Partonic Structure from LQCD

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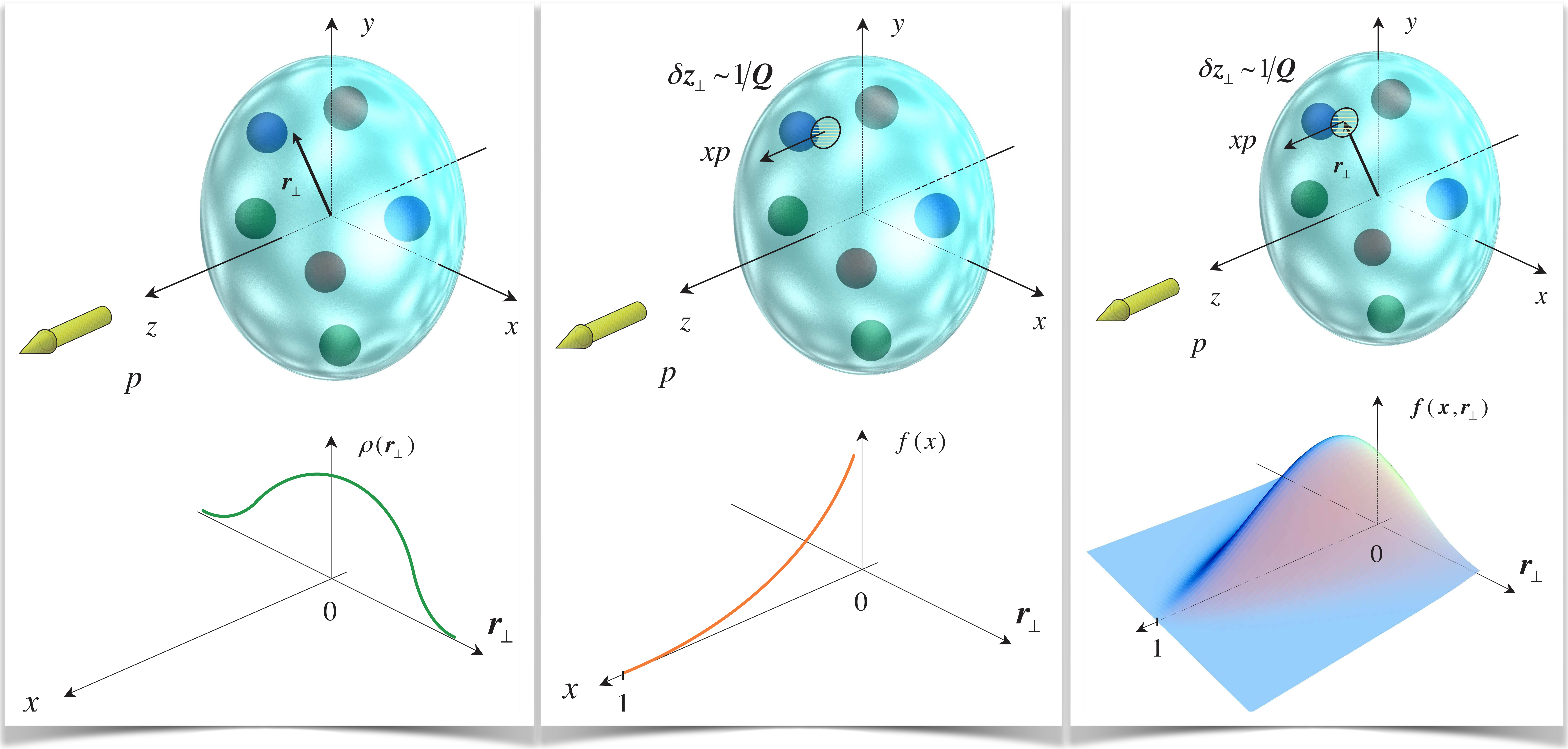
William & Mary / JLab

Xmas Theoretical Physics Workshop @ Athens 2022, 22-23 Dember, 2022

X. Ji, D. Muller, A. Radyushkin (1994-1997)

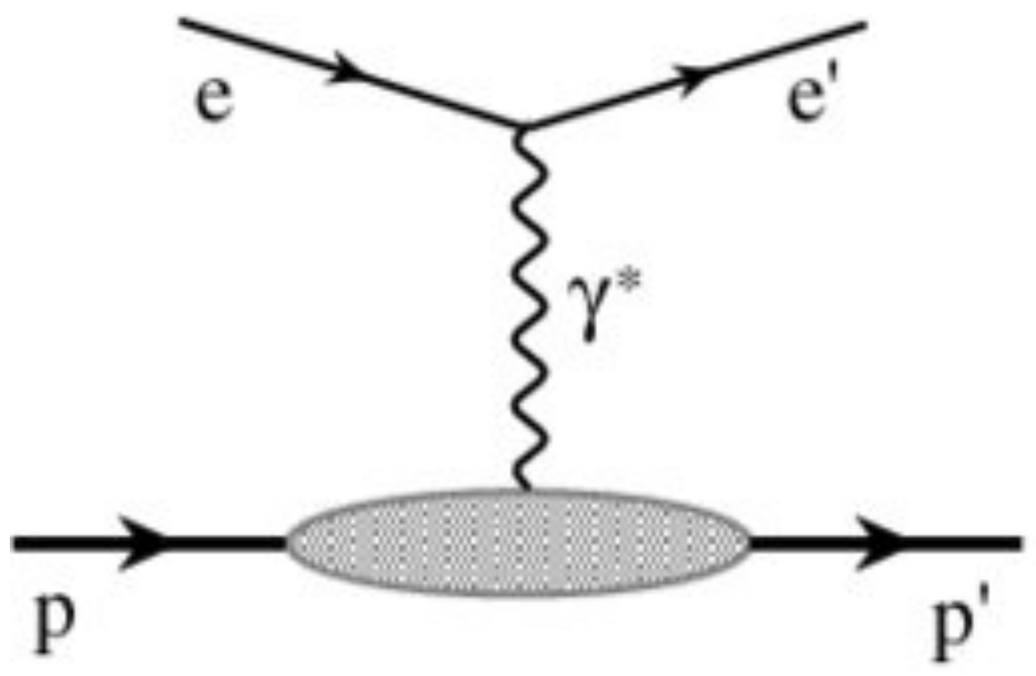


Form Factors

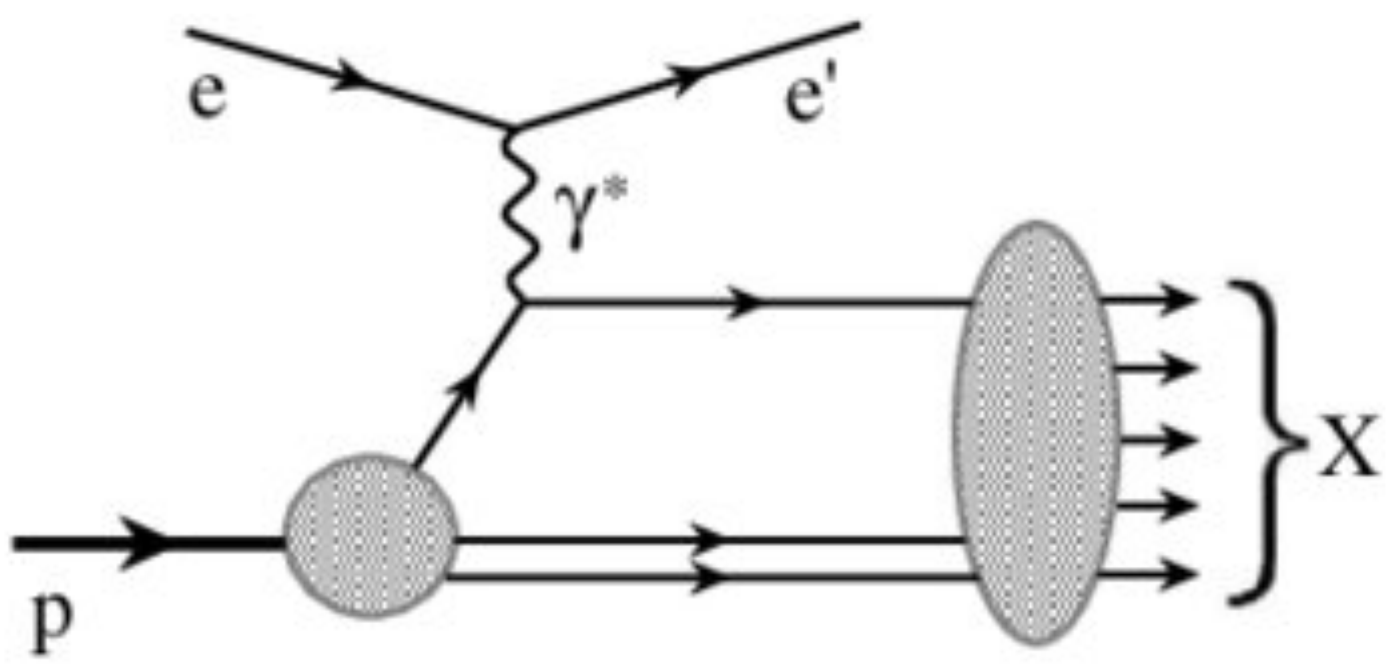
Parton Distribution functions

Generalized Parton Distribution functions

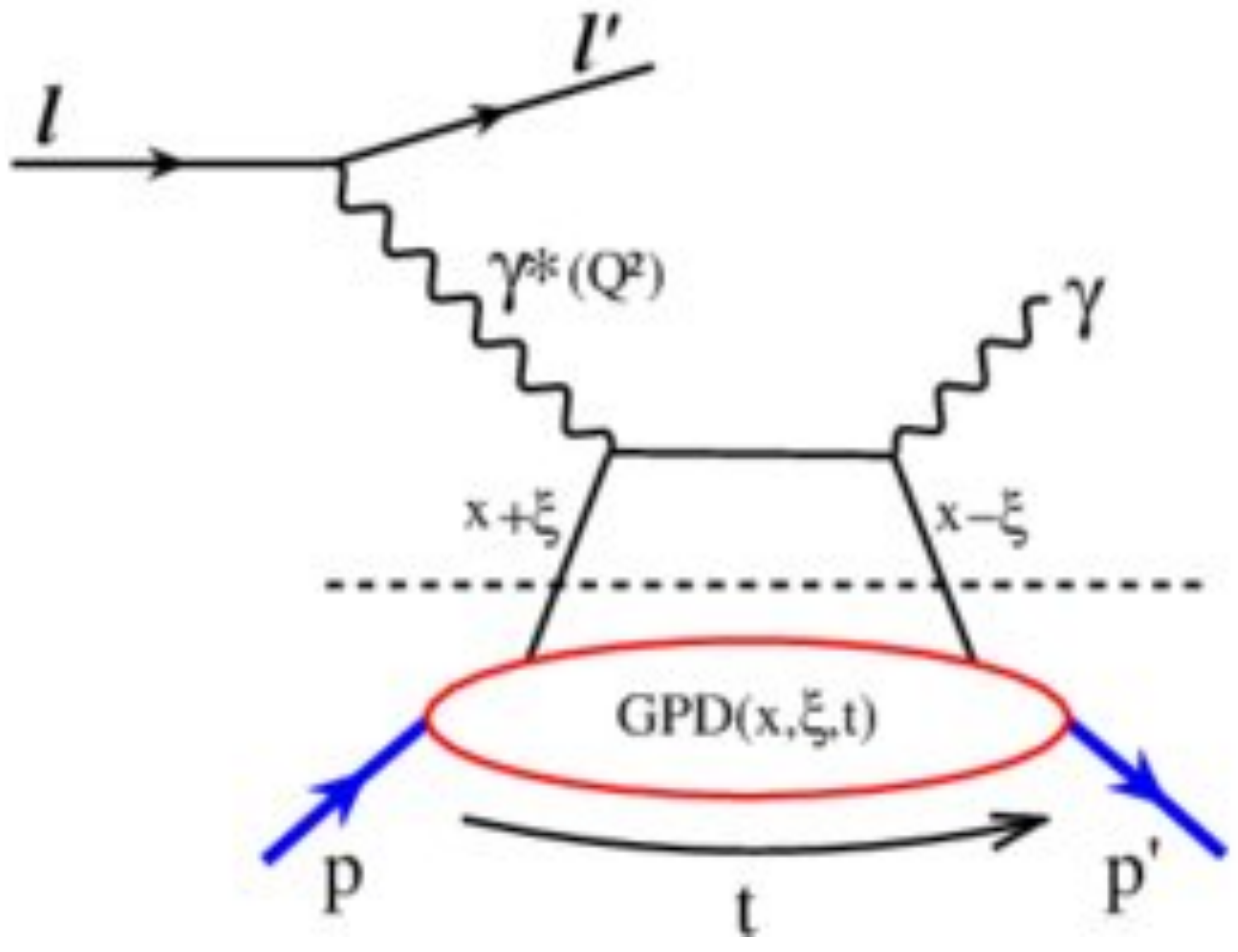
Factorization → non-perturbative structure

