Carroll, Cotton and Ehlers

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Starring

Lewis Carroll (1832 – 1898)

Poet and mathematician – Christ Church College, Oxford

Alice's Adventures in Wonderland & Through the Looking-Glass

Émile Cotton (1872 – 1950)

Professor of mathematics at the University of Grenoble

Cotton tensor

Jürgen Ehlers (1929 – 2008)

Max Planck Institute for Gravitational Physics – Potsdam

Ehlers group

Highlights

Framework

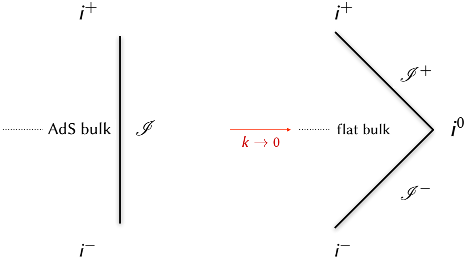
Asymptotically flat spacetimes

- Gravitational waves
- Charges and asymptotic symmetries
- Flat holography – as opposed to AdS/CFT

Defining feature of asymptotic flatness

From AdS$_n$ to flat$_n$ asymptotics

$\Lambda = -\frac{(n-1)(n-2)}{2} k^2 \to 0$



$\mathscr{I}^\pm$ is a null hypersurface $\to$ Carrollian geometry with $\mathfrak{ccarr}(n-1) \equiv \mathrm{BMS}_n$ conformal isometry group

Why Carrollian? [Lévy-Leblond '65; Sen Gupta '66]

Carroll group $=\lim_{c\to 0}$ Poincaré group

- light cone shrinks, causality fades $\rightarrow$ mad tea-party on the other side of the looking-glass
- Riemannian (relativistic) geometry $\rightarrow$ Carrollian geometry

What is BMS? [Bondi, van der Burg, Metzner, Sachs '62]

Asymptotic isometry group of asymptotically flat spacetimes
often ∞-dimensional

If flat holography exists the dual theory should be defined on a *Carrollian spacetime* and be $\mathrm{BMS}_n \equiv \mathfrak{ccarr}(n-1)$-invariant

Reconstructing the bulk from the boundary

Einstein spacetimes – time-like conformal boundary

- boundary metric
- boundary energy–momentum tensor

Ricci-flat spacetimes – null conformal boundary

- boundary Carrollian geometric data
- boundary Carrollian momenta
- boundary Chthonian (infinite) data

Prominent role of the Cotton/Carrollian-Cotton tensor

- enters explicitly the bulk-metric expansion
- defines magnetic duals to energy and momenta

Hidden vs. visible symmetries in Ricci-flat spaces

Isometries or asymptotic isometries are *visible* and act *locally*

Reductions along isometric orbits exhibit *hidden* symmetries

- from 4 to 3 dimensions: Ehlers' Möbius group [Ehlers '62]
- larger reduction: bigger group – even exceptional

acting *non-locally* in the parent space

Punch line [Mittal, Petropoulos, Rivera-Betancour, Vilatte '22]

Ehlers group is realized on the boundary and acts locally on the Carrollian data including the Carrollian-Cotton attributes

Carrollian geometry [Lévy-Leblond '65; Sen Gupta '66; Duval et al. '14; Bekaert et al. '16]

Basic ingredients in $d+1$ dimensions (coordinates $t, \mathbf{x}$)

- degenerate metric: $\mathrm{d}s^2 = 0 \times \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- field of observers: $\frac{1}{\Omega}\partial_t$ (t should be spelled u)
- clock form: $\mu = -\Omega \mathrm{d}t + b_i \mathrm{d}x^i$ (Ehresmann connection)

General covariance (in the present parameterization)

Carrollian diffeomorphisms: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$

Dynamics – relativistic

General-covariant action and energy–momentum tensor

Pseudo-Riemannian spacetimes in $d+1$ dim $T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$

- Weyl invariance $\to T^{\mu}{}_{\mu} = 0$
- general covariance ($\xi = \xi^{\mu}(t, \mathbf{x})\partial_{\mu}$ diffeos) $\to \nabla_{\mu} T^{\mu\nu} = 0$

