beta$$

$$A_i = A^i \qquad\qquad \eta^{-1}\eta = \mathbb{1}$$

▷ Kinematics of particle

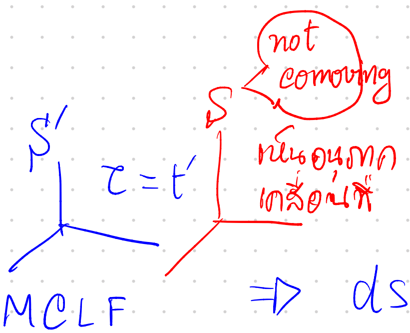
$$x^\alpha = r^\alpha(\lambda)$$




Choose $\lambda = \tau$ (L.I.)

$$x^\alpha = r^\alpha(\tau)$$

measured by the particle's momentarily comoving Lorentz frame (MCLF)




$$\Rightarrow \quad ds^2 = -(c\,dt')^2 = -c^2 d\tau^2$$

$$d\tau = c^{-1}\sqrt{-ds^2}$$

From the 4-position vector (on world line)

$$x^\alpha = r^\alpha(\tau)$$

the 4-velocity $\quad u^\alpha = \dfrac{dr^\alpha}{d\tau}$

$$u^\alpha = \frac{dt}{d\tau}\frac{dr^\alpha}{dt}$$

$$= \gamma(c, \vec{v}) \quad ; \quad \gamma = \frac{dt}{d\tau}$$

Ex Show that $\gamma = \frac{dt}{d\tau}$ = Lorentz factor $\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$

$$\underline{\underline{\eta_{\alpha\beta} u^\alpha u^\beta}} = \frac{\eta_{\alpha\beta}\, dr^\alpha dr^\beta}{(d\tau)^2}$$

(only instantaneously)

$$= \frac{ds^2}{d\tau^2} = \underline{\underline{-c^2}}$$

use $u^\alpha = \gamma(c, \vec{v})$

$$\gamma^2(-c^2 + \vec{v}^2) = -c^2$$

$$\therefore \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

▷ Momentum and energy

4-momentum $\quad p^\alpha = m u^\alpha$

$$= m\gamma(c, \vec{v})$$

using $\left.\begin{array}{l} E = \gamma m c^2 \\ \vec{p} = \gamma m \vec{v} \end{array}\right\} \Rightarrow \quad p^\alpha = \left(\frac{E}{c}, \vec{p}\right)$

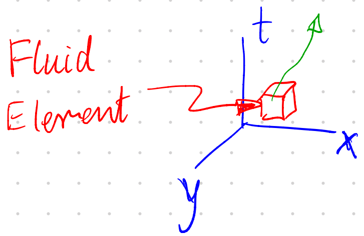
Note Photon

$$p^\alpha = \frac{\hbar\omega}{c}(1, \hat{k})$$

$$p^\alpha p_\alpha = 0$$

↑ massless

## Energy-Momentum Tensor (set $c=1$)




frame $S$
(Lab frame)

Consider in MCLF, $S'$, we can define proper energy density

$$\mu = \rho c^2 + \epsilon$$

$\rho$: proper mass density

$\epsilon$: proper internal energy density

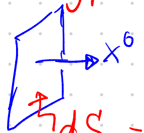
in the Lab frame ($S$)

$$j^\alpha = (j^0, j^k)$$

$j^0 :=$ fluid mass density

$j^k :=$ mass flux in the $x^k$-direction mass current = mass per unit 3D-hypersurface $\perp$ $\alpha$-direction

$$j^0 = \frac{m}{\Delta V_{xyz}} = \rho$$

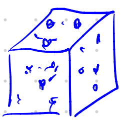



$$d_\rho S_0 = dV_{xyz}$$

$$j^k = \frac{m}{\Delta A_{\perp k}\Delta t} = \frac{m}{\Delta A_{\perp k}\Delta x^k}\frac{\Delta x^k}{\Delta t} = \rho v^k$$

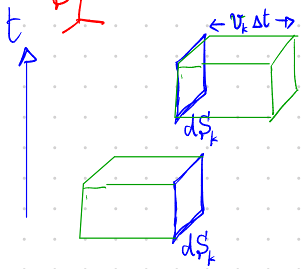
$S$

$$j^0 = \rho$$



(non-relativistic)

$S$




$$j^k = \rho v^k$$

(non-relativistic)

$S$ $\quad$ $S' \to v$

(MCLF) $\to S'$

$$j^0 = \frac{\Sigma m}{A_{yz}\Delta x_p} = \rho \quad , \quad j^k = 0$$

(Lab frame) $\to S$ , from Length contraction

$$j^0 = \frac{\Sigma m}{A_{yz}\Delta x} = \frac{\Sigma m}{A_{yz}\left(\frac{\Delta x_p}{\gamma}\right)} = \rho\gamma, \quad j^k = \rho\gamma v^k$$