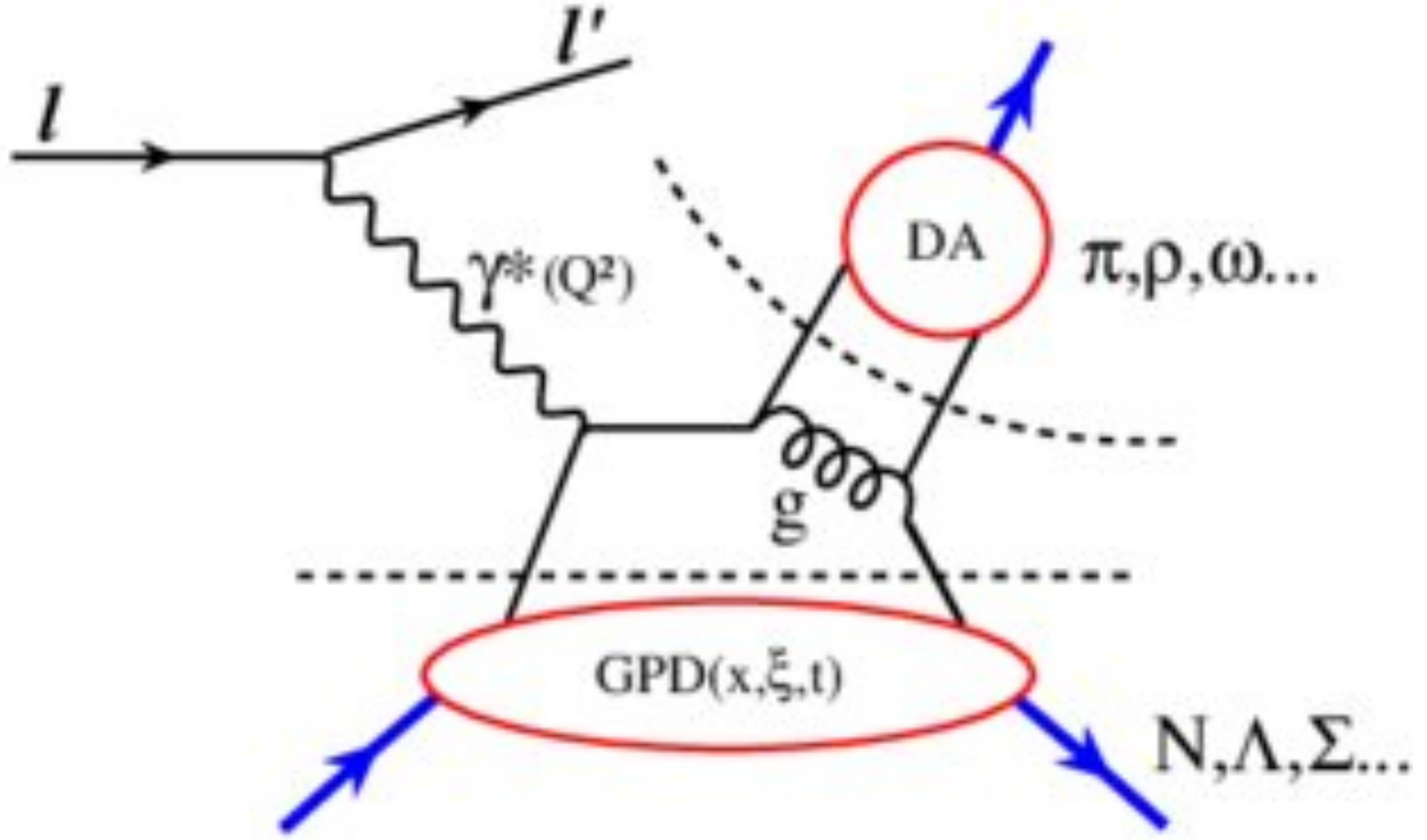


Elastic scattering: Form factor



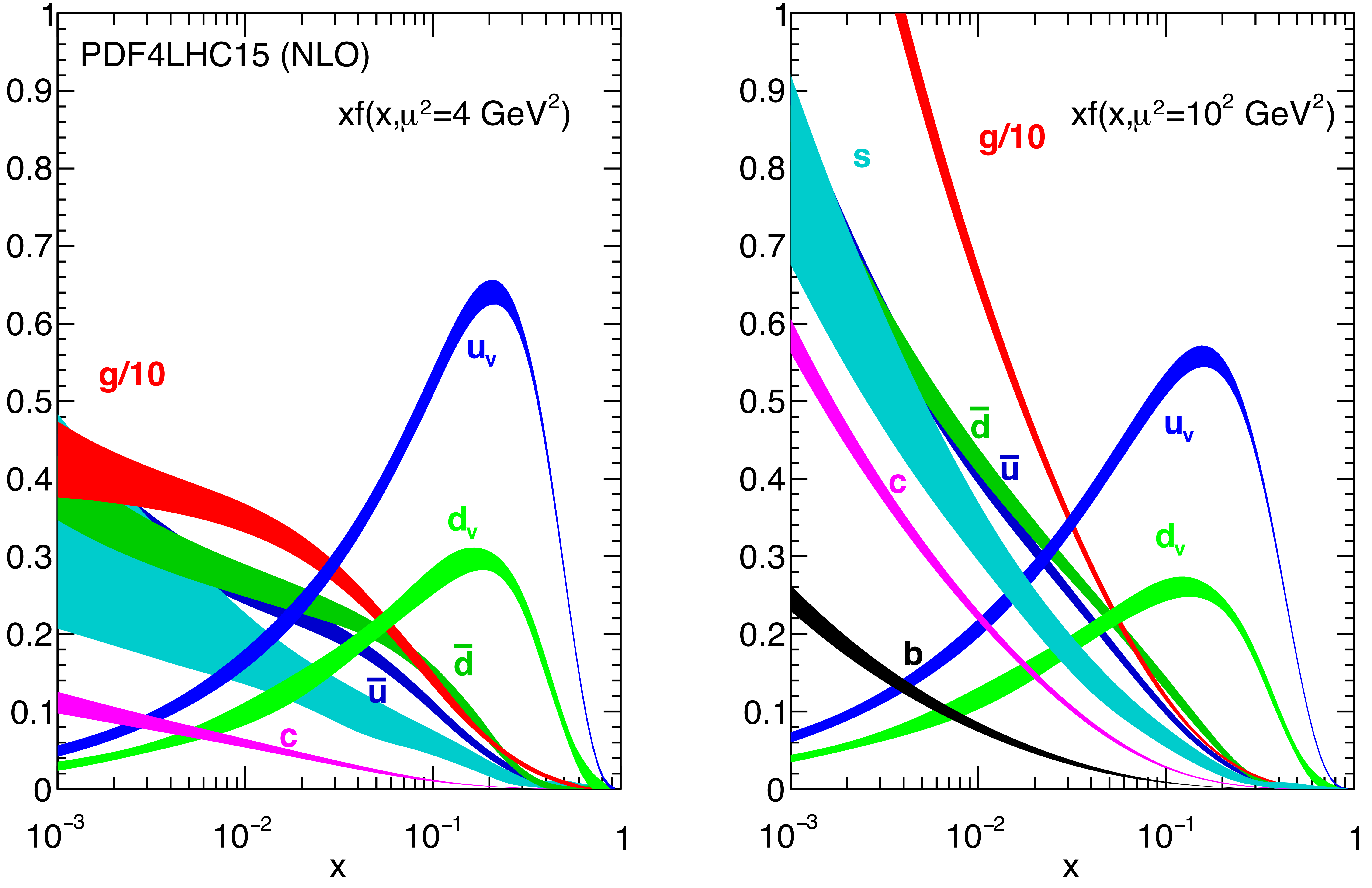
DIS: Parton distributions





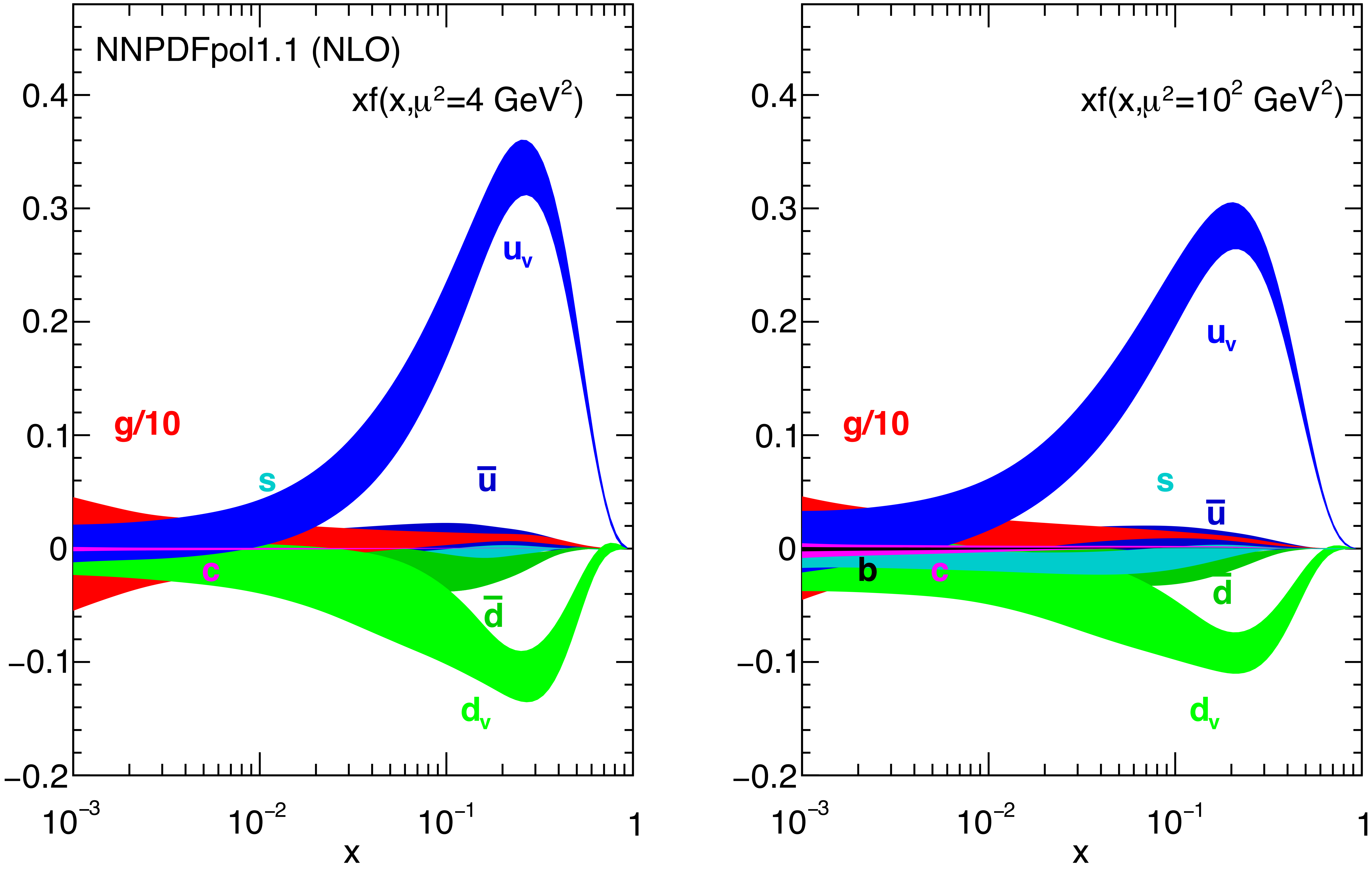
DVCS or DVMP: Generalized Parton distributions

Determination of Parton distribution functions from Experiment



Fits to experimental cross section data

Determination of Parton distribution functions from Experiment



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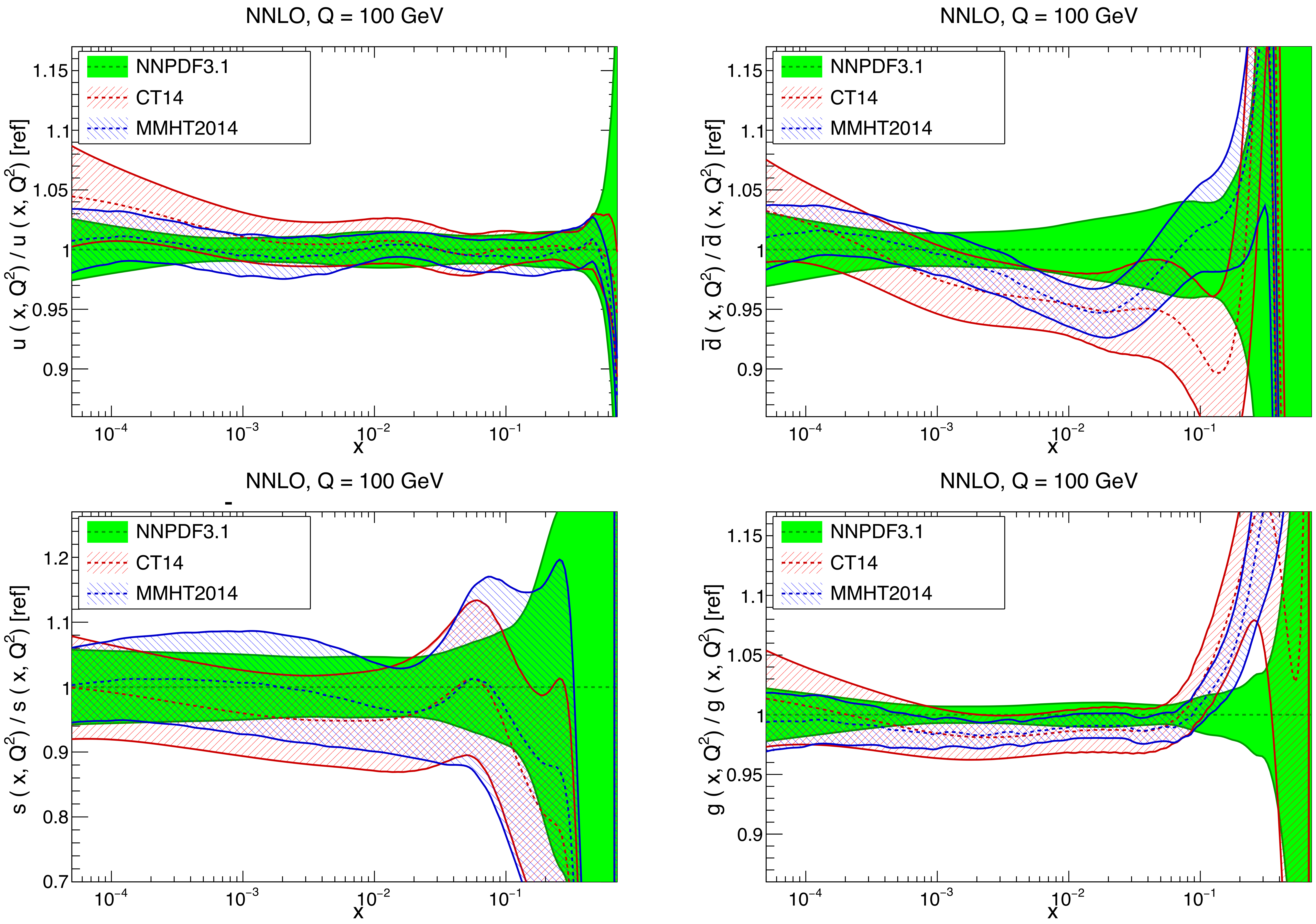
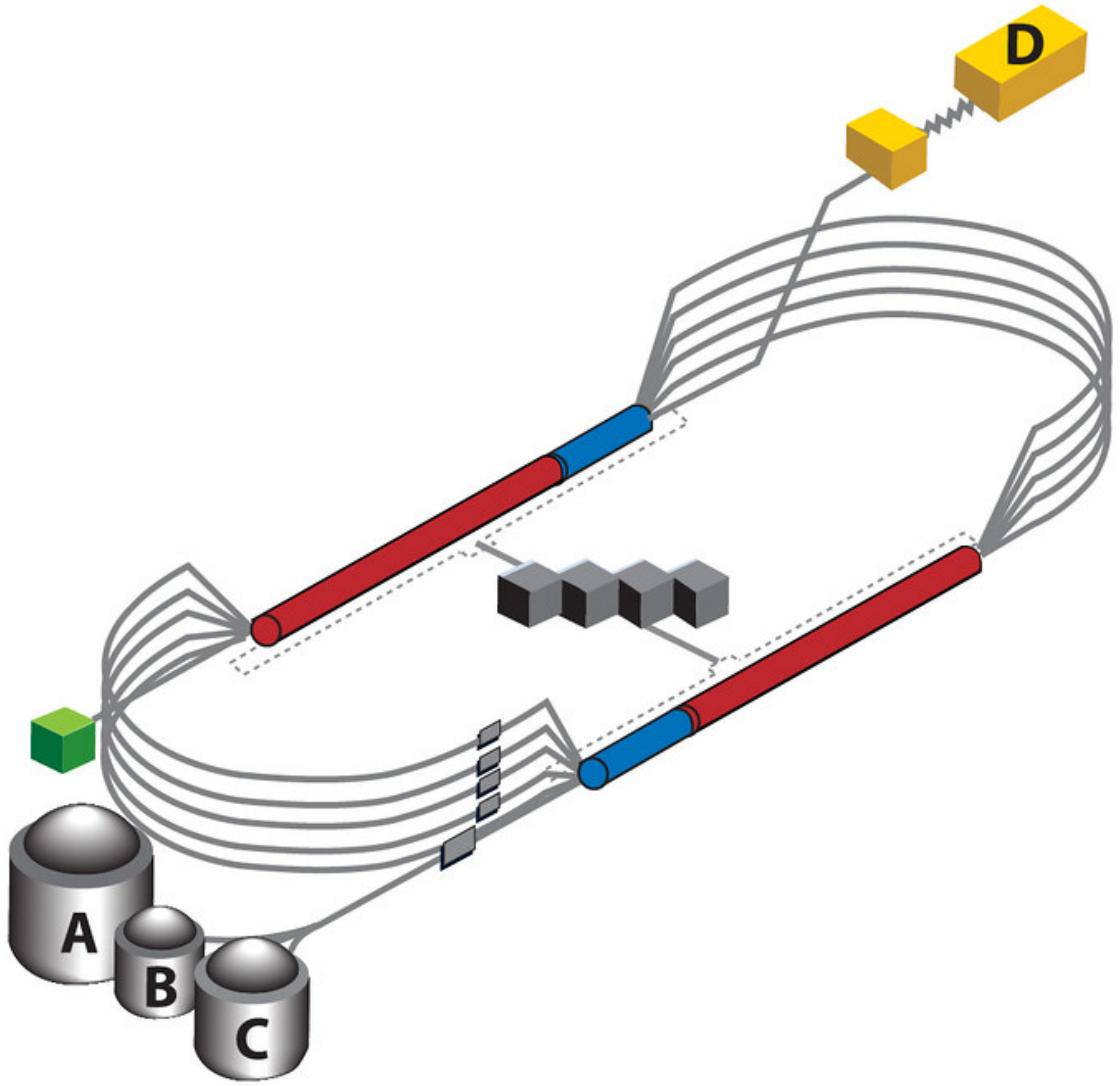


Parton distributions and lattice QCD calculations: a community white paper

arXiv:1711.07916

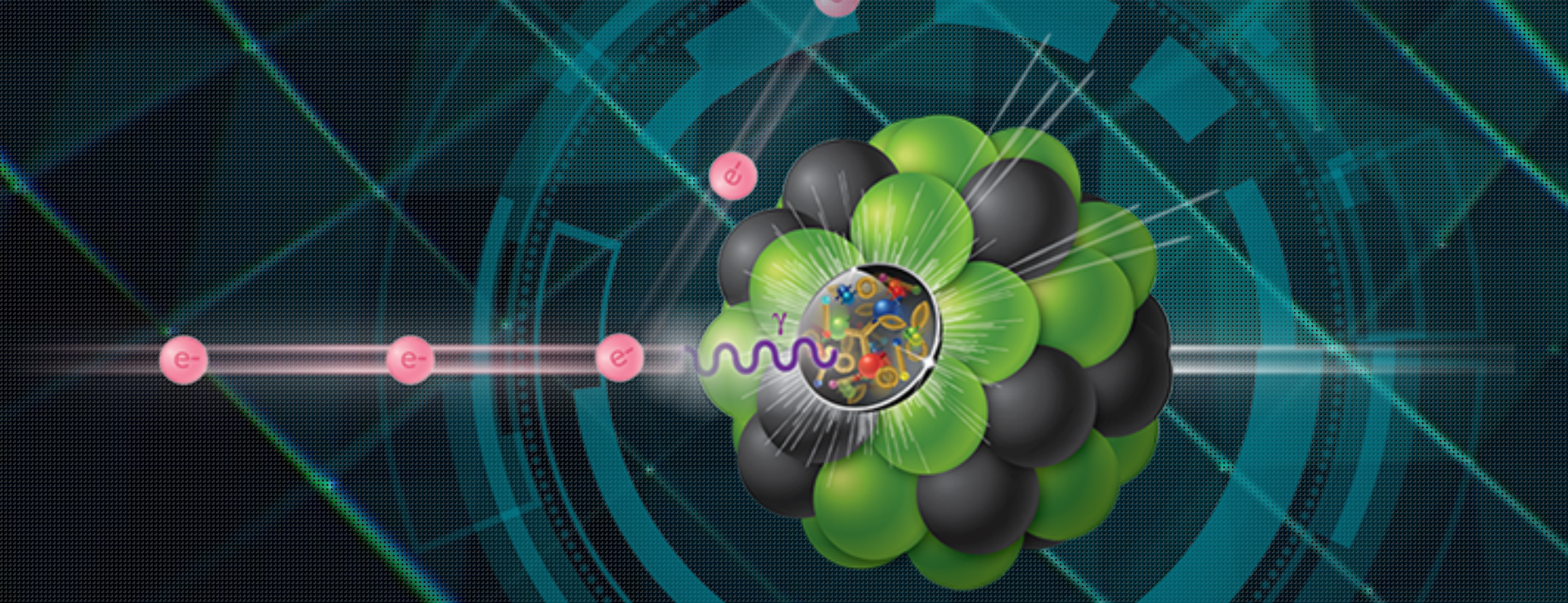
JLab 12 GeV

Generalized Parton Distributions



The Electron-Ion Collider

A machine that will unlock the secrets of the strongest force in Nature

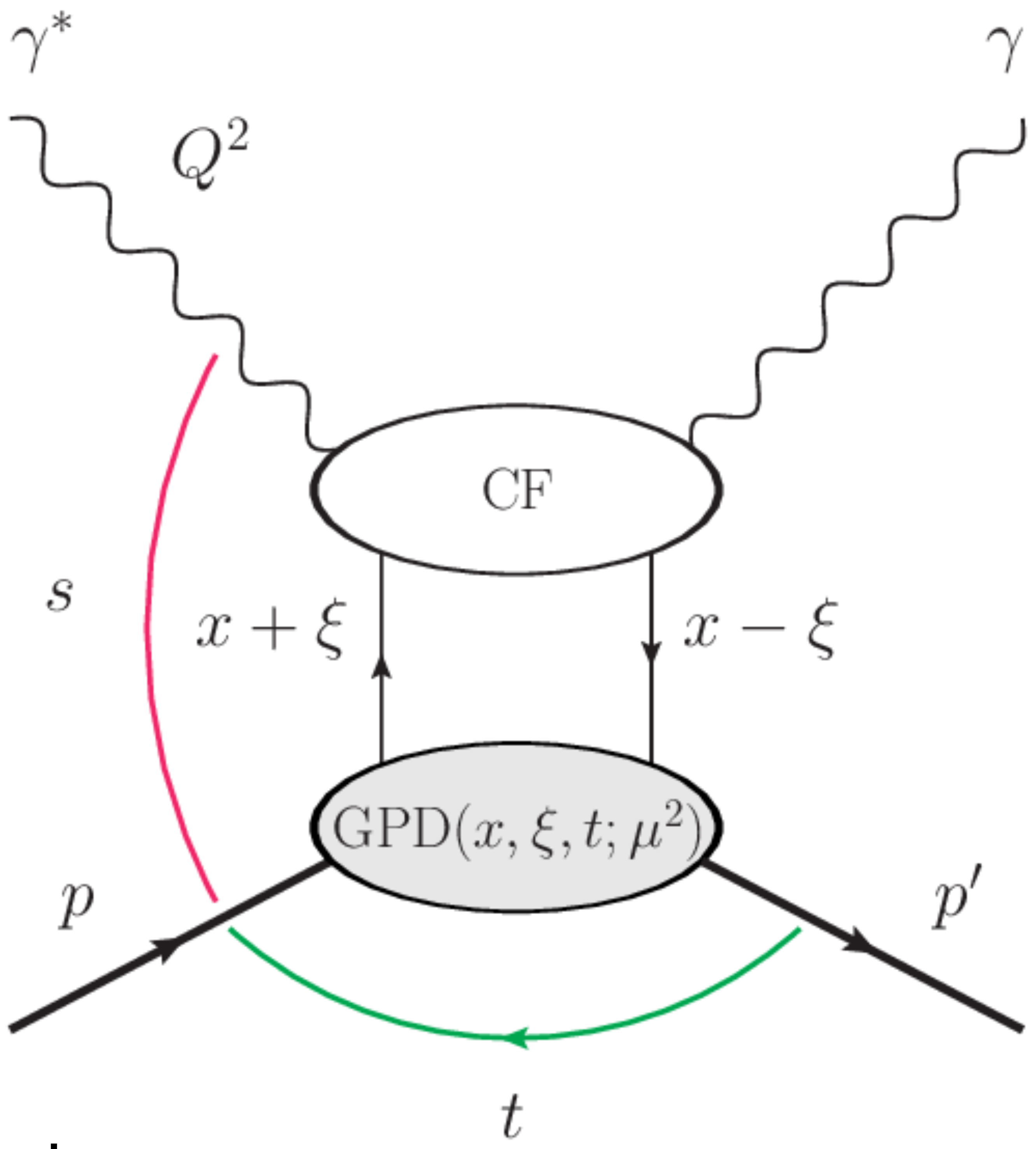


The computers and smartphones we use every day depend on what we learned about the atom in the last century. All information technology—and much of our economy today—relies on understanding the electromagnetic force between the atomic nucleus and the electrons that orbit it. The science of that force is well understood but we still know little about the microcosm within the protons and neutrons that make up the atomic nucleus. That's why Brookhaven Lab is building a new machine—an Electron-Ion Collider, or EIC—to look *inside* the nucleus and its protons and neutrons.