Dynamics – Carrollian

Carrollian-covariant action, energy and momenta

$$\begin{cases} \Pi^{ij} = \frac{2}{\sqrt{a}\Omega}\frac{\delta S}{\delta a_{ij}} & \text{energy–stress tensor} \\ \Pi^{i} = \frac{1}{\sqrt{a}\Omega}\frac{\delta S}{\delta b_{i}} & \text{energy flux} \\ \Pi = -\frac{1}{\sqrt{a}}\left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega}\frac{\delta S}{\delta b_i}\right) & \text{energy density} \end{cases}$$

Conservation equations in Carrollian spacetimes

- Carollian covariance ($\xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$ diffeos)

$$\to \begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0 & \text{time} \\ \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)P_i + \hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} = 0 & \text{space} \end{cases}$$

$\to$ momentum P_i

- Weyl covariance $\to \Pi^i{}_i = \Pi$

Applications

Black-hole horizons

Membrane paradigm – wrongly confused with Galilean Navier–Stokes equations

Here: Ricci-flat spacetimes

- Null boundaries in asymptotically flat spacetimes are *Carrollian geometries* – zero speed of light
- These Carrollian geometries have an *infinite tower of conformal Killings* [Ciambelli, Leigh, Marteau, Petropoulos '19]

$$\boxed{d+1=3 \to \mathfrak{ccarr}(3) \equiv \mathfrak{so}(3,1) \ltimes \text{supertransls.} \equiv \text{BMS}_4}$$

in line with expected asymptotic symmetries

Highlights

Pure gravity – asymptotically flat or AdS

Solving Einstein' equations in $n = d + 2$ dim for g_{AB}

$\{r, t, x^i\}$, $i = 1, \ldots, d$ plus *gauge fixing* ($n = d + 2$ conditions) $\rightarrow$ find g_{AB} as $\mathrm{O}\left({}^{1}/{r^{\ell}}\right)$ with coefficients $f(t, \mathbf{x})$ obeying a set of remaining Einstein' eqs.

Solution space $\equiv$ collection of data $f(t, \mathbf{x})$

Spirit: organize $f(t, \mathbf{x})$ and their dynamics tensorially wrt a *covariant structure on the boundary*

Gauge choice

- Fefferman–Graham covariant but singular at $k \rightarrow 0$
- Bondi valid at $k \rightarrow 0$ but not boundary-covariant

Einstein 4-dim spacetimes in a nutshell

Covariant incomplete modified Newman–Unti gauge

- boundary metric $\mathrm{d}s^2$ (6)
- conformal boundary energy–momentum tensor $T_{\mu\nu}$ (5)
- congruence u_μ (2) $\to$ boundary local Lorentz invariance

remaining Einstein' equations $\nabla_\mu T^{\mu\nu} = 0$ ("fluid")

The boundary Cotton tensor appears *explicitly* in g_{AB} [FG '85; de haro '08; Mansi et al. '09; de Freitas et al. & Bakas et al. '14; Gath et al. '15; Ciambelli et al. '18]

$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right)$ symmetric, traceless, $\nabla_\mu C^{\mu\nu} = 0$

$C_{\mu\nu} \neq 0 \Leftrightarrow$ non-conformally flat bry. $\leftrightarrow$ asymptotically *locally* AdS bulk (ex. Taub–NUT)

Cotton spin-offs

ξ bry. conformal Killing $\to I^\mu = \xi_\nu T^{\mu\nu}$ and $I^\mu_{\text{Cot}} = \xi_\nu C^{\mu\nu}$

$$Q_\xi = \int_{\Sigma_2} *\mathrm{I} \quad \text{and} \quad Q_{\text{Cot}\,\xi} = \int_{\Sigma_2} *\mathrm{I}_{\text{Cot}}$$

electric and *magnetic* dual conserved charges (bulk mass vs. nut)

- $Q_{\text{Cot}\,\xi} \sim$ magnetic Komar charges: off–shell conservation
- Limitation in AdS: at most 10 conformal Killings $(d+1=3)$

