Tutorial problems on Gravitational Perturbation and Gravitational Waves

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1 Gravitational Perturbation

1.1 Problem 1: Equivalent between linearized Einstein tensor and Lagragian of the small metric perturbation $h_{\mu\nu}$

Considering the metric decomposition $g_{\mu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu} = \eta^{\mu\nu} - h^{\mu\nu}$, show that

• At the first order of $h_{\mu\nu}$, the Christoffel symbol is given by

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda}\Big(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big) \\ &= \frac{1}{2} \eta^{\rho\lambda}\Big(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\Big). \end{aligned} \tag{1}$$

• At the first order of $h_{\mu\nu}$, the Riemann tensor is given by

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho} \\ &= \frac{1}{2}\Big(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\Big). \end{aligned} \tag{2}$$

• At the first order of $h_{\mu\nu}$, the Ricci tensor and Ricci scalar are given by

$$R_{\mu\nu} = \frac{1}{2}\Big(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \partial_\sigma\partial^\sigma h\Big), \tag{3}$$

where $h = h^{\nu}{}_{\nu} = \eta^{\mu\nu} h_{\mu\nu}$.

$$R = \Big(\partial_\mu\partial_\nu h^{\mu\nu} - \partial_\sigma\partial^\sigma h\Big). \tag{4}$$

• Then, the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$ is given by

$$G_{\mu\nu} = \frac{1}{2}\Big(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h - \partial_\sigma\partial^\sigma h - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\partial_\sigma\partial^\sigma h\Big). \tag{5}$$

• By varying with respect to the $h^{\mu\nu}$, show that the equation of motion of the following Lagrangian density is equivalent to the linearized Einstein tensor in Eq.(5)

$$\mathcal{L} = \frac{1}{2}\Big[\big(\partial_\mu h^{\mu\nu}\big)\big(\partial_\nu h\big) - \big(\partial_\mu h^{\rho\sigma}\big)\big(\partial_\rho h^{\mu}{}_{\sigma}\big) + \frac{1}{2}\eta^{\mu\nu}\big(\partial_\mu h^{\rho\sigma}\big)\big(\partial_\nu h_{\rho\sigma}\big) - \frac{1}{2}\eta^{\mu\nu}\big(\partial_\mu h\big)\big(\partial_\nu h\big)\Big]. \tag{6}$$

1.2 Problem 2: Gauge invariant transformation of the small metric perturbation $h_{\mu\nu}$

Show that the linearized Riemann tensor in Eq.(2) is invariance under the gauge transformation of the small metric perturbation, $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$.

1.3 Problem 3: Gauge invariance of the Bardeen variables

Consider the perturbed FRW metric written as the following line element,

$$ds^2 = a^2(\tau)\Big[-(1+2A)d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + 2E_{ij})dx^i dx^j\Big], \tag{7}$$

where

$$B_i = \partial_i B + \hat{B}_i\,, \tag{8}$$

$$E_{ij} = \delta_{ij} C + \big(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\partial_k\partial_k\big)E + \tfrac{1}{2}\big(\partial_i \hat{E}_j + \partial_j \hat{E}_i\big) + \hat{E}_{ij}\,. \tag{9}$$

The coordinate transformation at the same physical point, p is given by

$$x^\mu \to \tilde{x}^\mu(p) = x^\mu(p) + \xi^\mu(p)\,, \tag{10}$$

where

$$\xi^0 = T\,,$$

$$\xi^i = L^i = \partial^i L + \hat{L}^i\,. \tag{11}$$

The Scalar-Vector-Tensor decomposition variables are transformed as

$$A \to A - T' - \mathcal{H}T\,, \tag{12}$$

$$B \to B + T - L'\,, \qquad \hat{B}_i \to \hat{B}_i - L'_i\,, \tag{13}$$

$$C \to C - \mathcal{H}T - \frac{1}{3}\,\partial_j\partial^j L\,, \tag{14}$$

$$E \to E - L\,, \qquad \hat{E}_i \to \hat{E}_i - \hat{L}_i\,, \qquad \hat{E}_{ij} \to \hat{E}_{ij}\,, \tag{15}$$

where $\mathcal{H} \equiv a'/a$ and $X' = dX/d\tau$.
The Bardeen variables are defined to avoid the gauge problem of the metric

perturbations that do not invariant under the the change of coordinates. They are defined by

$$\begin{aligned}
\Psi &\equiv A + \mathcal{H}(B - E') + (B - E')' \,, \\
\hat{\Phi}_i &\equiv \hat{B}_i - \hat{E}'_i \,, \\
\Phi &\equiv -C + \frac{1}{3}\partial_i \partial^i E - \mathcal{H}(B - E') \,. \qquad (16)
\end{aligned}$$

Show that the Bardeen variables, Ψ, $\hat{\Phi}_i$ and Φ are gauge invariance.

2 Gravitational Waves

2.1 Lorenz gauge of the trace reversed perturbation

The trace reversed perturbation, $\widetilde{h}_{\mu\nu}$ is defined by

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \,. \tag{17}$$

Having use the Lorenz gauge $\partial_\mu \widetilde{h}^{\mu\nu} = 0$, show that the linearized Einstein tensor in Eq.(5) can be written in the following form,

$$G_{\mu\nu} = -\frac{1}{2}\partial_\rho \partial^\rho \widetilde{h}_{\mu\nu} \,. \tag{18}$$

2.2 Gravitational waves from cosmological background

Consider the following line element,

$$ds^2 = a^2(\tau)\Big[- d\tau^2 + \big(\delta_{ij} + h_{ij}\big) dx^i dx^j \Big] \,, \tag{19}$$

The non-vanished Christoffel symbols of the line element are given by

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} \,, \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + 2\mathcal{H} h_{ij} + h'_{ij} \,, \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j + \delta^{ik} h'_{kj} \,, \\
\Gamma^i_{jk} &= \partial_j h^i{}_k + \partial_k h^i{}_j - \delta^{il}\partial_l h_{jk} \,,
\end{aligned} \tag{20}$$

where $\mathcal{H} \equiv a'/a$ and $X' = dX/d\tau$.

• Show that the spatial part to the perturbed Einstein tensor is

$$\delta G_{ij} = h''_{ij} - \partial_k \partial^k h_{ij} + 2\mathcal{H} h'_{ij} - 2h_{ij}\big(2\mathcal{H}' + \mathcal{H}^2\big) \,. \tag{21}$$

• Combining the result of the perturbed Einstein tensor above and using $\delta T_{ij} = 2a^2 h_{ij}\bar{P}$ where $\bar{P}$ is pressure in the background. Show that, the perturbed Einstein equation ($\delta G_{ij} = 8\pi G \delta T_{ij}$) is written by

$$h''_{ij} + 2\mathcal{H} h'_{ij} - \partial_k \partial^k h_{ij} = 0 \,. \tag{22}$$

Hint. One can rewrite the pressure in the background $\bar{P}$ in terms of the geometrical quantities by considering the spatial component of the *unperturbed Einstein equation.*