The EIC will be a particle accelerator that collides electrons with protons and nuclei to produce snapshots of those particles' internal structure—like a CT scanner for atoms. The electron beam will reveal the arrangement of the quarks and gluons that make up the protons and neutrons of nuclei. The force that holds quarks together, carried by the gluons, is the strongest force in Nature. The EIC will allow us to study this "strong nuclear force" and the role of gluons in the matter within and all around us. What we learn from the EIC could power the technologies of tomorrow.

taken from https://www.bnl.gov/eic/

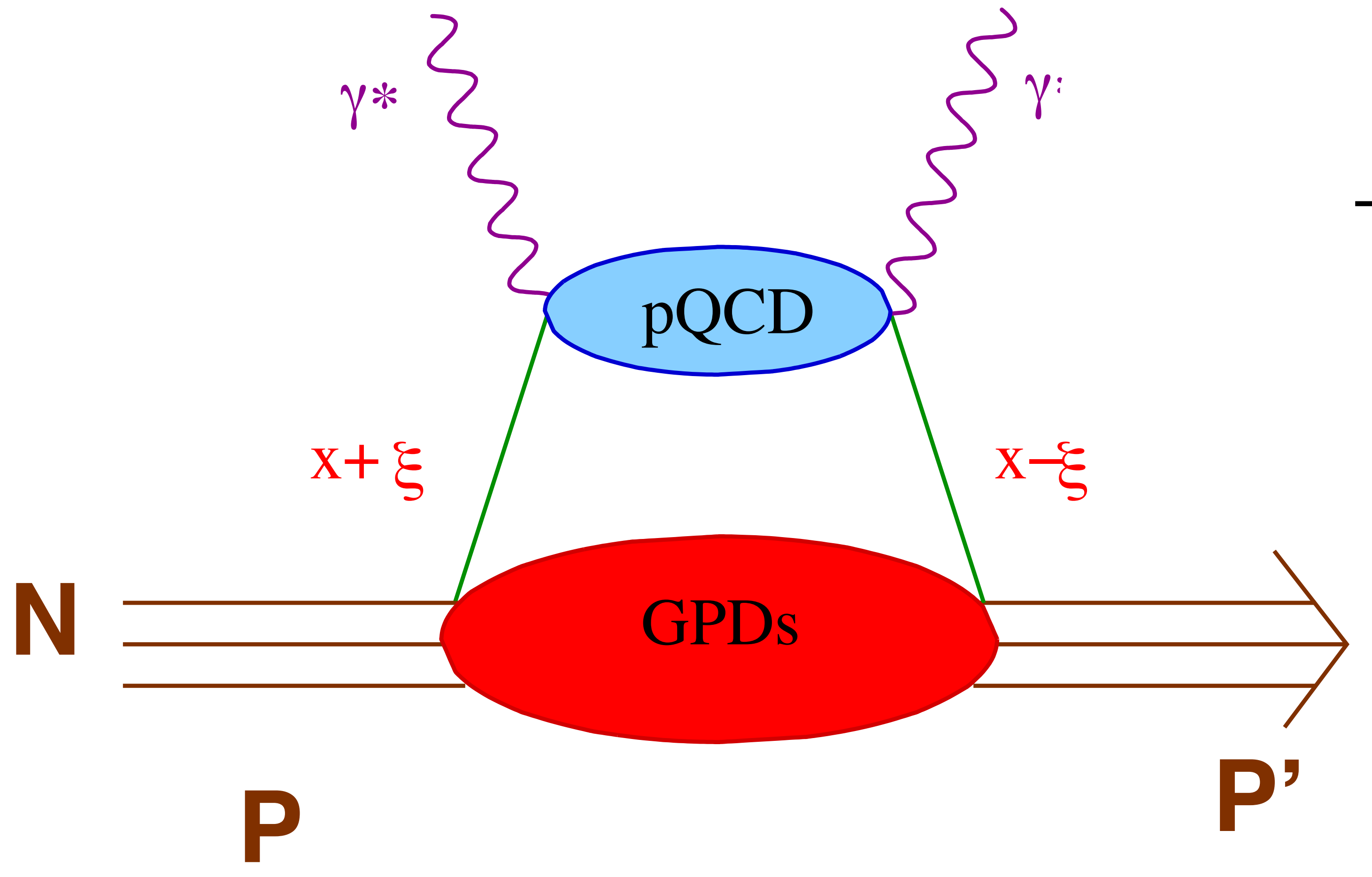
DVCS factorization

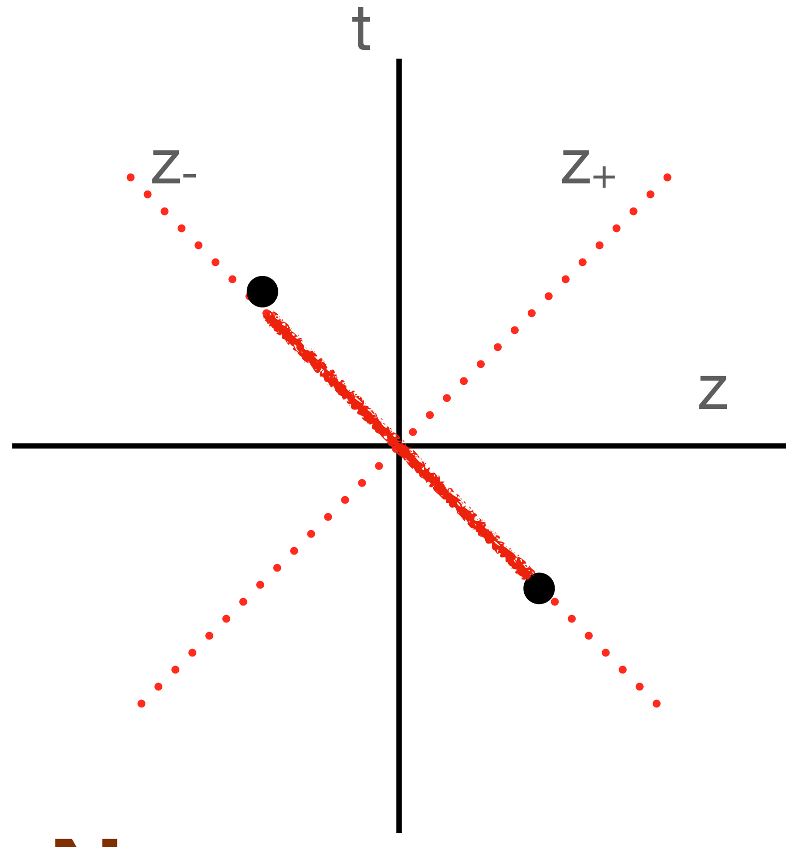


$$\mathcal{F}(\xi,t,Q^2) = \int_{-1}^{1} dx\, C(x,\xi,a_s(\mu),Q/\mu) G(x,\xi,t,\mu)$$

Ill-defined inverse problem —-> Lattice QCD computations are essential

Generalized Parton Distributions





$$\mathcal{O}_\Gamma(x) = \int \frac{d\lambda}{4\pi} e^{i\lambda x}\bar{q}\left(\frac{-\lambda n}{2}\right)\boldsymbol{\Gamma}\,\mathcal{P}e^{-ig\int_{-\lambda/2}^{\lambda/2} d\alpha\, n\cdot A(\alpha n)}\, q\left(\frac{\lambda n}{2}\right)$$

$$\Gamma = \not{n} \text{ or } \Gamma = \not{n}\gamma_5 \text{ or } \Gamma = n_\mu \sigma^{\mu\nu}\gamma_5$$

$$\Delta = P' - P \qquad \xi = -n\cdot\Delta/2 \qquad t = \Delta^2$$

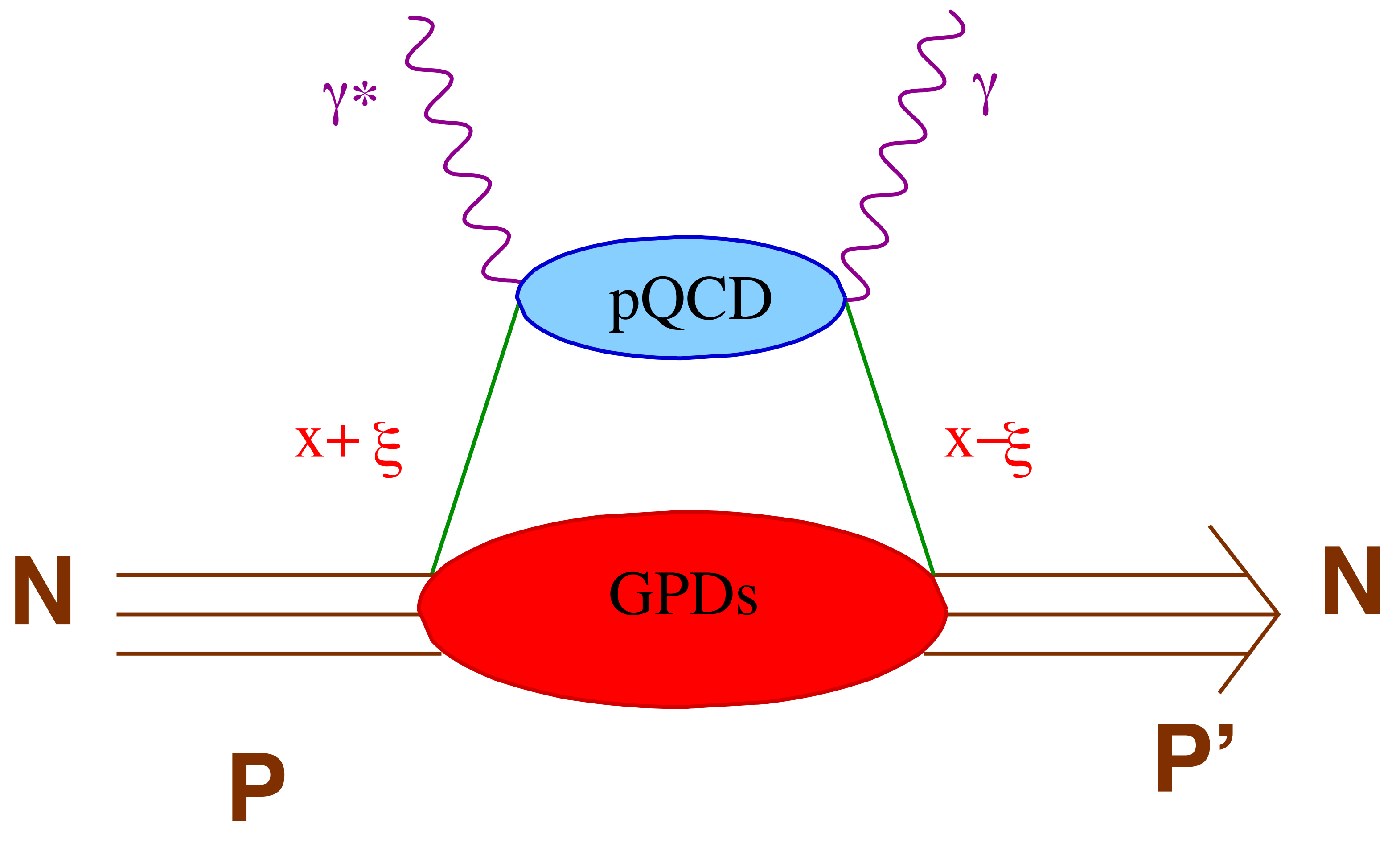
Vector: $$\langle P, s|\,\mathcal{O}_{\not{n}}(x)\,|P', s'\rangle = \bar{u}(p,s)\left[\not{n}\, H(x,\xi,t) + \frac{n_\mu \Delta_\nu}{2m} i\sigma^{\mu\nu}\, E(x,\xi,t)\right] u(p', s')$$

Axial Vector: $$\langle P, s|\,\mathcal{O}_{\not{n}\gamma_5}(x)\,|P', s'\rangle = \bar{u}(p,s)\left[\not{n}\gamma_5\, \tilde{H}(x,\xi,t) + \frac{n\cdot\Delta}{2m}\gamma_5\, \tilde{E}(x,\xi,t)\right] u(p', s')$$

Tensor:

$$\begin{aligned}\langle P, s|\,\mathcal{O}_{5T}(x)\,|P', s'\rangle &= \bar{u}(p,s)\left[n_\mu \sigma^{\mu k}\gamma_5 \left(H_T(x,\xi,t) - \frac{t}{2m^2}\tilde{H}_T\right) + \frac{\epsilon^{\mu\nu\alpha\beta}\Delta_\alpha \gamma_\beta}{2m}\left(E_T(x,\xi,t) + 2\tilde{H}_T(x,\xi,t)\right)\right. \\ &+ \left.\frac{n_\mu \Delta^{[\mu}\sigma^{\nu]\alpha}\gamma_5 \Delta_\alpha}{2m^2}\tilde{H}_T(x,\xi,t) + \frac{\epsilon^{\mu\nu\alpha\beta}P_\alpha\gamma_\beta}{m}\tilde{E}_T(x,\xi,t)\right] u(p', s')\end{aligned}$$

Generalized Parton Distributions



$$\mathcal{O}_\Gamma(x) = \int \frac{d\lambda}{4\pi} e^{i\lambda x} \bar{q}(\frac{-\lambda n}{2}) \mathbf{\Gamma}\, \mathcal{P} e^{-ig\int_{-\lambda/2}^{\lambda/2} d\alpha\, n\cdot A(\alpha n)} q(\frac{\lambda n}{2})$$

$$\Gamma = \not{n} \text{ or } \Gamma = \not{n}\gamma_5 \text{ or } \Gamma = n_\mu \sigma^{\mu\nu}\gamma_5$$

$$\Delta = P' - P \qquad \xi = -n\cdot\Delta/2 \qquad t = \Delta^2$$

Vector: → $H(x,\xi,t) \quad E(x,\xi,t)$

Axial Vector: → $\tilde{H}(x,\xi,t) \quad \tilde{E}(x,\xi,t)$

Tensor: → $H_T(x,\xi,t) \quad E_T(x,\xi,t)$
$\tilde{H}_T(x,\xi,t) \quad \tilde{E}_T(x,\xi,t)$

Generalized Parton Distributions

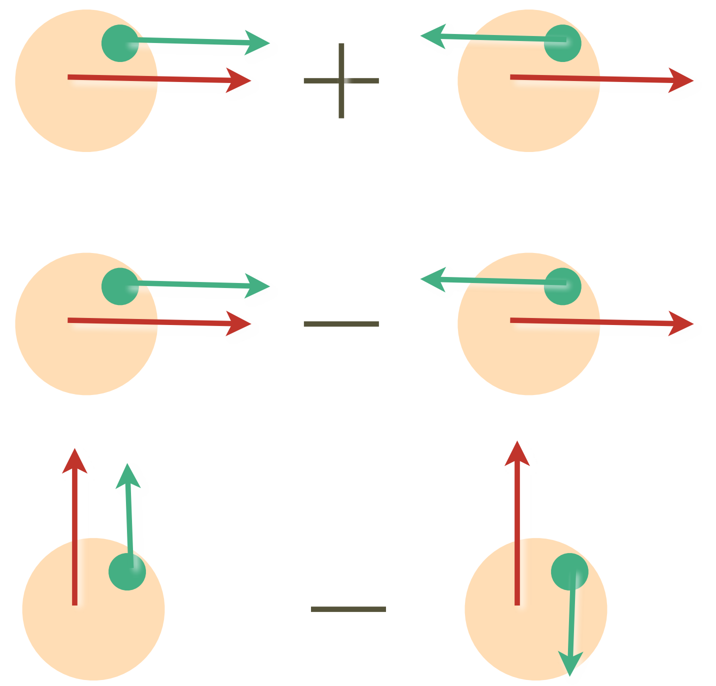
Unified Hadronic structure

Forward limit: t=0

$$H(x,0,0) = q(x)$$

$$\tilde{H}(x,0,0) = \Delta q(x)$$

$$H_T(x,0,0) = \delta q(x)$$



Local limit

Vector	$\int dx H(x,\xi,t) = F_1(t)$	$\int dx E(x,\xi,t) = F_2(t)$
Axial Vector	$\int dx \tilde{H}(x,\xi,t) = g_A(t)$	$\int dx \tilde{E}(x,\xi,t) = g_P(t)$
Tensor	$\int dx H_T(x,\xi,t) = g_T(t)$	

Moments of GPDs

Operator Product Expansion

Off forward Matrix elements of local operators $\langle P, S|\mathcal{O}|P', S'\rangle$

Unpolarized $$\mathcal{O}^{q}_{\{\mu_1\mu_2\cdots\mu_n\}} = \bar{q}\left[\left(\frac{i}{2}\right)^{n-1}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n} - trace\right]q$$

Polarized $$\mathcal{O}^{5q}_{\{\mu_1\mu_2\cdots\mu_n\}} = \bar{q}\left[\left(\frac{i}{2}\right)^{n-1}\gamma_5\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n} - trace\right]q$$

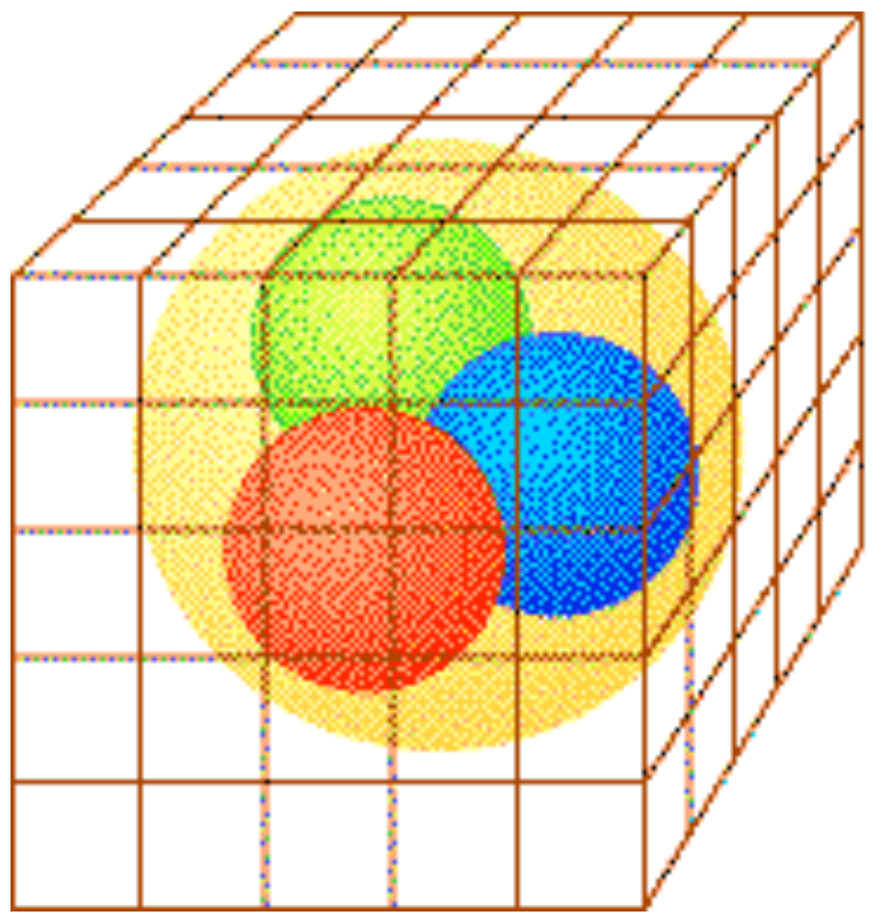
Transversity $$\mathcal{O}^{\sigma q}_{\rho\nu\mu_1\mu_2\cdots\mu_n} = \bar{q}[\left(\frac{i}{2}\right)^{n}\gamma_5\sigma_{\rho\nu}\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n} - traces]q$$

- Generalized form factors: $A_{nk}(t)$ $B_{nk}(t)$ $C_{nk}(t)$
- Moments of GPDs are polynomials in ξ with coefficients the generalized form factors.
- Breaking of rotational symmetry: Mixing with lower dimensional operators
  - Only first few moments can be computed on the lattice
- Perturbative renormalization has been used extensively
- Non-perturbative (ex. Rome-Southampton RI-MOM)

Lattice QCD

Defined on a Euclidean Lattice

- Lattice QCD: QCD on discrete Euclidean space time
    - The lattice regulates UV divergences
- QCD: the continuum limit of Lattice QCD



- Provides a numerical, non-perturbative method for computing correlation functions : Monte Carlo evaluation of integrals

$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\bar{\psi}]\mathcal{D}[\psi] \; e^{-\bar{\psi}D(U)\psi - S_g(U)}$$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U]\mathcal{D}[\bar{\psi}]\mathcal{D}[\psi] \; \mathcal{O}(\bar{\psi}, \psi, U) \; e^{-\bar{\psi}D(U)\psi - S_g(U)}$$

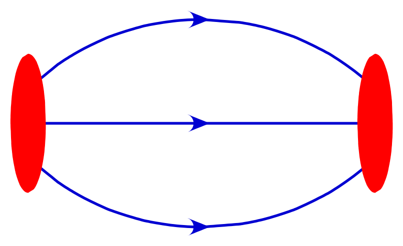
Computation of equal time matrix elements
LQCD

At sufficiently large T and t we get

$$C_{2pt} = \langle N(p,s,T)\bar{N}(p,s,0)\rangle = \langle 0|N,p,s\rangle \frac{e^{-E_p T}}{2E_p}\langle N,p,s,|0\rangle$$

Computation of ground state energy and overlap factors

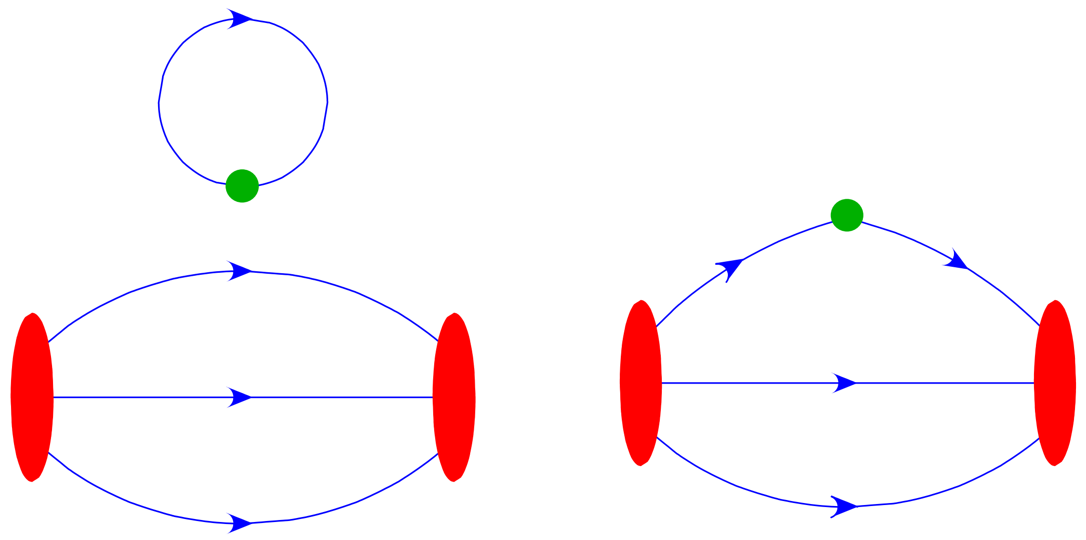
Two point function



$$C_{3pt} = \langle 0|N,p,s\rangle \frac{e^{-E_p(T-t)}}{2E_p}\langle N,p,s|\mathcal{O}|N,p',s'\rangle \frac{e^{-E'_p t}}{2E'_p}\langle N,p',s',|0\rangle$$