Extendable in Ricci-flat spacetimes – more interesting

Ricci-flat in covariant Newman–Unti gauge

Full solution space in $n = 4$ [Brussels & Paris groups]

$\mathrm{d}s^2_{\text{Ricci-flat}}$ described in terms of $2+1$ *Carrollian boundary data*

- Carrollian geometry (6) with zero geometrical shear ξ_{ij}
  - degenerate metric (3) $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
  - Ehresmann connection (3) $\mu = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$
- Carrollian conformal "fluid" (5)
  - energy (1) $\varepsilon \to$ Bondi mass aspect
  - momenta – heat current (2) and stress tensor (2) E_{ij} & $\pi_i \to$ angular momentum aspect
- Carrollian-fluid "velocity" (2) hydro-frame freedom
- Carrollian dynamical shear (2) $\mathscr{C}_{ij}$
- *infinite* number of further Carrollian data – at every $\mathrm{O}\left(1/r^n\right)$: *Chthonian* challenges flat "hologtaphy"

obeying Carrollian dynamics

$3+1$ Ricci-flat bulk from its $2+1$ Carrollian boundary

- Here $C_{\mu\nu} \to c, \psi_i, \chi_i, \Psi_{ij}, X_{ij}$ Carrollian Cotton tensors
- Resummability condition $\pi^i = \frac{1}{8\pi G} * \psi^i \to$ algebraic Petrov

$$\mathrm{d}s^2_{\text{res. Ricci-flat}} = \upmu \left[2\mathrm{d}r + 2\left(r\varphi_j - *\hat{\mathscr{D}}_j * \varpi\right)\mathrm{d}x^j - \left(r\theta + \hat{\mathscr{K}}\right)\upmu\right] + \rho^2 \mathrm{d}\ell^2 + \frac{\upmu^2}{\rho^2}\left[8\pi G\varepsilon r + *\varpi c\right]$$

$\rho^2 = r^2 + *\varpi^2$

$\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{8\pi G}\hat{\mathscr{D}}_i * \chi^i = 0 \quad \hat{\mathscr{D}}_j\varepsilon - \frac{1}{8\pi G} * \hat{\mathscr{D}}_j c = 0 \quad \frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0$

- Assuming ∂_t be a Killing $\to \Omega = 1, \theta = \varphi_i = 0, \hat{\mathscr{K}} = K$

$$\mathrm{d}\ell^2 = \frac{2}{P^2\left(\zeta, \bar\zeta\right)}\mathrm{d}\zeta\mathrm{d}\bar\zeta$$

$-2c = \hat\tau(\zeta) + \bar{\hat\tau}\left(\bar\zeta\right) \quad 16\mathrm{i}\pi G\varepsilon = \hat\tau(\zeta) - \bar{\hat\tau}\left(\bar\zeta\right) \quad 2K = \hat k(\zeta) + \bar{\hat k}\left(\bar\zeta\right)$

Example: Kerr–Taub–NUT family

$$P = A\zeta\bar\zeta + D \quad \hat k = K = 2AD \quad \hat\tau = 2\mathrm{i}(M + \mathrm{i}Kn),$$

Ehlers and Geroch – Ricci-flat spaces [Ehlers '62; Geroch '71]

Bulk reduction along a Killing ζ: $\mathcal{M} \to \mathcal{S} = \mathcal{M}/_{\text{orb}(\zeta)}$

$$\lambda = \zeta^A \zeta_A \Rightarrow h_{AB} = g_{AB} - \frac{\zeta_A \zeta_B}{\lambda}$$

Further ingredients: twist and its on-shell potential

$$w_A = \eta_{ABCD} \zeta^B \nabla^C \zeta^D \Rightarrow \mathrm{w} = \mathrm{d}\omega$$

Three-dimensional dynamics (sigma-model)

DOFS. $\tau = \omega + \mathrm{i}\lambda \quad \tilde{h}_{AB} = \lambda h_{AB}$

EQS. $\tilde{\mathcal{R}}_{AB} = -\frac{2}{(\tau-\bar{\tau})^2} \tilde{\mathcal{D}}_{(A}\tau \tilde{\mathcal{D}}_{B)}\bar{\tau} \quad \tilde{\mathcal{D}}^2\tau = \frac{2}{\tau-\bar{\tau}} \tilde{\mathcal{D}}_M\tau \tilde{\mathcal{D}}_N\tau \tilde{h}^{MN}$

