โจทย์ tutorial

1. Using $\eta_{\alpha\beta} u^\alpha u^\beta = -c^2$ and $p^\alpha = m\gamma(c, \vec{v})$,
Derive the well-known formula $E^2 = m^2c^4 + p^2c^2$

2. Show that

$$P^\alpha_{\ \beta} u^\beta = 0 = u_\alpha P^\alpha_{\ \beta}$$

$$P^\alpha_{\ \mu} P^\mu_{\ \beta} = P^\alpha_{\ \beta}$$

$$P^\alpha_{\ \alpha} = 3$$

$$P^{\alpha\beta} = e^\alpha_{(1)} e^\beta_{(1)} + e^\alpha_{(2)} e^\beta_{(2)} + e^\alpha_{(3)} e^\beta_{(3)}$$

3. (Particle Dynamics)

Defining 4-acceleration

$$a^\alpha = \frac{du^\alpha}{d\tau}$$

a) Show that $a^\alpha u_\alpha = 0$

b) Using $u^\alpha = \gamma(c, \vec{v})$, show that $a^0 = \vec{a} \cdot \frac{\vec{v}}{c}$

<u>Hint</u> $\vec{v} = \frac{d\vec{r}}{dt}$, $\underbrace{\vec{u}}_{\text{spatial components of 4-velocity}} = \frac{d\vec{r}}{d\tau}$, $\underbrace{\vec{a}}_{\text{spatial components of 4-acceleration}} = \frac{d^2\vec{r}}{d\tau^2}$

c) Relativistic version of Newton's Law

$$F^\alpha = m a^\alpha$$

show that

$$F^0 = \frac{\vec{F} \cdot \vec{v}}{c}$$

and combining this result with that from b) to discuss that $F^0 = m a^0$ is a relativistic statement of the work-energy theorem

(from Poisson & Will, 2014, Gravity)

4. The Lorentz transformation for an arbitrary velocity whose components in S are v^j is given by

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta ,$$

where the components of Λ^α_β are given by

$$\begin{aligned}
\Lambda^0_0 &= \gamma , \\
\Lambda^0_j &= -\gamma v_j / c , \\
\Lambda^j_0 &= -\gamma v^j / c , \\
\Lambda^j_k &= \delta^j_k + (\gamma - 1) n^j n_k ,
\end{aligned}$$

where $n^j = v^j / |\boldsymbol{v}|$.

(a) Show that this reduces to Eq. (4.1) when $\boldsymbol{v}$ is aligned with the x-direction.

(b) By considering the invariance of the interval ds^2, show that the Minkowski metric in S' is related to the Minkowski metric in S by

$$\eta_{\alpha\beta} = \eta'_{\gamma\delta} \Lambda^\gamma_\alpha \Lambda^\delta_\beta .$$

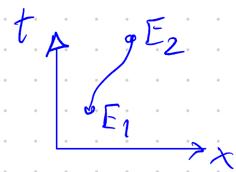
(c) Verify using the general Lorentz transformation that $\eta_{\alpha\beta}$ has the same diagonal form with entries $(-1,\ 1,\ 1,\ 1)$ as it had in S'. You can do this using matrix multiplication by recognizing that the transformation of the metric can be expressed as the matrix equation $\eta = \Lambda^{\mathrm{T}} \eta' \Lambda$.

5. Show that

$$\eta^{\alpha\beta} = -e^\alpha_{(0)} e^\beta_{(0)} + e^\alpha_{(1)} e^\beta_{(1)} + e^\alpha_{(2)} e^\beta_{(2)} + e^\alpha_{(3)} e^\beta_{(3)}$$

by checking its components (a) in a frame where the basis vectors are attached to a particle at rest, and (b) in a frame where the vectors are attached to a particle moving in the x-direction with velocity v.

6.



Consider a freely moving particle ($a^\alpha = 0$).

The action $$S = -mc^2 \int_1^2 d\tau$$

$$= \int_1^2 dt\, L$$

where $$L = -mc\sqrt{-\eta_{\alpha\beta}\frac{dr^\alpha}{dt}\frac{dr^\beta}{dt}}$$

Using the Euler-Lagrange equation (corresponding to $\delta S = 0$)

$$\frac{d}{dt}\frac{\partial L}{\partial v^\mu} - \frac{\partial L}{\partial r^\mu} = 0$$

Show that a uniform motion of a free particle, according to $\delta S = 0$, maximizes the elapsed proper time between E_1 and E_2