Computation of ground state matrix elements

Three point function

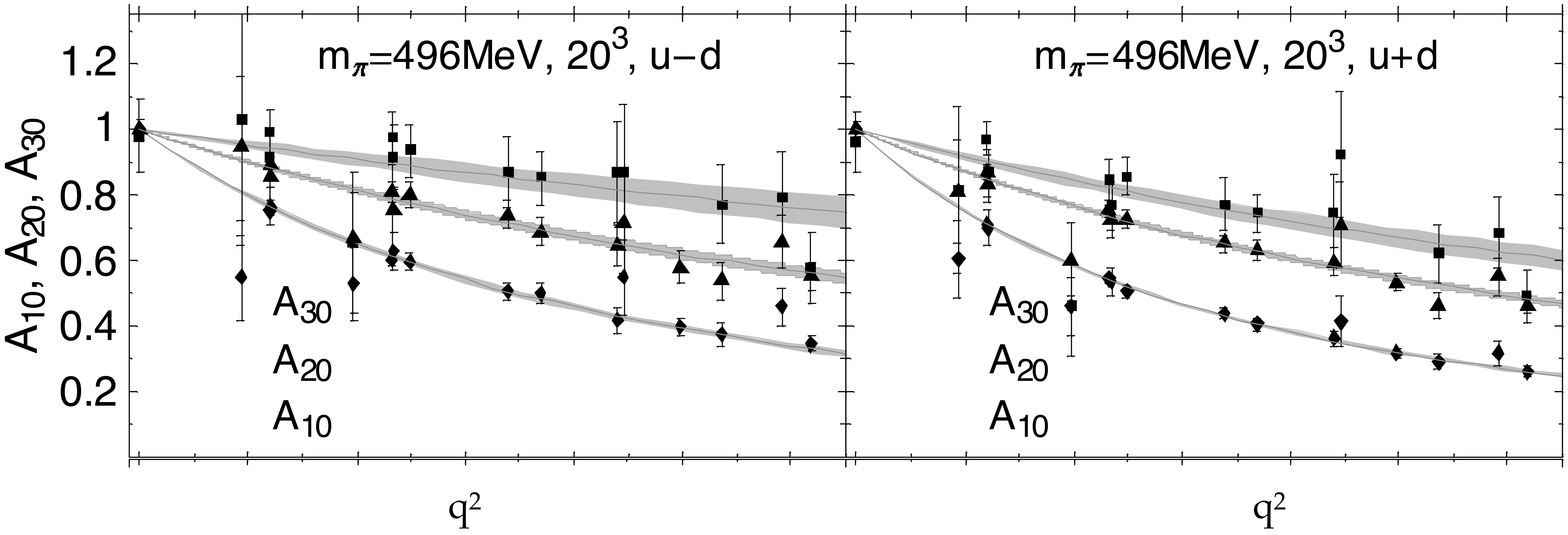


In practice we need to account for contributions from excited states

Energies and equal matrix elements are the same as those in Minkowski space

Briceno *et al* arXiv:1703.06072

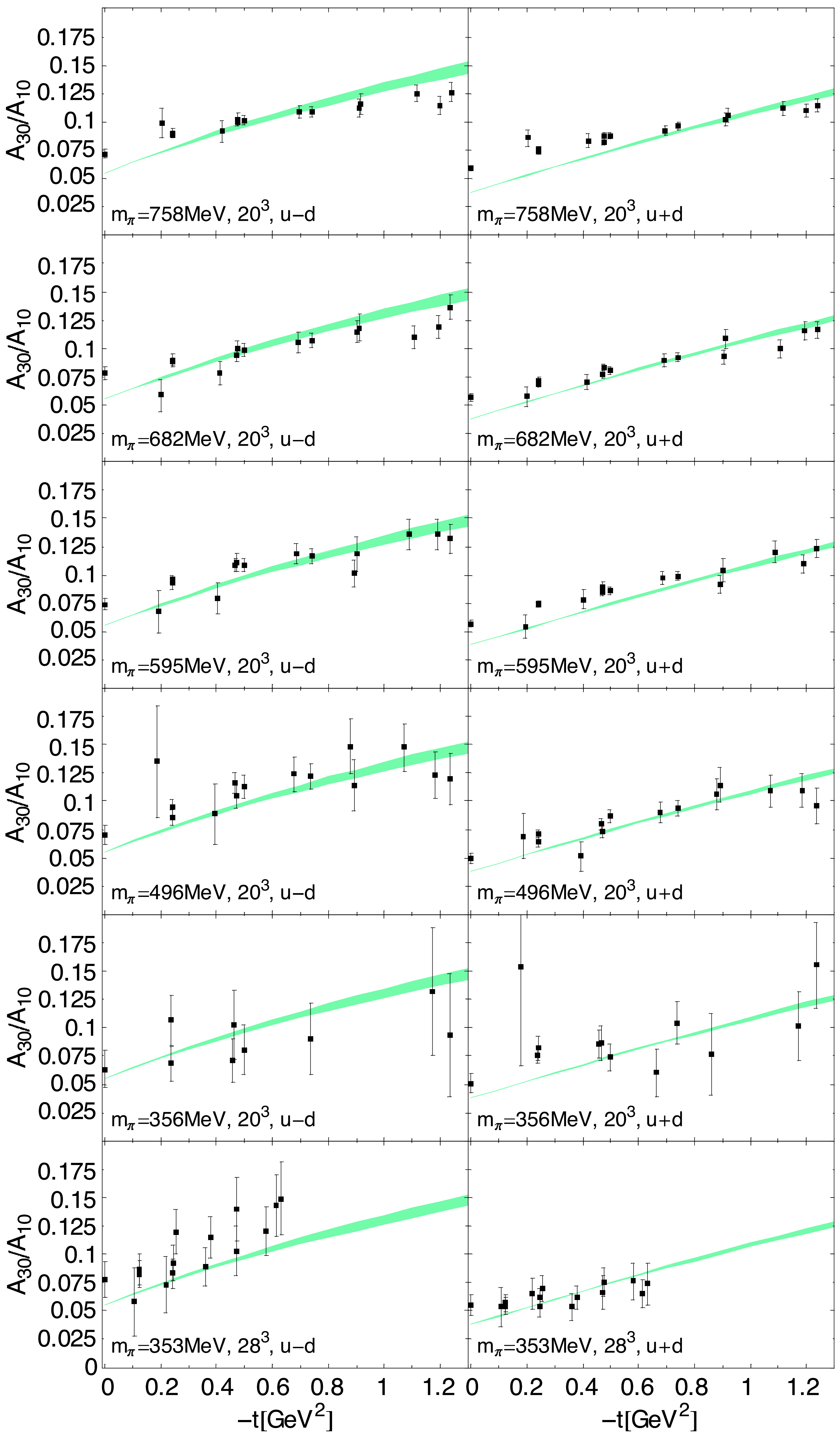
Moments of Generalized Parton Distributions



LHPC: Phys. Rev. D 77, 094502 (2008)
hep-lat/0705.4295

- Slope at small t decreases as we go to higher moments
- Higher moments dominated by higher x

Lattice computations of moments of GPDs



Unpolarized

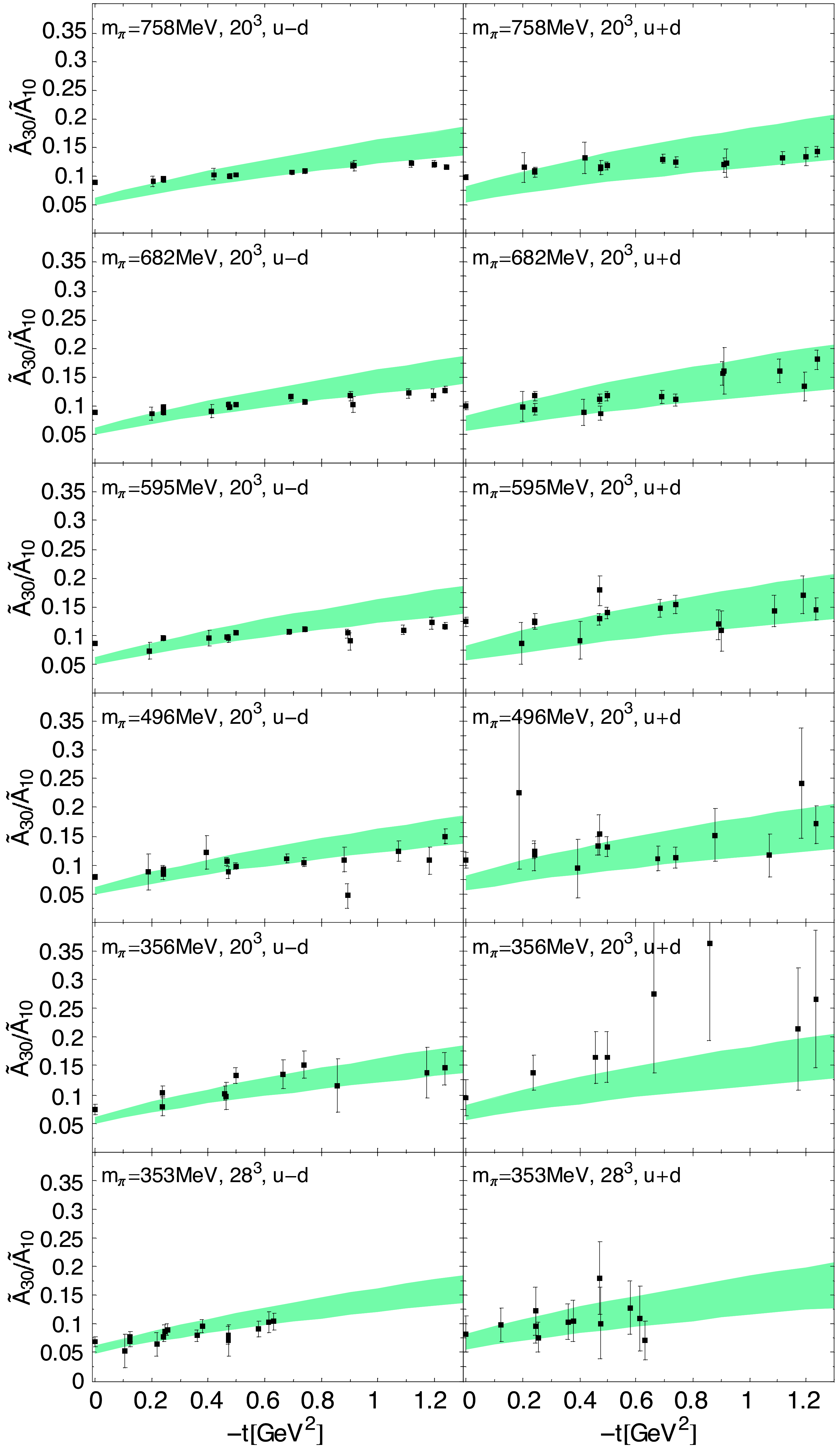
- Green band is a phenomenological parameterization using Form factor data, CTEQ patron distributions, and Regge Anzatz as input

[Deihl et.al. hep-ph/048173]

- As pion mass becomes smaller agreement is better.

LHPC: Phys. Rev. D 77, 094502 (2008)
hep-lat/0705.4295

Lattice computations of moments of GPDs



Polarized

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[Deihl et.al. hep-ph/048173]

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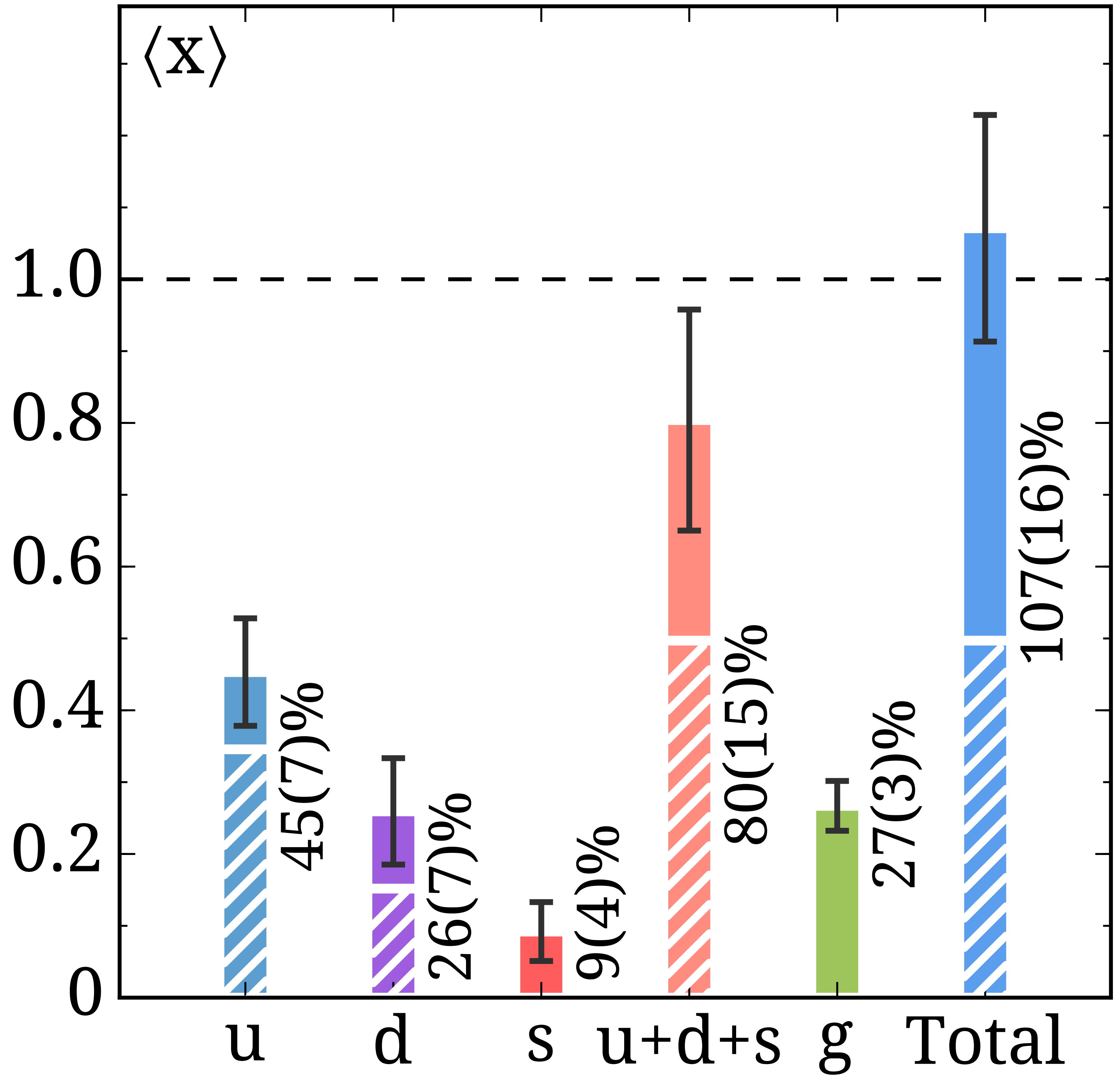
LHPC: Phys. Rev. D 77, 094502 (2008)
hep-lat/0705.4295

ETMC: *Phys. Rev. Lett.* 119 (2017) 142002

The Proton Momentum

sum rule

MSbar 2 GeV



$$A_{20}(0) = \langle x \rangle$$

Matrix element of the enegy momentum tensor operator

2013 revolution

Go beyond moments

- Goal: Compute full x-dependence (generalized) parton distribution functions (GPDFs)
- Operator product: Mellin moments are local matrix elements that can be computed in Lattice QCD
  - Power divergent mixing limits us to few moments
- X. Ji suggested an approach for obtaining PDFs from Lattice QCD
- First calculations quickly became available
- Older approaches based on the hadronic tensor

X. Ji, Phys.Rev.Lett. 110, (2013)

Y.-Q. Ma J.-W. Qiu (2014) 1404.6860

H.-W. Lin, J.-W. Chen, S. D. Cohen, and X. Ji, Phys.Rev. D91, 054510 (2015)

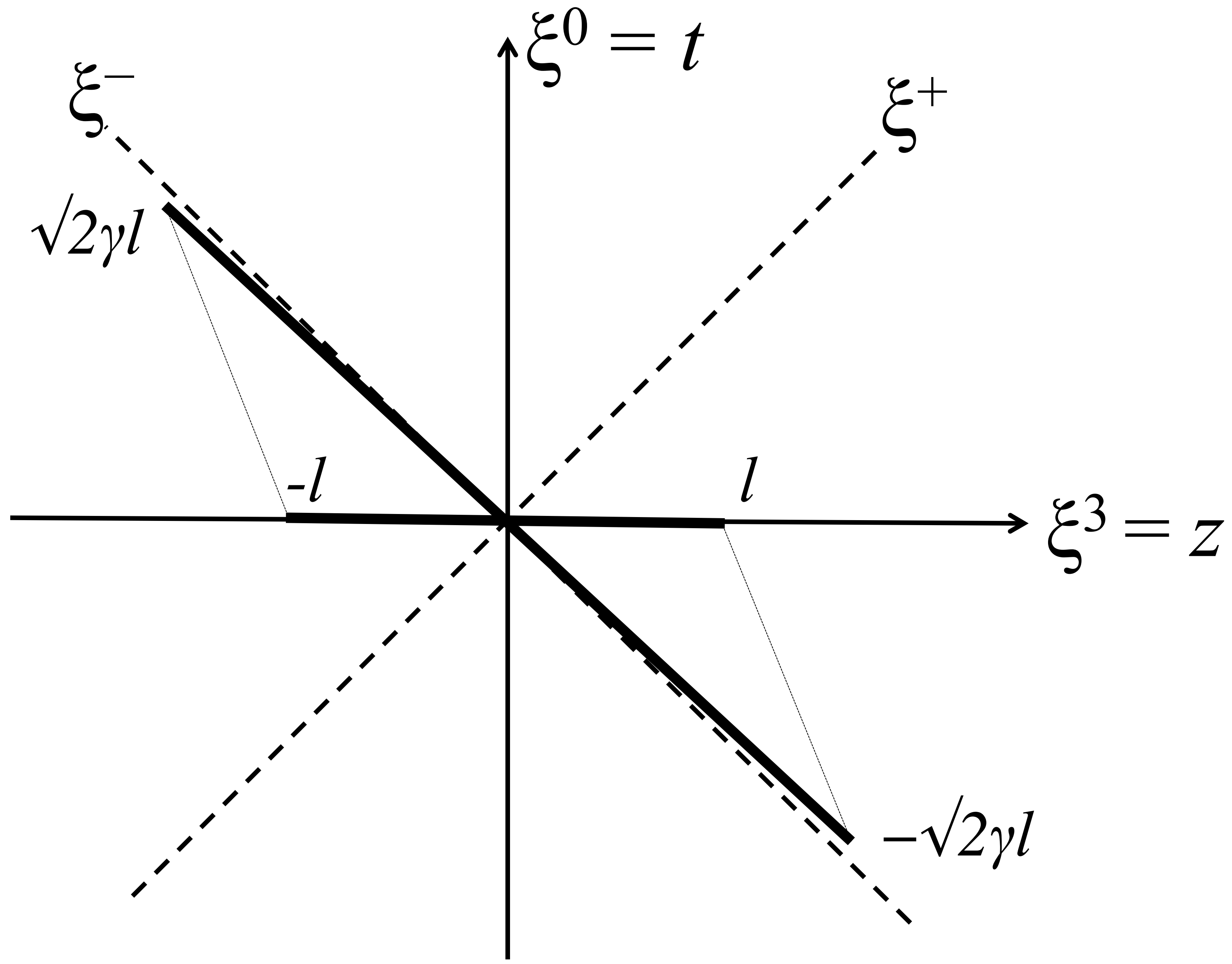
C. Alexandrou, et al, Phys. Rev. D92, 014502 (2015)

K-F Liu et al Phys. Rev. Lett. 72 (1994) , Phys. Rev. D62 (2000) 074501
Detmold and Lin 2005
M. T. Hansen et al arXiv:1704.08993.
UKQCD-QCDSF-CSSM Phys. Lett. B714 (2012), arXiv:1703.01153

X. Ji, Phys.Rev.Lett. 110, (2013)
X. Ji (2014) Sci. China Phys. Mech. Atron. 57 arXiv:1404.6680

Quasi-PDF
X. Ji's Basic idea

- Lattice QCD computes equal time matrix elements
- Displace quarks in space-like interval
- Boost states to infinite momentum
- On the frame of the proton displacement becomes light-like
- Infinite momentum not possible on the lattice
  - Perurbative matching from finite momentum
  - LaMET



Renormalization of UV divergences is required

Good Lattice Cross sections

Current-Current Correlators

Y.-Q. Ma J.-W. Qiu (2014) arXiv:1404.6860
Y.-Q. Ma J.-W. Qiu (2017) arXiv:1709.3018

4-quark bi-local matrix elements:

$$\sigma_n(\nu, z^2) = \langle P \mid T\{\mathcal{O}_n(z)\} \mid P\rangle$$

equal time matrix element

Ex.

$$\begin{aligned}
\mathcal{O}_S(z) &= (z^2)^2 Z_S^2 [\bar\psi_q \psi_q](z)[\bar\psi_q \psi](0) \\
\mathcal{O}_{V'}(z) &= z^2 Z_{V'}^2 [\bar\psi_q (z\cdot\gamma)\psi_{q'}](z)[\bar\psi_{q'} z\cdot\gamma\psi](0),
\end{aligned}$$