INV. $\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})$: Ehlers' group

not local in the bulk

Boundary Möbius action [Mittal, Petropoulos, Rivera-Betancour, Vilatte '22]

Algebraic Ricci-flat spacetimes with Killing ∂_t

$$\tau\left(r,\zeta,\bar\zeta\right)=\frac{\hat\tau\left(\zeta\right)}{r+\mathrm{i}*\varpi\left(\zeta,\bar\zeta\right)}-\mathrm{i}\hat k\left(\zeta\right)$$

Carrollian boundary *local* transformations

$$\boxed{P'=\frac{P}{\left|\gamma\hat k+\mathrm{i}\delta\right|}\quad \hat k'=\mathrm{i}\frac{\alpha\hat k+\mathrm{i}\beta}{\gamma\hat k+\mathrm{i}\delta}\quad \hat\tau'=-\frac{\hat\tau}{\left(\gamma\hat k+\mathrm{i}\delta\right)^2}}$$

Example: Kerr–Taub–NUT family

$$P=A\zeta\bar\zeta+D\quad \hat k=K=2AD\quad \hat\tau=2\mathrm{i}(M+\mathrm{i}Kn),$$

$$*\varpi(\zeta,\bar\zeta)=n+a-\frac{2Da}{P}$$

M & n $\leftrightarrow$ electric & magnetic charges – mixed under Möbius

Quotable facts

- Ricci-flat spacetimes are reconstructed from boundary Carrollian-covariant data including the Cotton tensors
- A bulk isometry reveals into a boundary Ehlers' Möbius local invariance involving the Cotton as a magnetic facet
- Origin of the localization: $^1\!/_r$-expansion of the bulk Weyl $\rightarrow$ energy–momentum & Cotton ($\rightarrow$ Petrov classification)

Further knowledge and investigation

- Towers of charges and dual charges are available from boundary Carrollian methods $\rightarrow$ organized wrt $SL(2,\mathbb{R})$
- Further comparison with bulk approaches for towers of charges and duality issues [Godazgar, Godazgar, Pope '18–21]
- Method generalizable beyond algebraic spaces or ∂_t Killing
- Hope for this boundary invariance without bulk isometry?
- What could we learn for flat holography?

6 Killings & charges

Killings and conserved charges

Relativistic conformal dynamics

- ξ conformal Killing $\rightarrow$ $I^{\mu} = \xi_{\nu} T^{\mu\nu}$ divergence-free
- $Q_{\xi} = \int_{\Sigma_d} *\mathrm{I}$ conserved

Conformal Carrollian dynamics [Petkou, Petropoulos, Rivera-Bet., Siampos '22]

- Conformal Killings via $\mathscr{L}_{\xi} a_{ij}$ and $\mathscr{L}_{\xi} \frac{1}{\Omega}\partial_t$ *but not* $\mathscr{L}_{\xi} \mu$
- ξ plus $\left\{\Pi, \Pi^{ij}, P^i, \Pi^i\right\} \rightarrow \left\{\kappa, K^i\right\}$
- Carrollian divergence: $\mathcal{K} = \frac{1}{\Omega}\hat{\mathscr{D}}_t \kappa + \hat{\mathscr{D}}_j K^j = 0$ if $\Pi^i = 0$
- Charge: $Q_K = \int_{\Sigma_d} \mathrm{d}^d x \sqrt{a} \left(\kappa + b_i K^i\right)$ conserved if $\mathcal{K} = 0$

Application for 4-dim Ricci-flat spacetimes

Main consequences

- Null boundaries in asymptotically flat spacetimes are *Carrollian geometries*
- Boundary Carrollian geometries with have an *infinite tower of conformal Killings* generating BMS_4
- Carrollian dynamics appears for the energy–momentum and the Cotton
- Infinite towers of conformal-Killing charges exist in electric and magnetic versions but *not all are always conserved*
- The Möbius group acts on these charges – only $\begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix}$ is a genuine transformation