Recall that $u^\alpha = \gamma(1, \vec{v})$ ; $c = 1$

i.e. $u^0 = \gamma$, $u^i = \gamma v^i$

$$j^0 = \rho u^0 , \qquad j^k = \rho u^k$$

$$\Downarrow$$

$$\boxed{j^\alpha = \rho u^\alpha}$$

mass 4-current in Lab frame ($S$)

it follows the local mass conservation

$$\boxed{\partial_\alpha j^\alpha = 0}$$

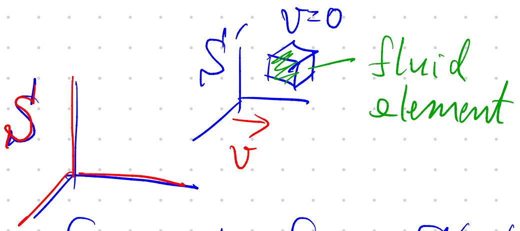
Remark $\quad j^\alpha \to$ vector field

mass current = (mass density, mass flux)

$$\Downarrow$$

energy density
energy flux
momentum density
momentum flux

tensor field !!!




proper energy density $\mu$

from the fram $S'$ to $S$

- ▷ energy increased by $\gamma$
- ▷ volume decreased by $\frac{1}{\gamma}$

$\boxed{\text{Consider Perfect Fluid w/ } P=0,}$ **Dust**

energy density

$$\mu \longrightarrow \mu\gamma^2 = \underbrace{\mu u^0 u^0}_{T^{00}} \qquad \boxed{(c=1)}$$

(frame $S'$) (fram $S$) $T^{00}$ → tensor

$u^\alpha = \gamma(1, \vec{v})$

energy flux in $x^j$ direction

$$0 \longrightarrow (\mu\gamma^2)v^j = \underbrace{\mu u^0 u^j}_{T^{0j}}$$

non-rela. $\mu v^j$

**momentum** density

$$\mu\gamma^2 v^j = \underbrace{\mu u^j u^0}_{T^{j0}}$$

$$\xrightarrow[\text{non-rela.}]{0}$$

$$\left(\frac{\mu}{c} v^j\right) = \mu v^j$$

**momentum** flux

$$\mu\gamma^2 v^j v^k = \underbrace{\mu u^j u^k}_{T^{jk}}$$

$$\xrightarrow[\text{non-rela.}]{0}$$

$$\underbrace{\left(\frac{\mu}{c} v^j\right)}_{\text{momentum density}} v^k = \mu v^j v^k$$

flux of $j$-momentum in $x^k$-direction

$\partial_\alpha j^\alpha = 0 \longrightarrow \boxed{\partial_\beta T^{0\beta} = 0}$

conservation of mass

energy conserv.

$\longrightarrow \boxed{\partial_\beta T^{j\beta} = 0}$

momentum conserv.

Thus, we have

$$\boxed{\partial_\beta T^{\alpha\beta} = 0}$$

* $T^{\alpha\beta}$ is divergence free when energy and momentum are conserved.

# the energy-momentum tensor

(stress-energy tensor)

$T^{\alpha\beta}$ : energy density

pressure

shear stress

$p^0$ ทิศทางของ flux

$T^{00}$ : energy density $\qquad \dfrac{E}{\Delta x \Delta y \Delta z}$

$p^0$

$T^{0i}$ : energy flux $\qquad \dfrac{E}{\Delta y \Delta z \Delta t}$

$p^i$

$T^{i0}$ : momentum density $\qquad \dfrac{P_x}{\Delta x \Delta y \Delta z}$

$T^{ii}$ : momentum flux $\qquad F_x \quad \dfrac{P_x}{\Delta t\, \Delta y \Delta z} = P$

$p^i$ ทิศทาง flux

$T^{ij}$ : —— $\qquad F_x \quad \dfrac{P_x}{\Delta t\, \Delta x \Delta y} =$ shear stress

Consider perfect fluid (no viscosity, no heat conduct$^{n}$)

$\downarrow$

no shear stress

Ex 1

fluid in MCLF $\leftarrow$ fluid at rest

without pressure

$$T^{00} = \mu$$

$$T^{i0} = T^{0i} = T^{ij} = 0$$

$$u^{\alpha} = (1, 0, 0, 0)$$

$$T^{\alpha\beta} = \mu u^{\alpha} u^{\beta} \Rightarrow T^{00} = \mu u^0 u^0 = \mu$$

$$\{T^{\alpha\beta}\} = \begin{pmatrix} \mu & & & \\ & 0 & & 0 \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$$

Ex 2

moving fluid

without pressure

$$u^{\alpha} = \gamma(1, \vec{v})$$

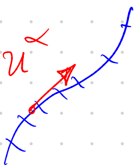
$$T^{\alpha\beta} = \mu u^{\alpha} u^{\beta} \Rightarrow \begin{cases} T^{00} = \mu\gamma^2 \\ T^{0i} = T^{i0} = \mu\gamma^2 v^i \\ T^{ii} = p = 0 \\ T^{ij} = 0 \end{cases}$$

Ex 3

perfect fluid
with pressure > 0

MCLF $\rightarrow$ $$T^{ii} = \frac{F_i}{\text{area} \perp \text{direction } i} = p$$

$$\therefore \quad T^{ij} = p\,\delta^{ij}$$




in the frame $S'$

$$T^{ij} = p\,\underbrace{P^{\alpha\beta}}_{\eta^{\alpha\beta}+u^\alpha u^\beta}$$

$$= p\,(\eta^{\alpha\beta}+u^\alpha u^\beta)$$

$$T^{\alpha\beta} = \mu u^\alpha u^\beta + p\,(\eta^{\alpha\beta}+u^\alpha u^\beta)$$
$$= (\mu+p)\,u^\alpha u^\beta + p\,\eta^{\alpha\beta}$$

Note that $$\boxed{T^{\alpha\beta} = T^{\beta\alpha}}$$

(symmetric)

this property is in fact very general.