Short distance factorization:

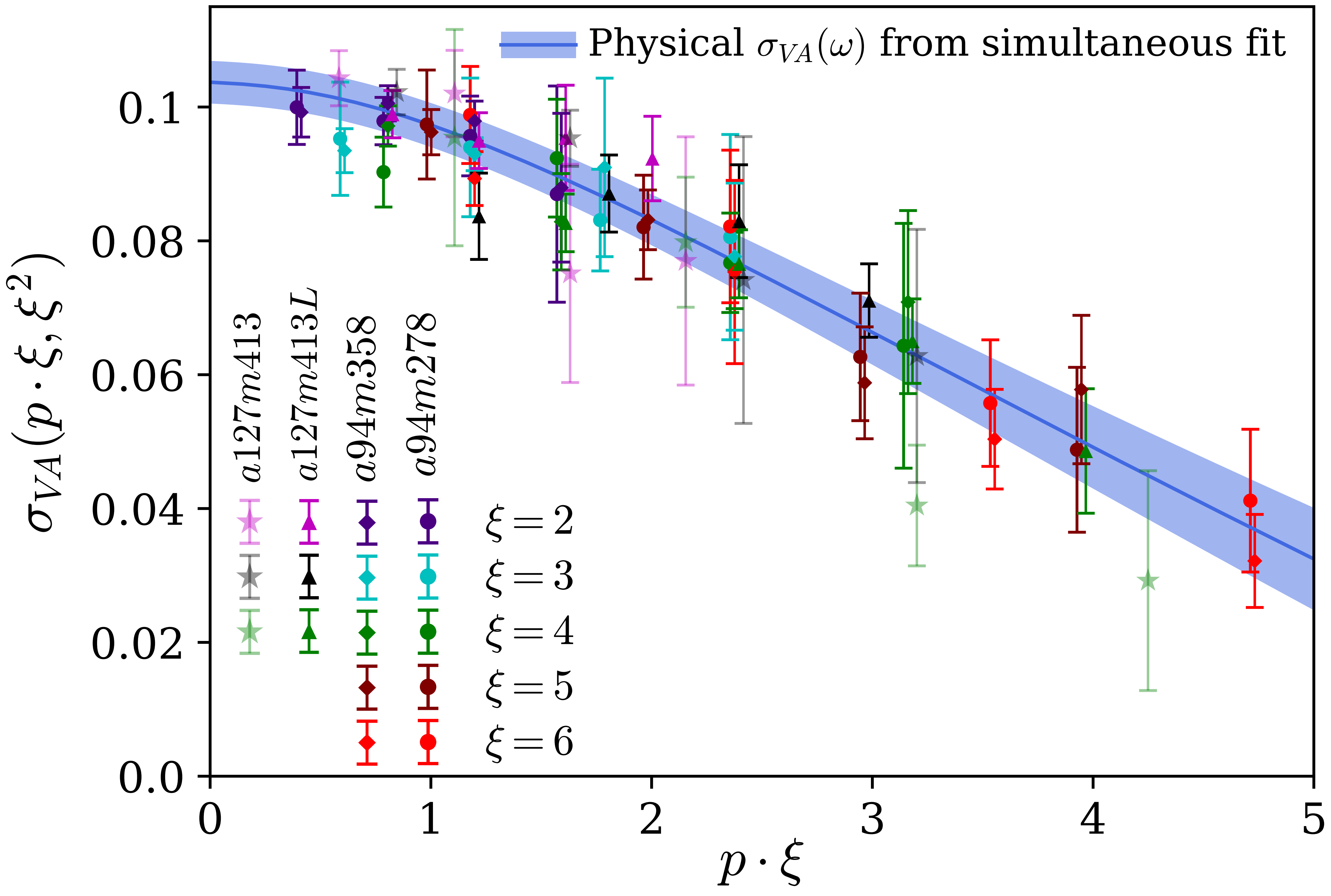
$$\sigma_n(\nu, z^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2)\, K_n^a(x\nu, z^2\mu^2) + \mathcal{O}(z^2\Lambda_{\rm QCD}^2),$$

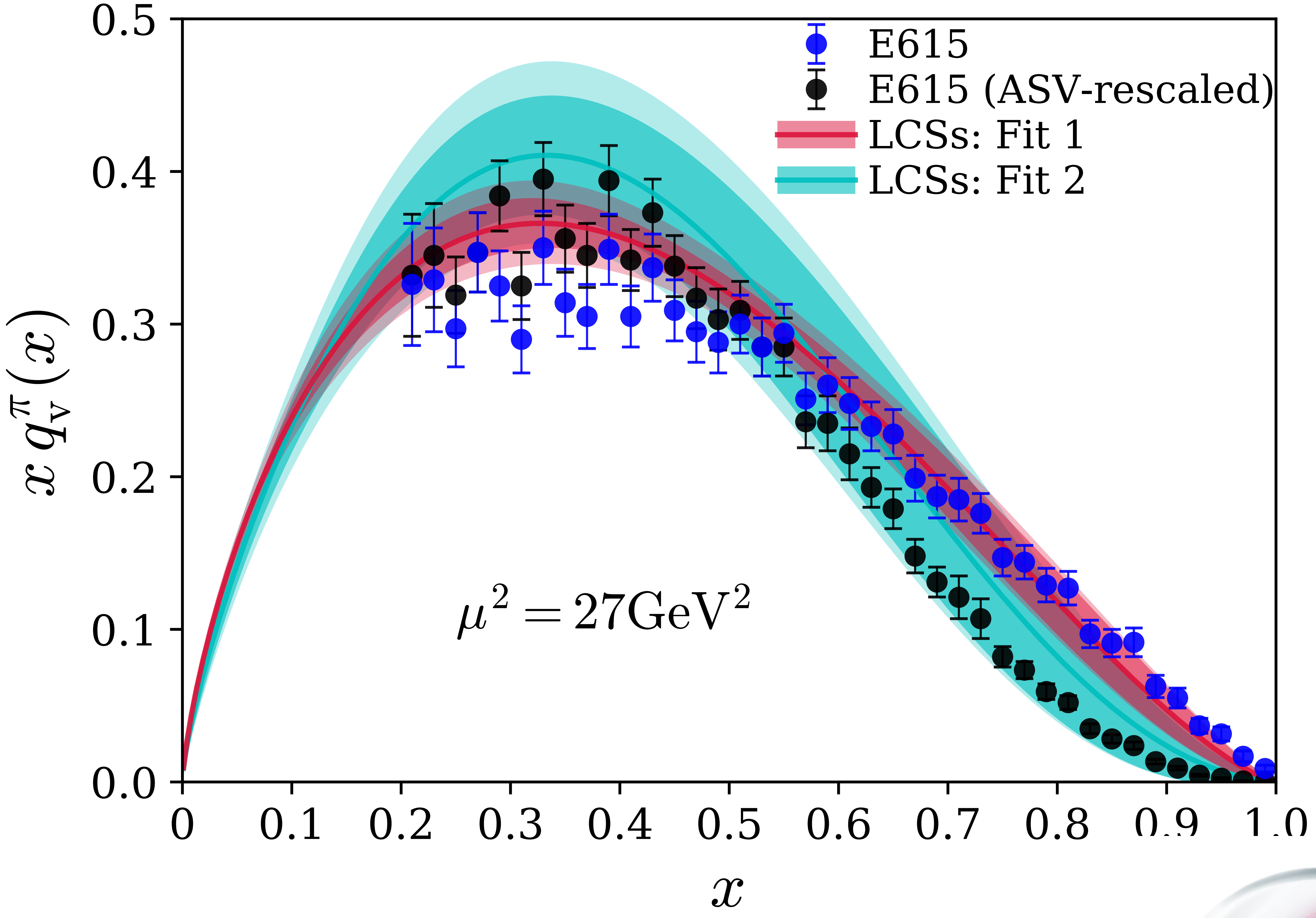
PDFs can be obtained

Renormalization of UV divergences of local operators is required

Pion PDF from current-current correlators

- R. Sufian *et al* , e-Print: 2001.04960 [hep-lat]





3 pion masses, 2 lattice spacings, 2 volumes

Pseudo-PDFs

An alternative point of view

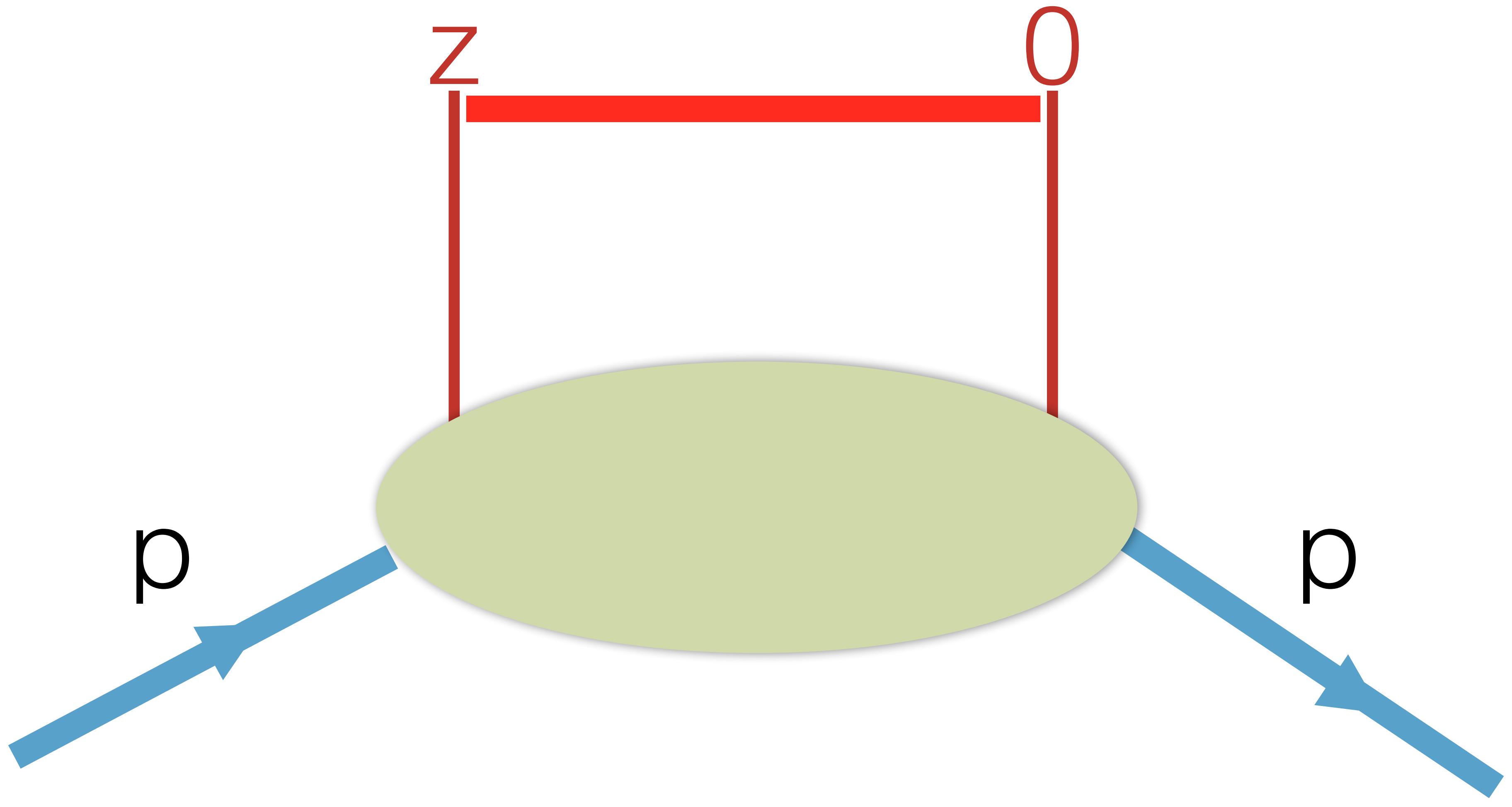
A. Radyushkin Phys.Lett. B767 (2017)

Unpolarized PDFs proton:

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^\alpha\,\hat{E}(0,z;A)\psi(z)|p\rangle$$

$$\hat{E}(0,z;A) = \mathcal{P}\exp\left[-ig\int_0^z \mathrm{d}z'_\mu\, A^\mu_\alpha(z')T_\alpha\right]$$

space-like separation of quarks



Lorentz decomposition:

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha\mathcal{M}_p(-(zp),-z^2) + z^\alpha\mathcal{M}_z(-(zp),-z^2)$$

A. Radyushkin Phys.Lett. B767 (2017)

Pseudo-PDFs

Connection to light cone PDFs

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha \mathcal{M}_p(-(zp),-z^2) + z^\alpha \mathcal{M}_z(-(zp),-z^2)$$

Collinear PDFs: Choose

$$z = (0, z_-, 0)$$
$$p = (p_+, 0, 0)$$
$$\gamma^+$$

$$\mathcal{M}^+(z,p) = 2p^+ \mathcal{M}_p(-p_+ z_-, 0)$$

Definition of PDF:

$$\mathcal{M}_p(-p_+ z_-, 0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+ z_-}$$

Lorentz invariance allows for the computation of invariant form factors in any frame

Use equal time kinematics for LQCD

A. Radyushkin Phys.Lett. B767 (2017)

Lattice QCD calculation:

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar\psi(0)\,\gamma^\alpha\,\hat{E}(0,z;A)\psi(z)|p\rangle$$

Choose $\quad p = (p_0, 0, 0, p_3) \quad z = (0, 0, 0, z_3) \quad \gamma^0$

On shell equal time matrix element
computable in Euclidean space

Briceno *et al* arXiv:1703.06072

Obtaining only the relevant

$$\mathcal{M}_p(\nu, z_3^2) = \frac{1}{2p_0}\mathcal{M}^0(z_3, p_3)$$

$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, \mathcal{M}_p(\nu, -z^2)e^{-ix\nu}$$

the pseudo-PDF $\quad x \in [-1, 1]$

Radyusking Phys.Lett. B767 (2017) 314-320

Chosing γ^0 was also suggested also by M. Constantinou at GHP2017 based
on an operator mixing argument for the renormalized matrix element

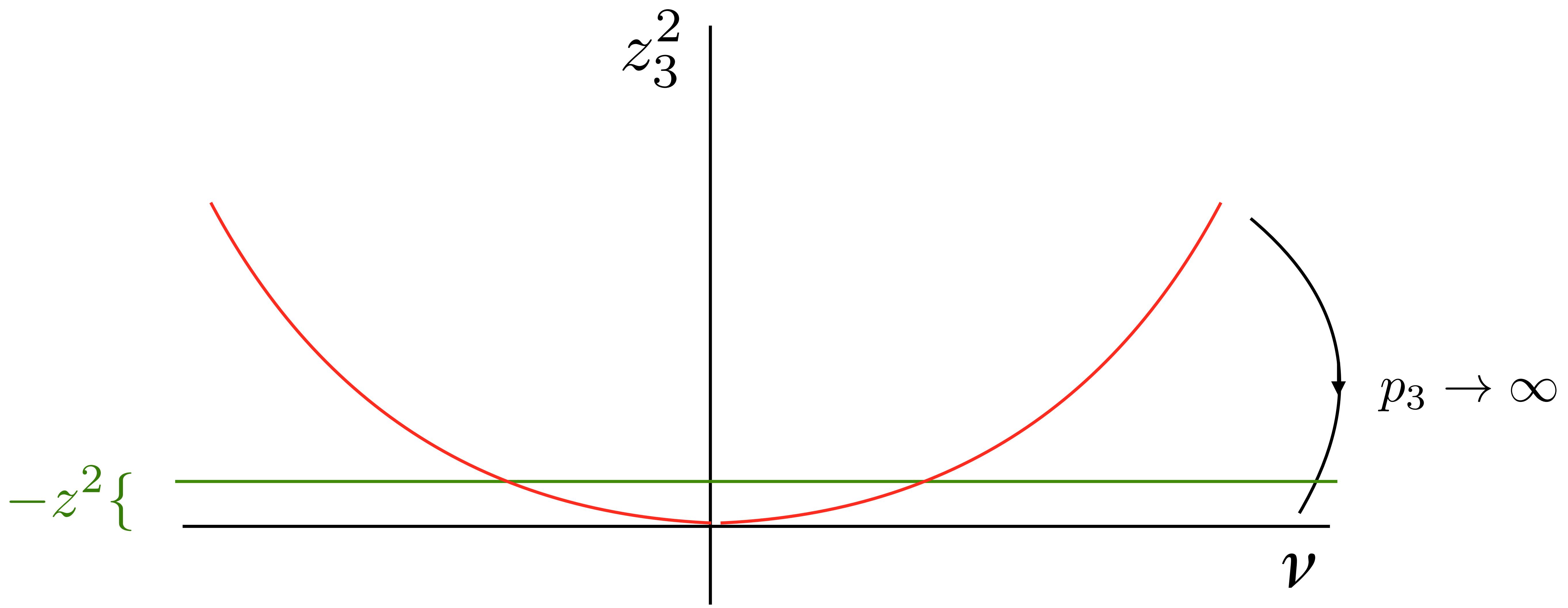
Alexandrou *et al* arXiv:1706.00265

$$Q(y,p_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu \mathcal{M}_p(\nu, \nu^2/p_3^2) e^{-iy\nu}$$

Ji's quasi-PDF

Large values of $z_3 = \nu/p_3$ are problematic

Alternative approach to the light-cone:



$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, \mathcal{M}_p(\nu, -z^2) e^{-ix\nu}$$

PDFs can be recovered $-z^2 \to 0$

Note that $x \in [-1, 1]$

Radyusking Phys.Lett. B767 (2017) 314-320

Collinear singularity at $-z^2 \to 0$

Factorization of collinear divergence at

$$\mathcal{M}_p(\nu, z^2) = \int_0^1 d\alpha\, \mathcal{C}(\alpha, z^2\mu^2, \alpha_s(\mu))\mathcal{Q}(\alpha\nu, \mu) \ + \mathcal{O}(z^2\Lambda_{qcd}^2)$$

$\mathcal{Q}(\nu, \mu)$ is called the Ioffe time PDF

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

$$\mathcal{Q}(\nu, \mu) = \int_{-1}^1 dx\, e^{-ix\nu} f(x, \mu)$$

$\overline{MS}$ Calculation of the Kernel

Radyushkin Phys.Rev. D98 (2018) no.1, 014019
Izubuchi et al. Phys.Rev. D98 (2018) no.5, 056004
Zhang et al. Phys.Rev. D97 (2018) no.7, 074508

Ioffe time $-z \cdot p = \nu$

Statistical noise

Nucleon with momentum P two-point function:

$$C_{2p}(P,t) = \langle O_N(P,t) O_N^\dagger(P,0)\rangle \sim \mathcal{Z} e^{-E(P)t}$$

Variance of nucleon two-point function:

$$\mathrm{var}\left[C_{2p}(P,t)\right] = \langle O_N(P,t) O_N(P,t)^\dagger O_N(P,0) O_N^\dagger(P,0)\rangle \sim \mathcal{Z}_{3\pi} e^{-3m_\pi t}$$

Variance is independent of the momentum

$$\frac{\mathrm{var}\left[C_{2p}(P,t)\right]^{1/2}}{C_{ap}(P,t)} \sim \frac{\mathcal{Z}}{\mathcal{Z}_{3\pi}} e^{[E(P)-3/2m_\pi]t}$$

Statistical accuracy drops exponentially with the increasing momentum limiting the maximum achievable momentum.

Lattice QCD requirements

$$aP_{max} = \frac{2\pi}{4} \sim \mathcal{O}(1)$$

$$a \sim 0.1fm \to P_{max} = 10\Lambda$$
$$a \sim 0.05fm \to P_{max} = 20\Lambda$$

$$\Lambda \sim 300MeV$$

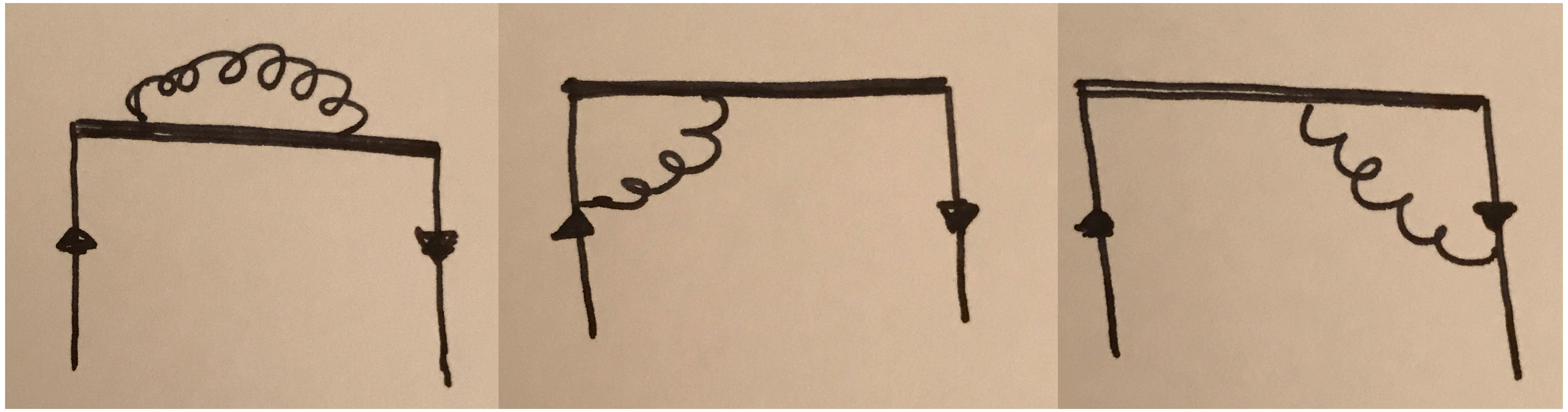
For practical calculations large momentum is needed

*Higher twist effect suppression (qpdfs)

*Wide coverage of Ioffe time ν

P= 3 GeV is already demanding due to statistical noise
achievable with easily accessible lattice spacings

P= 6 GeV exponentially harder
maybe intractable without new ideas



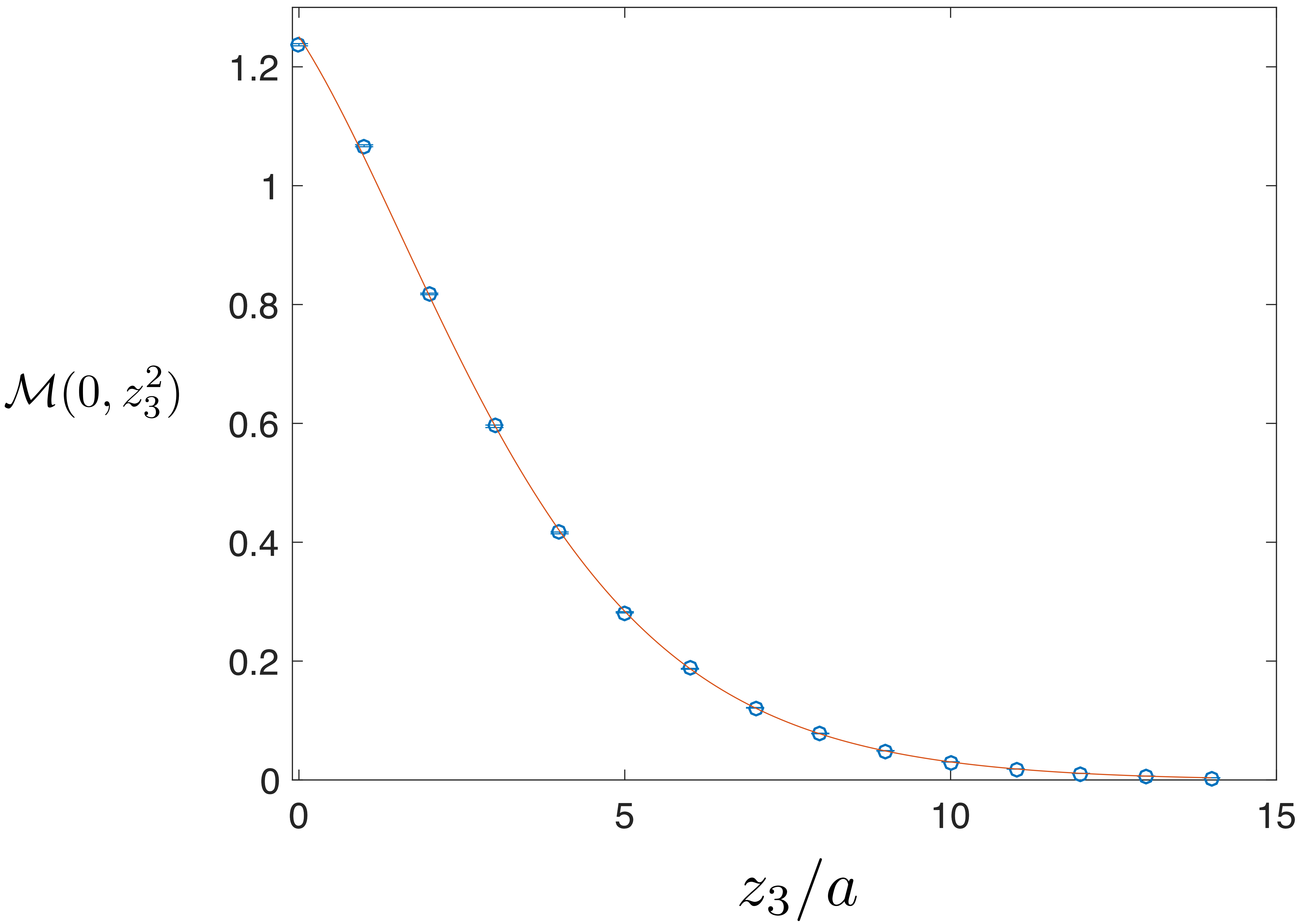
One loop calculation of the UV divergences results in

$$\mathcal{M}^0(z,P,a) \sim e^{-m|z|/a}\left(\frac{a^2}{z^2}\right)^{2\gamma_{end}}$$

after re-summation of one loop result resulting exponentiation

- J.G.M.Gatheral,Phys.Lett.133B,90(1983)
- J.Frenkel, J.C.Taylor,Nucl.Phys.B246,231(1984),
- G.P.Korchemsky, A.V.Radyushkin,Nucl.Phys.B283,342(1987).

UV divergences appear multiplicatively



Cusp indicates “linear” divergence of Wilson line

Consider the ratio $$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}_p(\nu, z_3^2)}{\mathcal{M}_p(0, z_3^2)}$$

UV divergences will cancel in this ratio resulting a renormalization group invariant (RGI) function

The collinear divergences at z_3^2=*0* limit only appear in the numerator

The lattice regulator can now be removed

$\mathfrak{M}^{cont}(\nu, z_3^2)$ Universal independent of the lattice

Its Fourier transformation with respect to ν is a particular definition of a PDF

$\mathcal{M}_p(0, 0) = 1$ Isovector matrix element

Continuum limit matching to $\overline{MS}$ computed at 1-loop

Radyushkin Phys.Rev. D98 (2018) no.1, 014019
Zhang et al. Phys.Rev. D97 (2018) no.7, 074508

$$\mathfrak{M}(\nu, z^2) = \int_0^1 dx\, q_v(x,\mu)\mathcal{K}(x\nu, z^2\mu^2) \ + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu)(z^2)^k \,.$$

$$\mathcal{K}(x\nu, z^2\mu^2) = \cos(x\nu) - \frac{\alpha_s}{2\pi} C_F \left[\ln(e^{2\gamma_E+1} z^2\mu^2/4)\tilde{B}(x\nu) + \tilde{D}(x\nu) \right]$$

$$\tilde{B}(x) = \frac{1-\cos(x)}{x^2} + 2\sin(x)\frac{x\mathrm{Si}(x)-1}{x} + \frac{3-4\gamma_E}{2}\cos(x) + 2\cos(x)\left[\mathrm{Ci}(x) - \ln(x)\right]$$

$$\tilde{D}(x) = x\mathrm{Im}\left[e^{ix}{}_3F_3(111;222;-ix)\right] - \frac{2-(2+x^2)\cos(x)}{x^2}$$

Polynomial corrections to the Ioffe time PDF may be suppressed

B. U. Musch, *et al* Phys. Rev. D 83, 094507 (2011)
M. Anselmino et al. 10.1007/JHEP04(2014)005
A. Radyushkin Phys.Lett. B767 (2017)

$$\mu^2 \frac{d}{d\mu^2} \mathcal{Q}(\nu,\mu^2) = -\frac{2}{3}\frac{\alpha_s}{2\pi}\int_0^1 du\, B(u)\, \mathcal{Q}(u\nu,\mu^2)$$

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

DGLAP kernel in position space

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

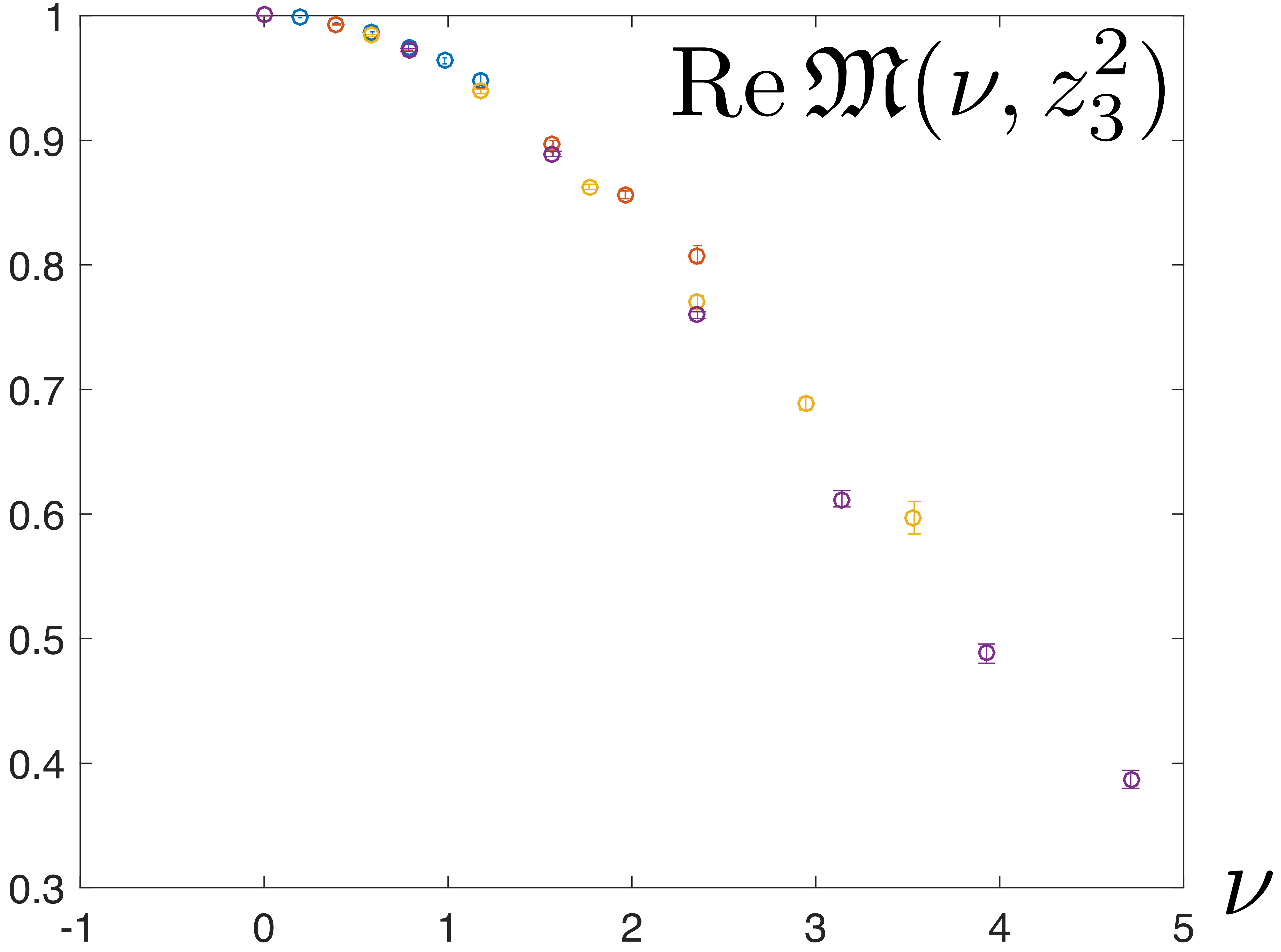
At 1-loop

$$\mathcal{Q}(\nu,\mu'^2) = \mathcal{Q}(\nu,\mu^2) \; -\frac{2}{3}\frac{\alpha_s}{2\pi}\ln(\mu'^2/\mu^2)\int_0^1 du\, B(u)\, \mathcal{Q}(u\nu,\mu^2)$$

Which implies (ignoring higher twist)

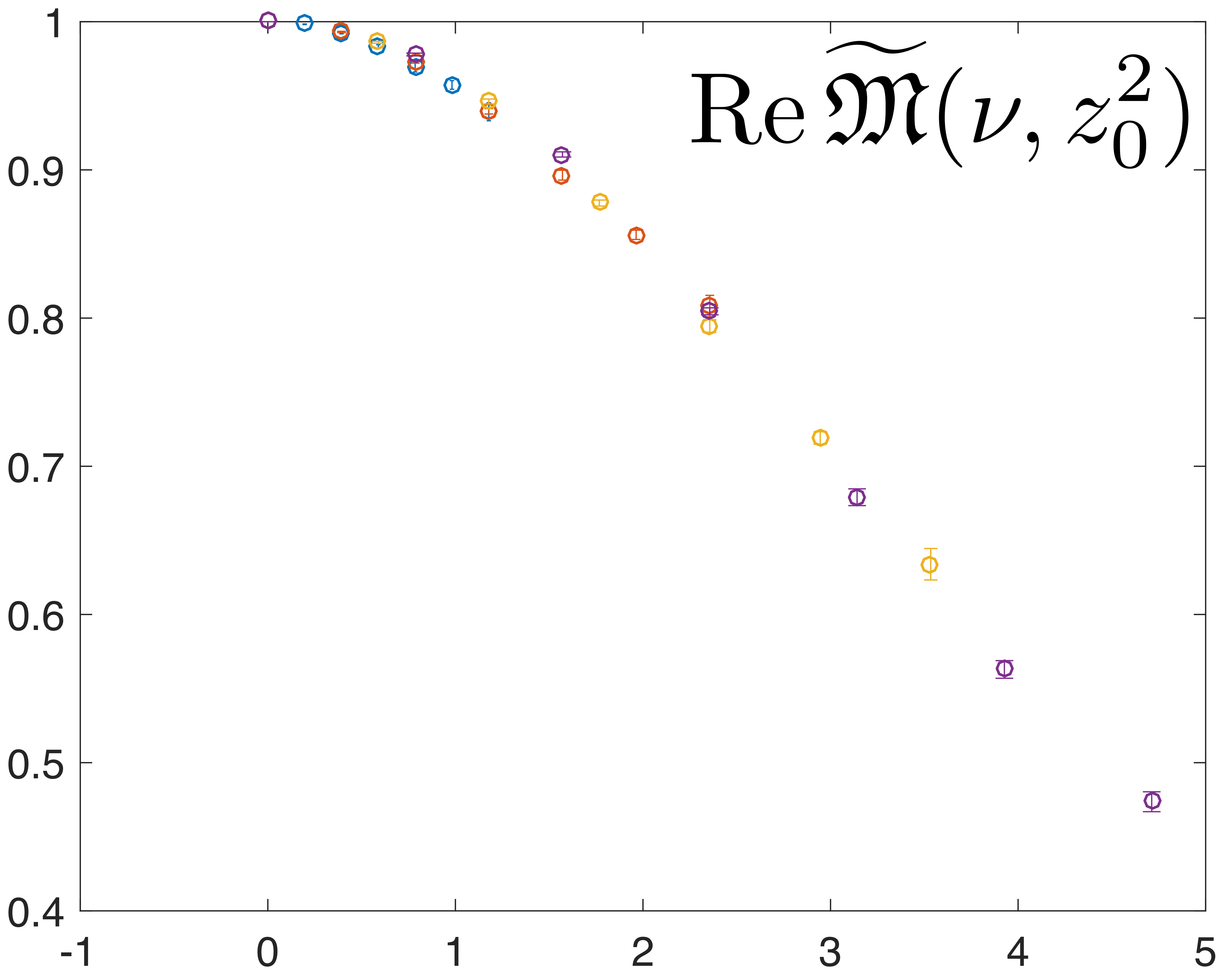
$$\mathfrak{M}(\nu,z'^2) = \mathfrak{M}(\nu,z^2) \; -\frac{2}{3}\frac{\alpha_s(z^2)}{\pi}\ln(z'^2/z^2)\int_0^1 du\, B(u)\, [\mathfrak{M}(u\nu,z^2)$$

Quenched QCD



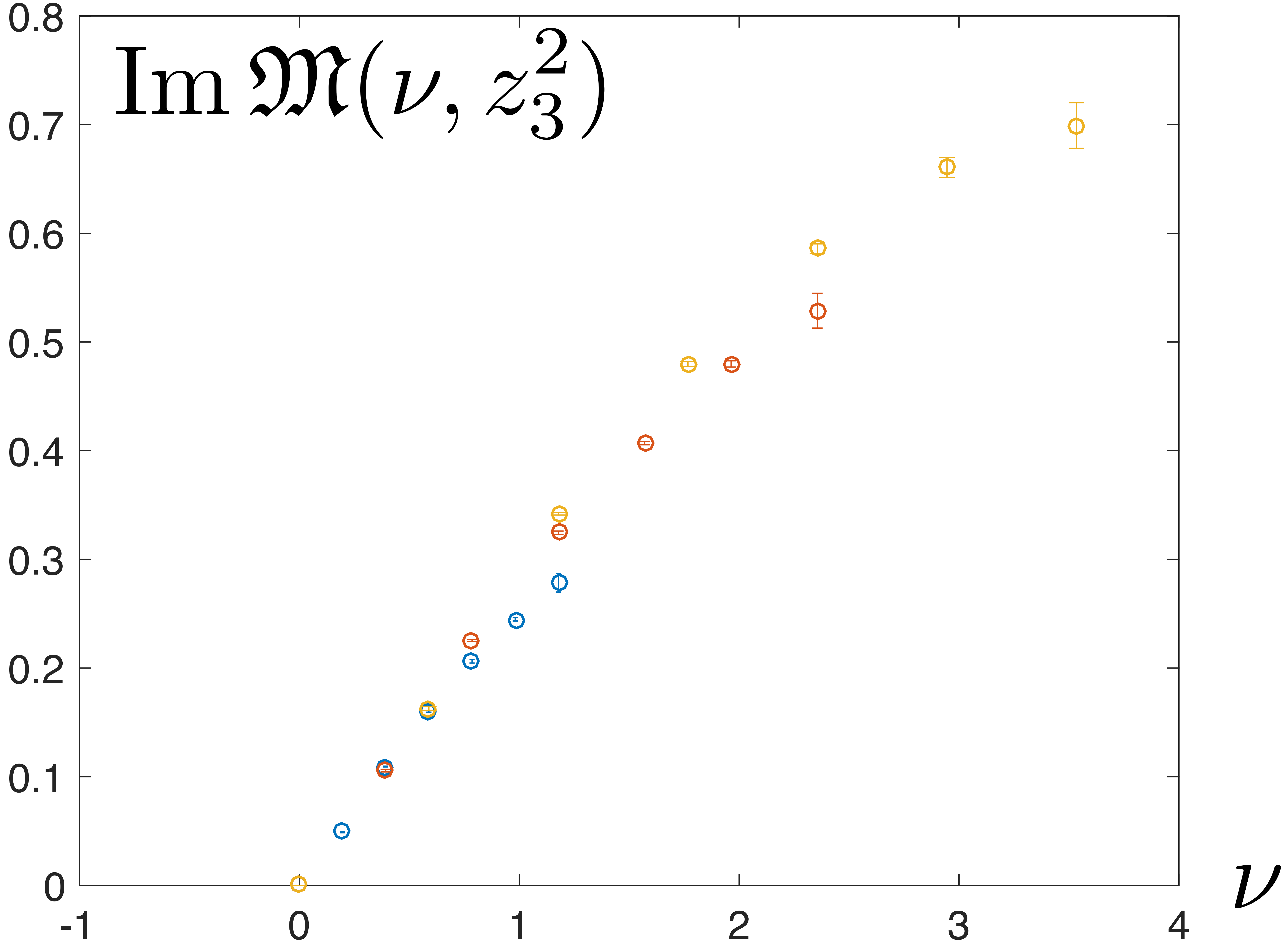
Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD



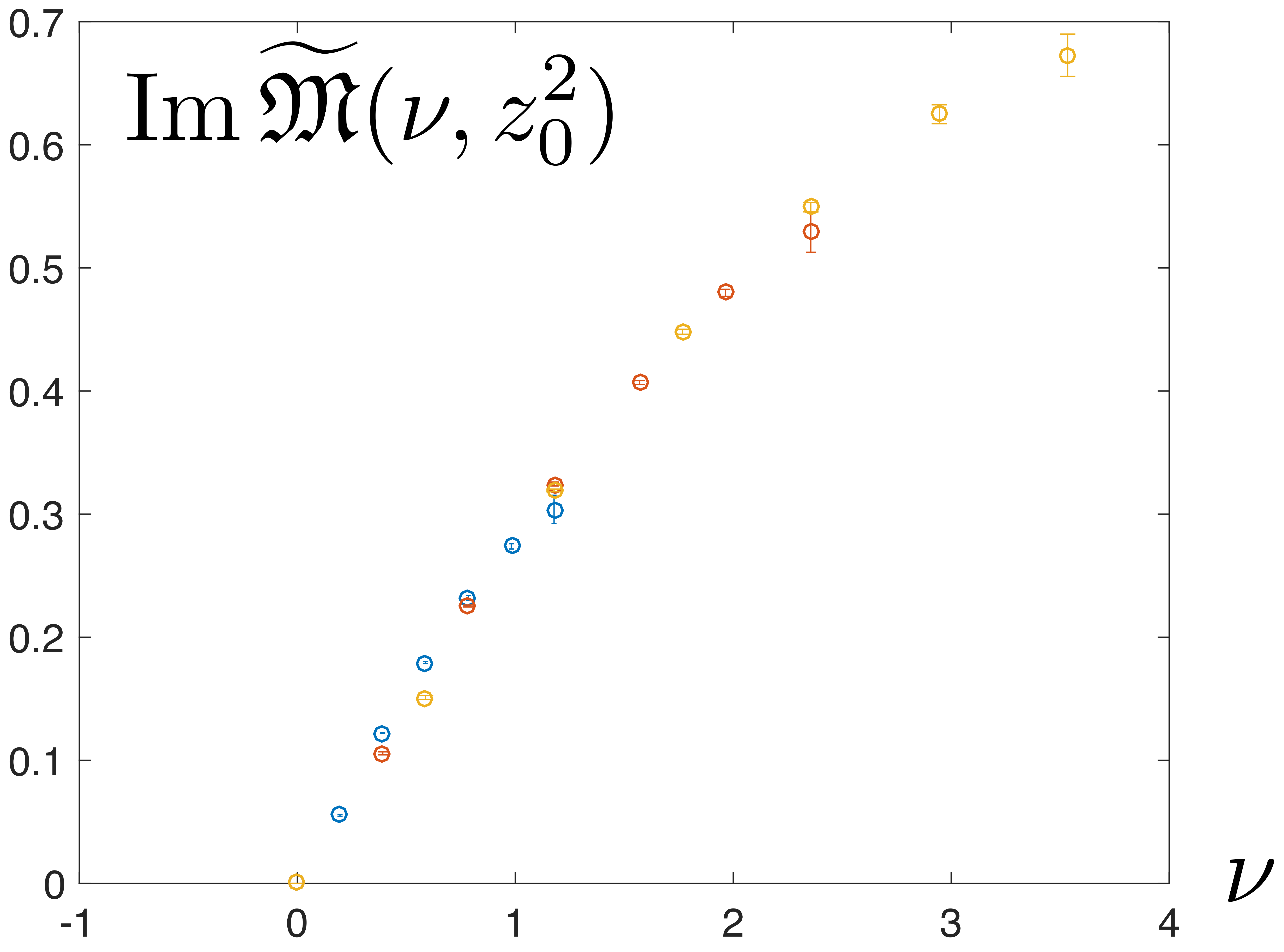
Evolved to 1GeV

Quenched QCD



Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD



Evolved to 1GeV

The Moments

Karpie et al. arXiv:1807.10933

Using OPE:

$$\mathfrak{M}(\nu, z^2) = 1 + \frac{1}{2p^0} \sum_{k=1}^{\infty} i^k \frac{1}{k!} z_{\alpha_1} \cdots z_{\alpha_k} c_k(z^2\mu^2) \langle p | \mathcal{O}_{(k)}^{0\alpha_1 \cdots \alpha_k} | p \rangle_\mu + \mathcal{O}(z^2)$$

$$\langle p | \mathcal{O}_{(k)}^{0\alpha_1 \cdots \alpha_k} | p \rangle_\mu = 2[p^0 p^{\alpha_1} \cdots p^{\alpha_k} - \text{traces}]_{\text{sym}} \, a_{k+1}(\mu) \, ,$$

Where $$a_n(\mu) = \int_{-1}^{1} dx \, x^{n-1} \, q(x, \mu) \, ,$$

are the moments of the PDFs

The Moments

Karpie et al. arXiv:1807.10933

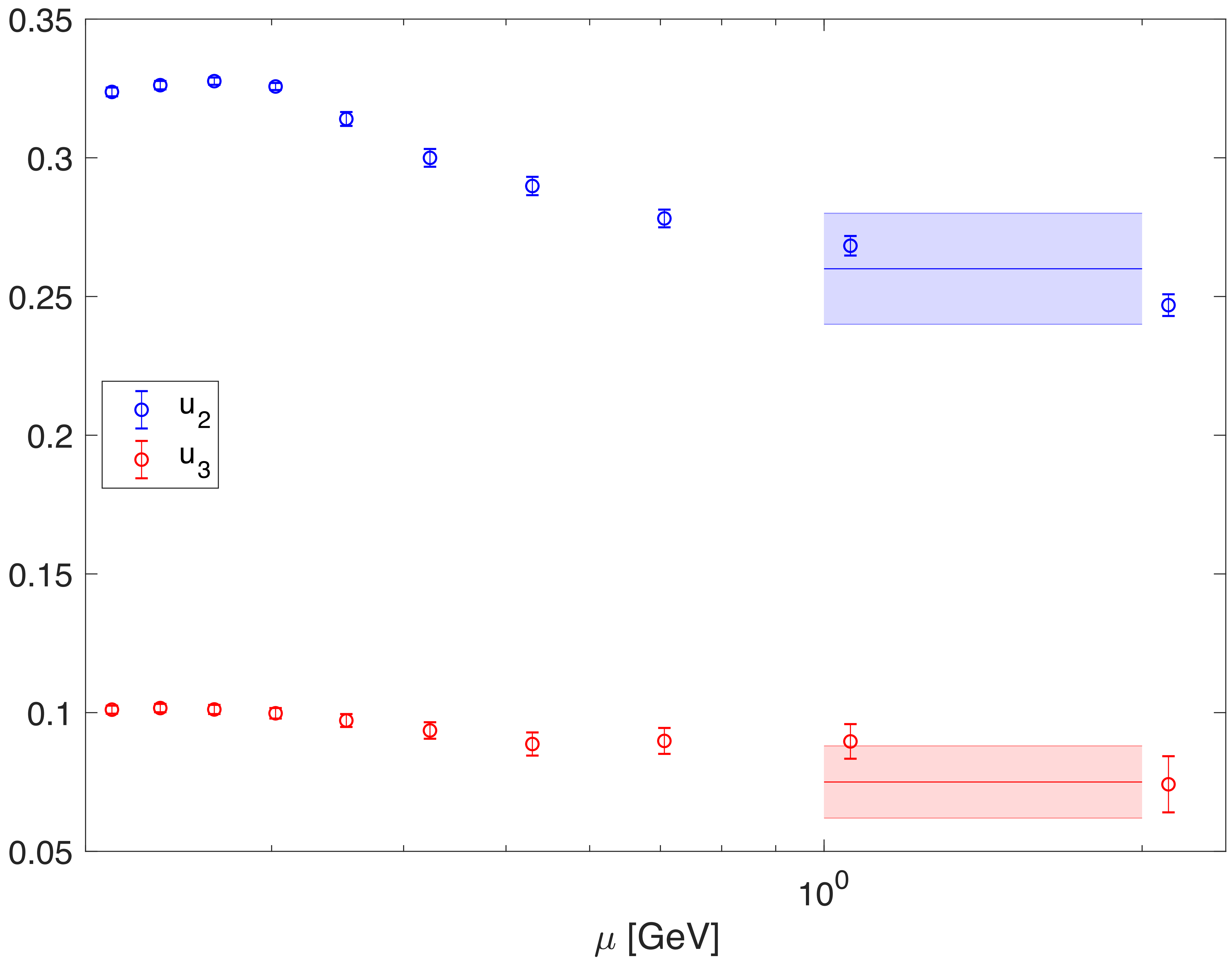
As a consequence:

$$(-i)^n \frac{\partial^n \mathfrak{M}(\nu, z^2)}{\partial \nu^n}\Big|_{\nu=0} = c_n(z^2\mu^2) a_{n+1}(\mu) + \mathcal{O}(z^2)\,.$$

Where the Wilson coefficients are

$$c_n(z^2\mu^2) = \int_0^1 d\alpha\, \mathcal{C}(\alpha, z^2\mu^2, \alpha_s(\mu))\alpha^n\,.$$

Quenched QCD: moments



QCDSF: Phys.Rev. D53 (1996) 2317-2325 — shown as shaded patches at μ=2 GeV

$$\overline{MS} \quad \mu^2 = (2e^{-\gamma_E}/z_3)^2$$

However on the Lattice after expanding in lattice spacing we have

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\text{cont}}(\nu, z^2) + \sum_{n=1}\left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\text{QCD}})^n R_n(\nu)\,.$$

$$\mathfrak{M}(\nu, z^2) = \int_0^1 dx\, q_v(x,\mu)\mathcal{K}(x\nu, z^2\mu^2) + \sum_{k=1}^{\infty}\mathcal{B}_k(\nu)(z^2)^k\,.$$

Ioffe time $-z\cdot p = \nu$

- All coefficient functions respect continuum symmetries
- Lattice spacing corrections to higher twist effects are ignored
- On dimensional ground a/z terms must exist
- Additional O(a) effects (last term)

The inverse problem to solve: Obtain q(x,μ) from the lattice matrix elements

see dicussion in J. Karpie *et al JHEP* 04 (2019) 057 and L. DelDebio *et al JHEP* 02 (2021) 138

Exploration of various methods for LO matching

Exploration of the NNPDF approach applied to lattice data

Our inverse problem

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\rm cont}(\nu,z^2) + \sum_{n=1}\left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\rm QCD})^n R_n(\nu)\,.$$

$$\mathrm{Re}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\,\mathcal{K}_R(x\nu,\mu^2z^2)q_-(x,\mu^2) + \mathcal{O}(z^2)$$

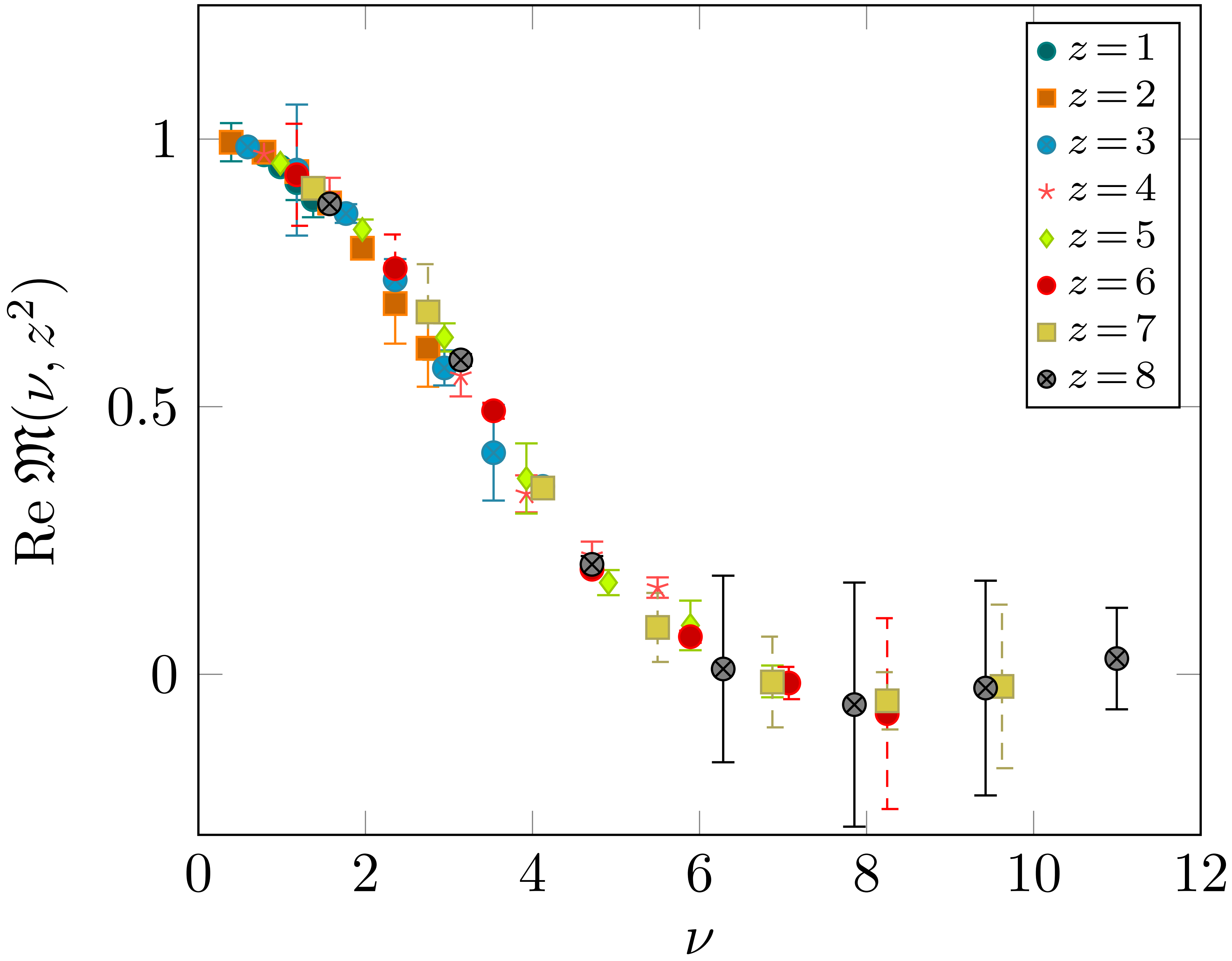
$$\mathrm{Im}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\,\mathcal{K}_I(x\nu,\mu^2z^2)q_+(x,\mu^2) + \mathcal{O}(z^2)\,,$$

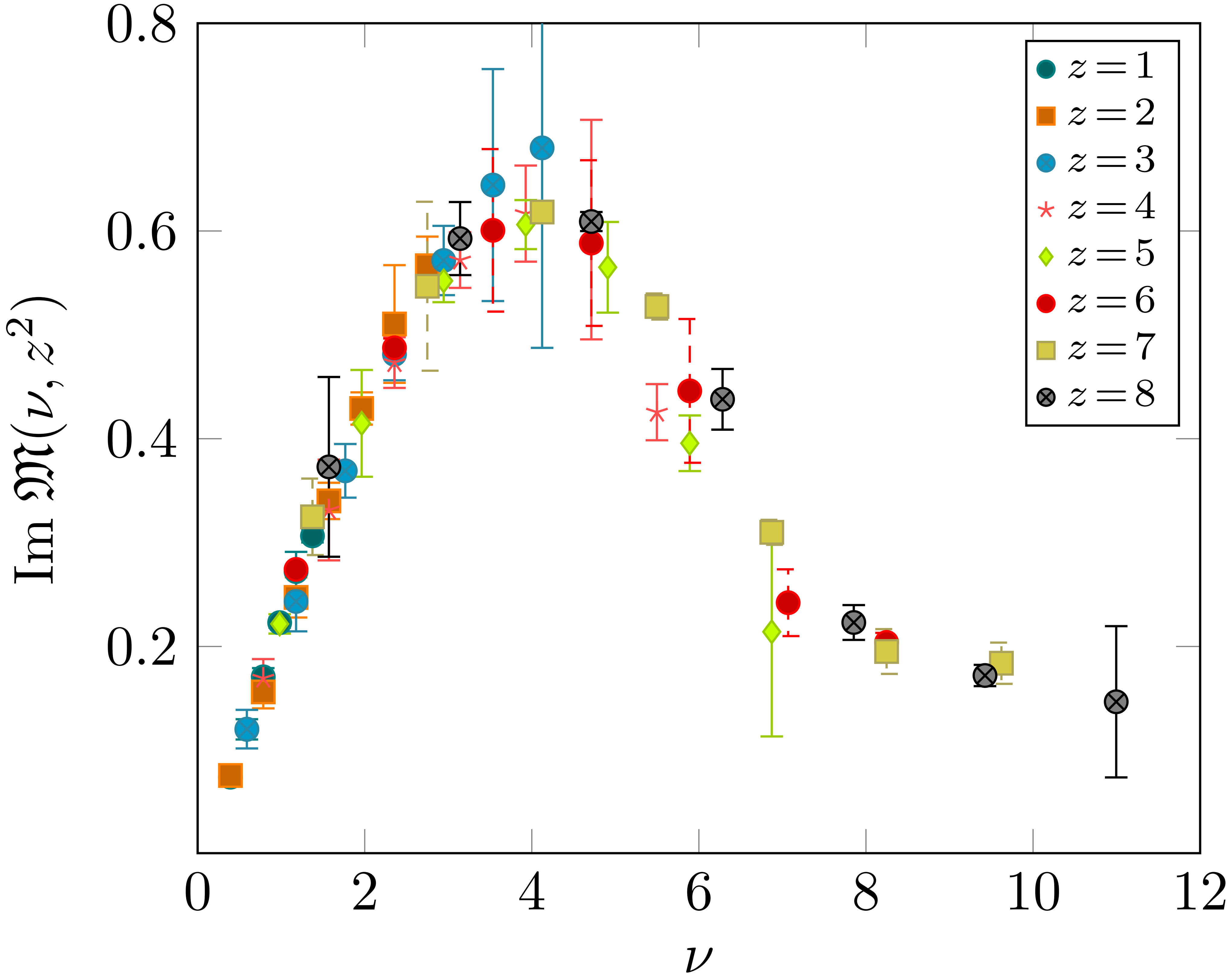
- Obtain the PDF from a limited set of matrix elements obtained from lattice QCD
- z^2 is a physical length scale sampled on discrete values
- z^2 needs to be sufficiently small so that higher twist effects are under control
- v is dimensionless also sampled in discrete values
- the range of v is dictated by the range of z and the range of momenta available and is typically limited
- Parametrization of unknown functions

Sample data

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*

ID	a(fm)	M_π(MeV)	β	c_{SW}	κ	$L^3 \times T$	N_{cfg}
$\widetilde{A}5$	0.0749(8)	446(1)	5.2	2.01715	0.13585	$32^3 \times 64$	1904
E5	0.0652(6)	440(5)	5.3	1.90952	0.13625	$32^3 \times 64$	999
N5	0.0483(4)	443(4)	5.5	1.75150	0.13660	$48^3 \times 96$	477





a= 0.075fm

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*

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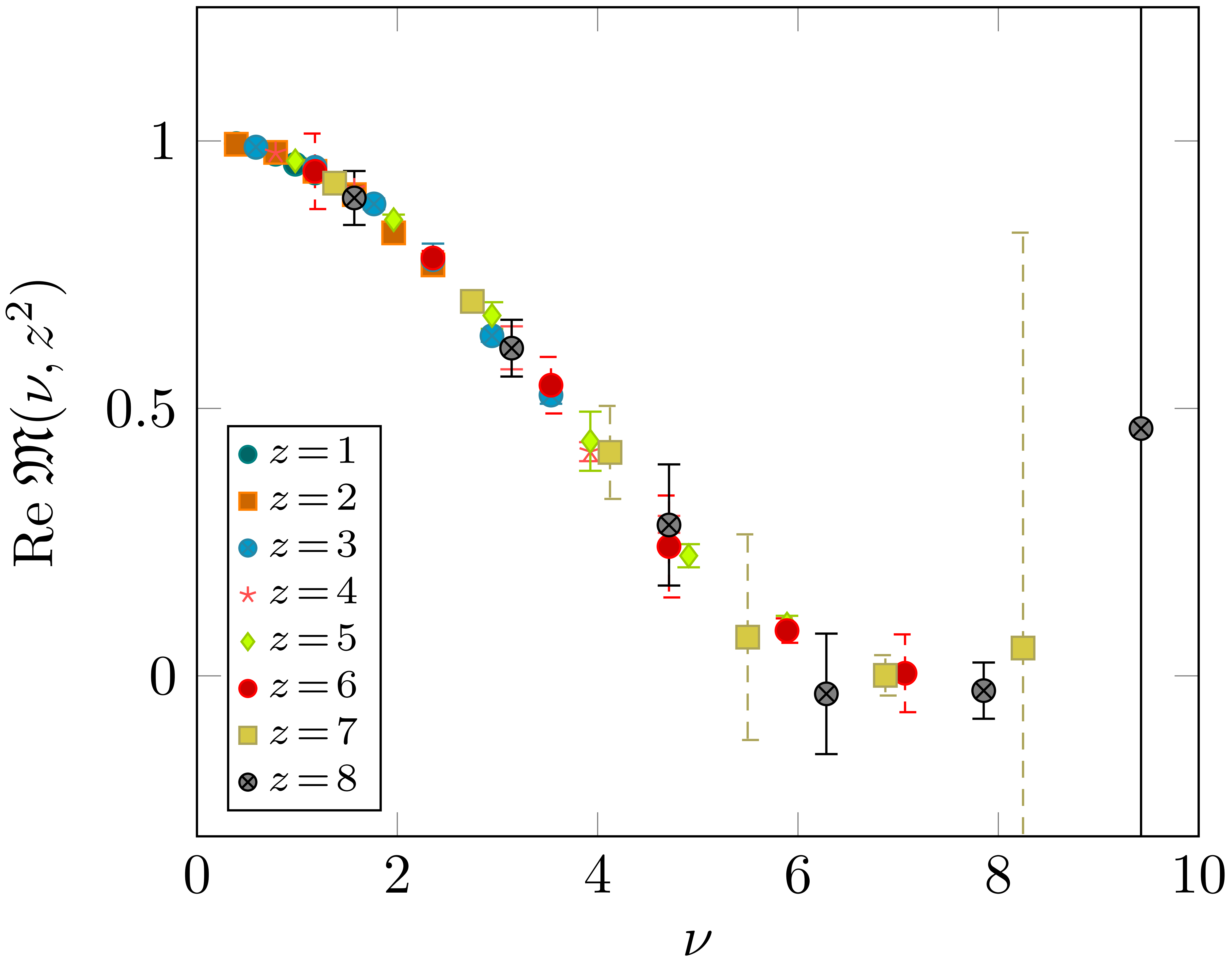
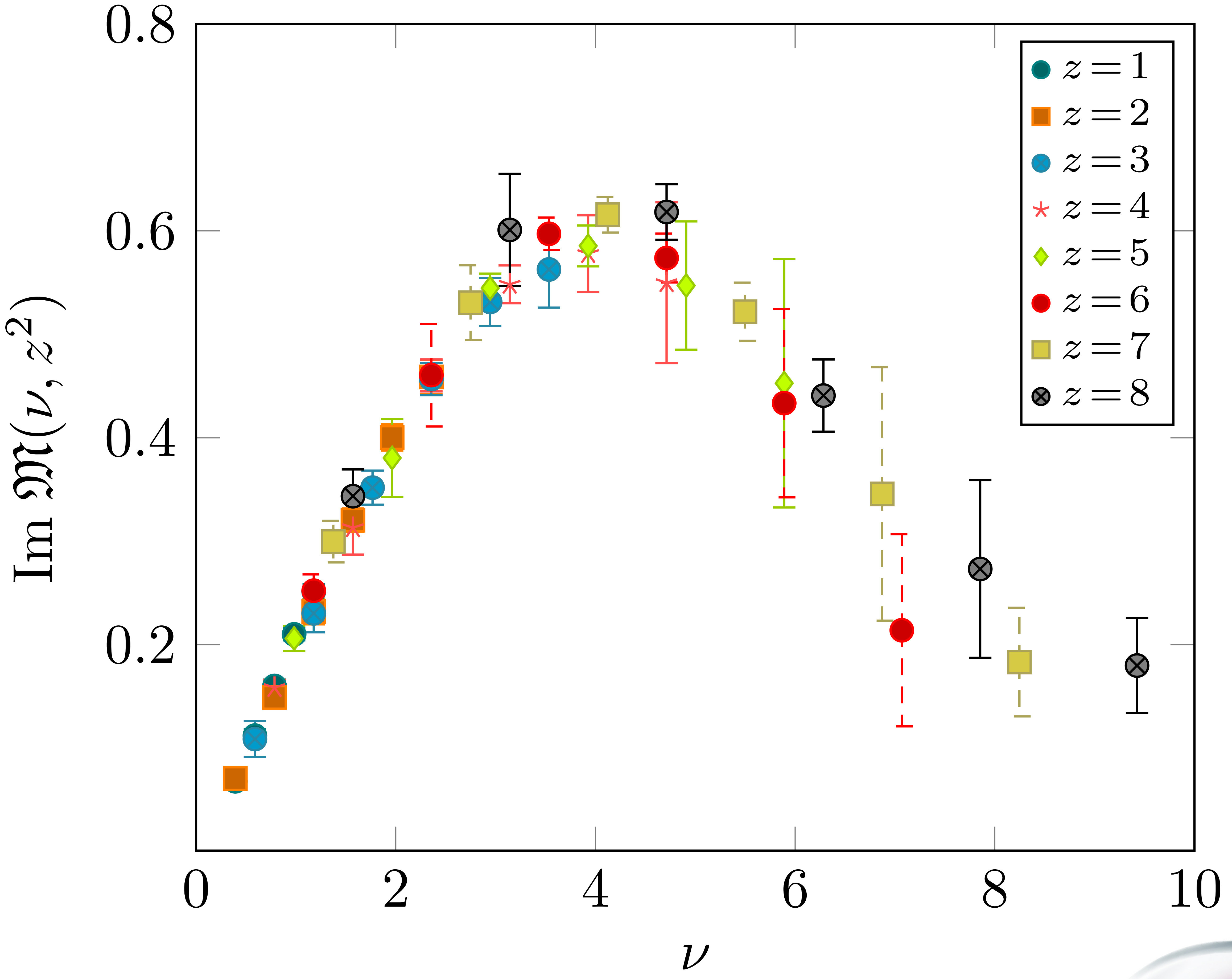
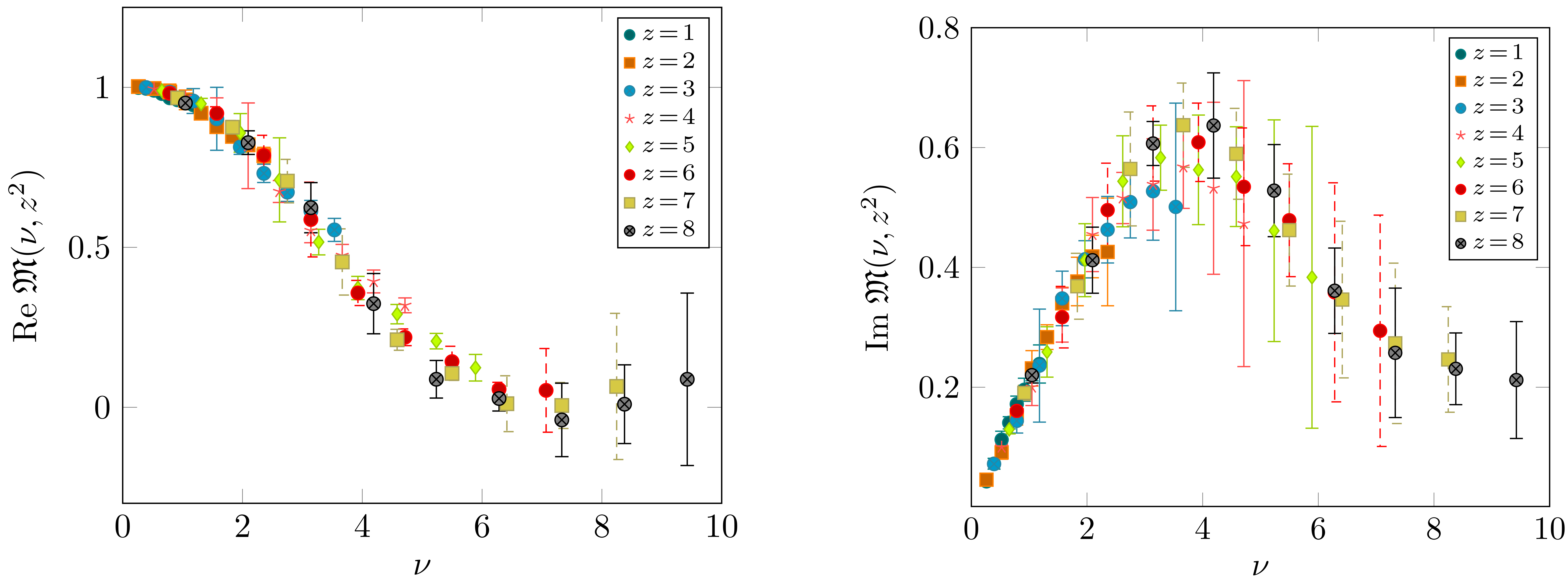




a= 0.065fm

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*



a= 0.048fm

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*

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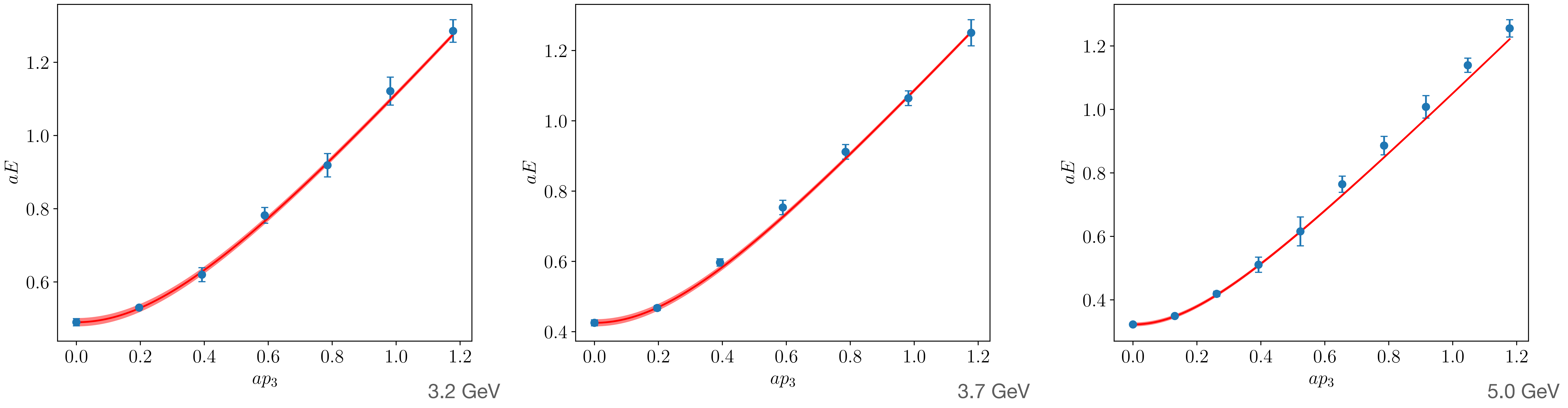
Nucleon Momentum scan

Energy *vs* momentum



a=0.075 fm

a=0.065 fm

a=0.048 fm

Maximum attainable momentum in lattice units can be up to $\mathcal{O}(1)$

Smaller lattice spacing allows for physically larger momentum

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*

Jacobi Polynomials

Inverse problem

$$q_+(x) = q(x) + \bar{q}(x)$$

$$q_-(x) = q(x) - \bar{q}(x)$$

PDF parametrization

$$q_\pm(x) = x^\alpha (1-x)^\beta \sum_{n=0}^{\infty} {}_\pm d_n^{(\alpha,\beta)} J_n^{(\alpha,\beta)}(x)$$

$J_n^{(\alpha,\beta)}(x)$ Jacobi Polynomials: Orthogonal and complete in the interval [0,1]

$$\int_0^1 dx\, x^\alpha (1-x)^\beta J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) = N_n^{(\alpha,\beta)} \delta_{n,m}$$

Complete basis of functions in the interval [0,1] for any α and β

Bayesian Inference

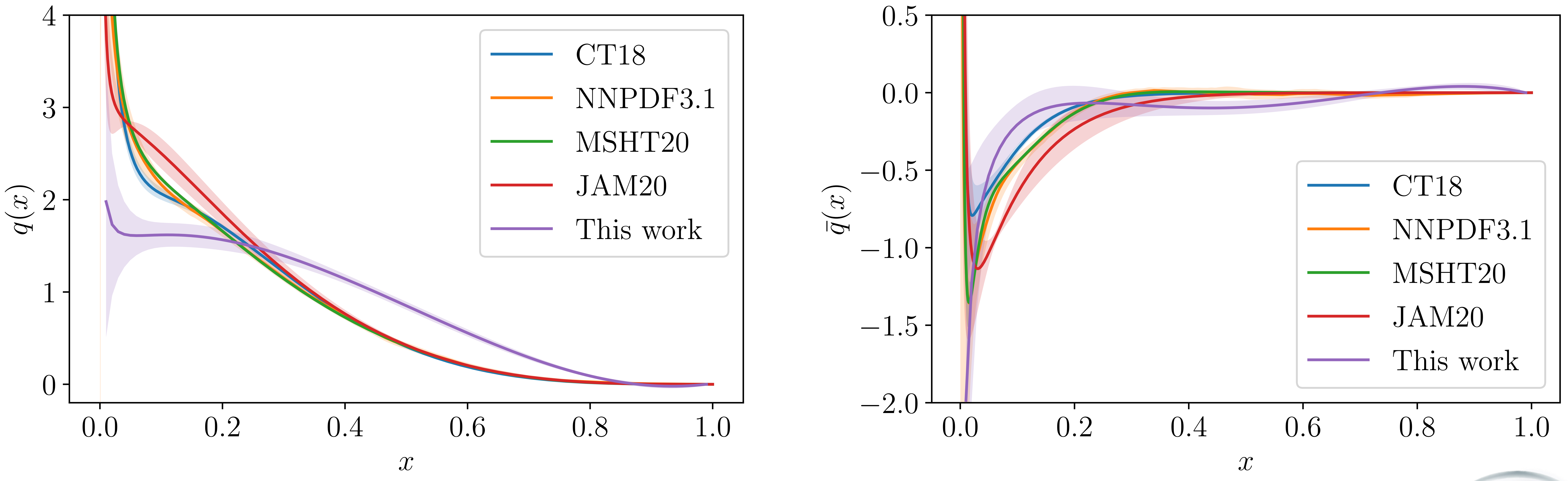
Optimize model parameters

- Fix the expansion order in the Jacobi polynomial expansion
- Optimize α,β and the expantion of coefficients by maximizing the posterior probability
- Average over models using AICc
- Note that one could fix α,β at a reasonable value and the vary the order of trancation in the Jacobi polynomial expansion

$$P\left[\theta|\mathfrak{M}^L, I\right] = \frac{P\left[\mathfrak{M}^L|\theta\right] P\left[\theta|I\right]}{P\left[\mathfrak{M}^L|I\right]} \, .$$

Isovector quark and anti-quark distributions

Comparison with phenomenology



2 flavor QCD in the continuum limit with 450 MeV pions

arXiv:2105.13313 **[hep-lat] J. Karpie** *et. al.*

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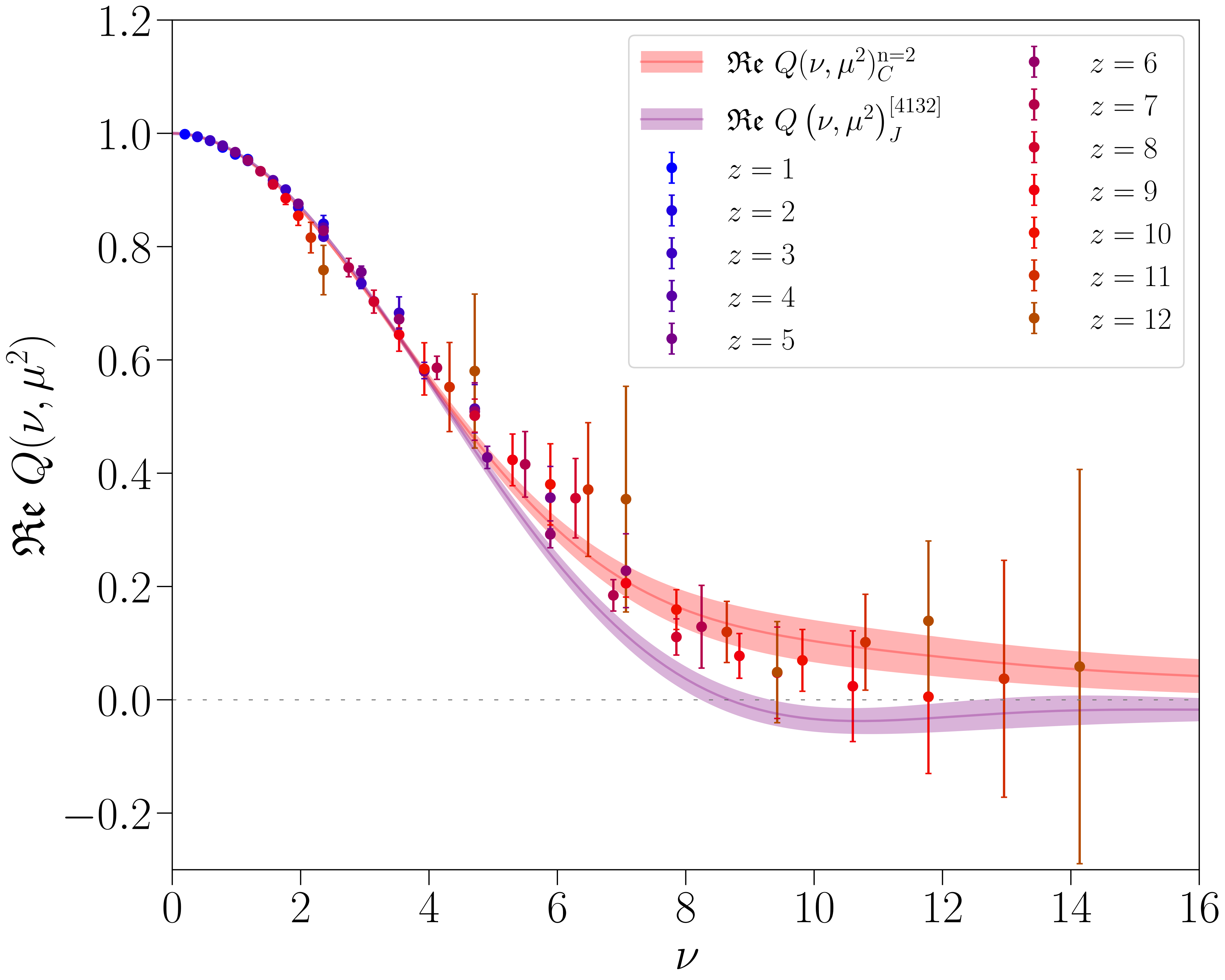
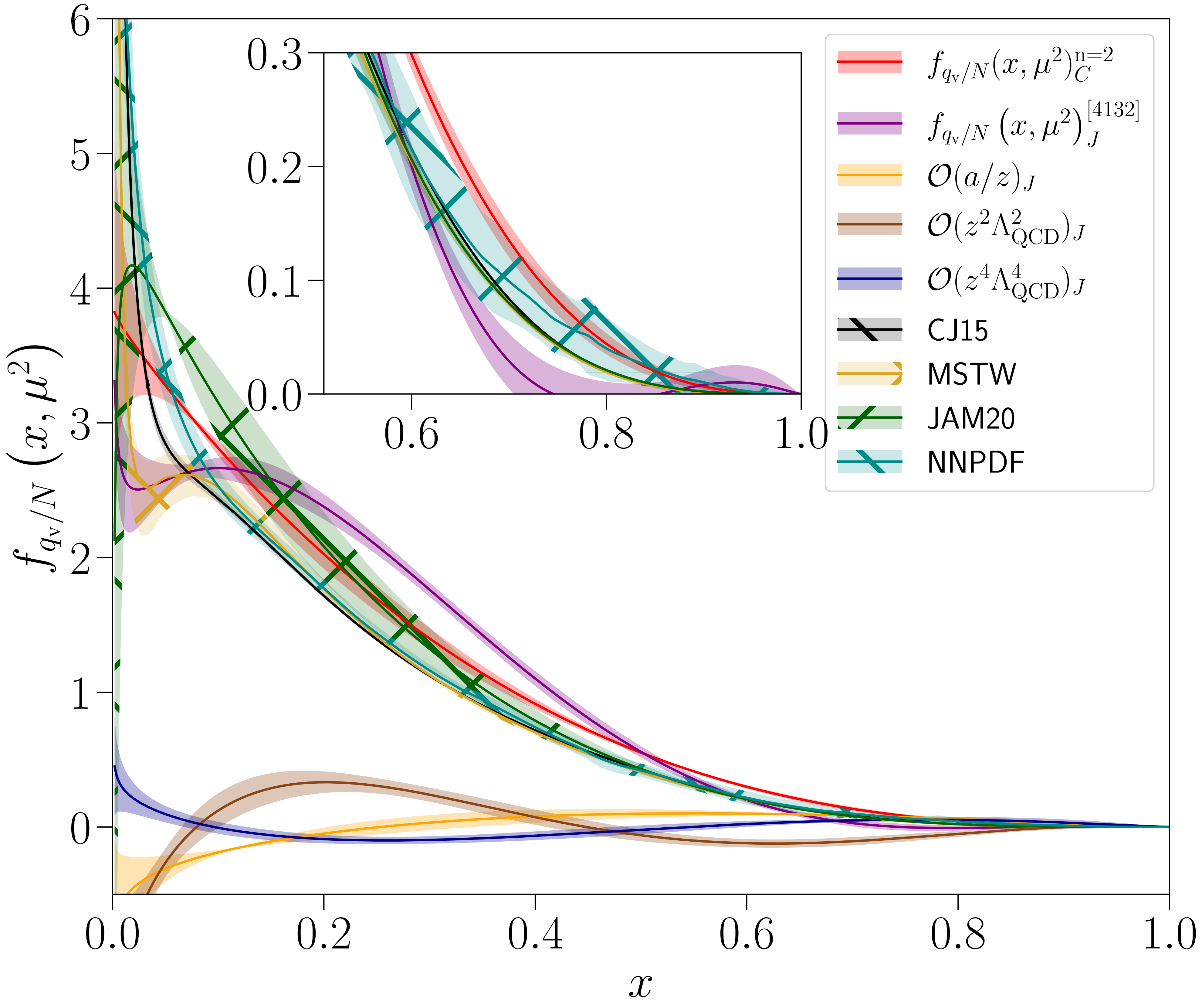
Unpolarized Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





arXiv:2107.05199 **[hep-lat] C. Egerer** *et. al.*

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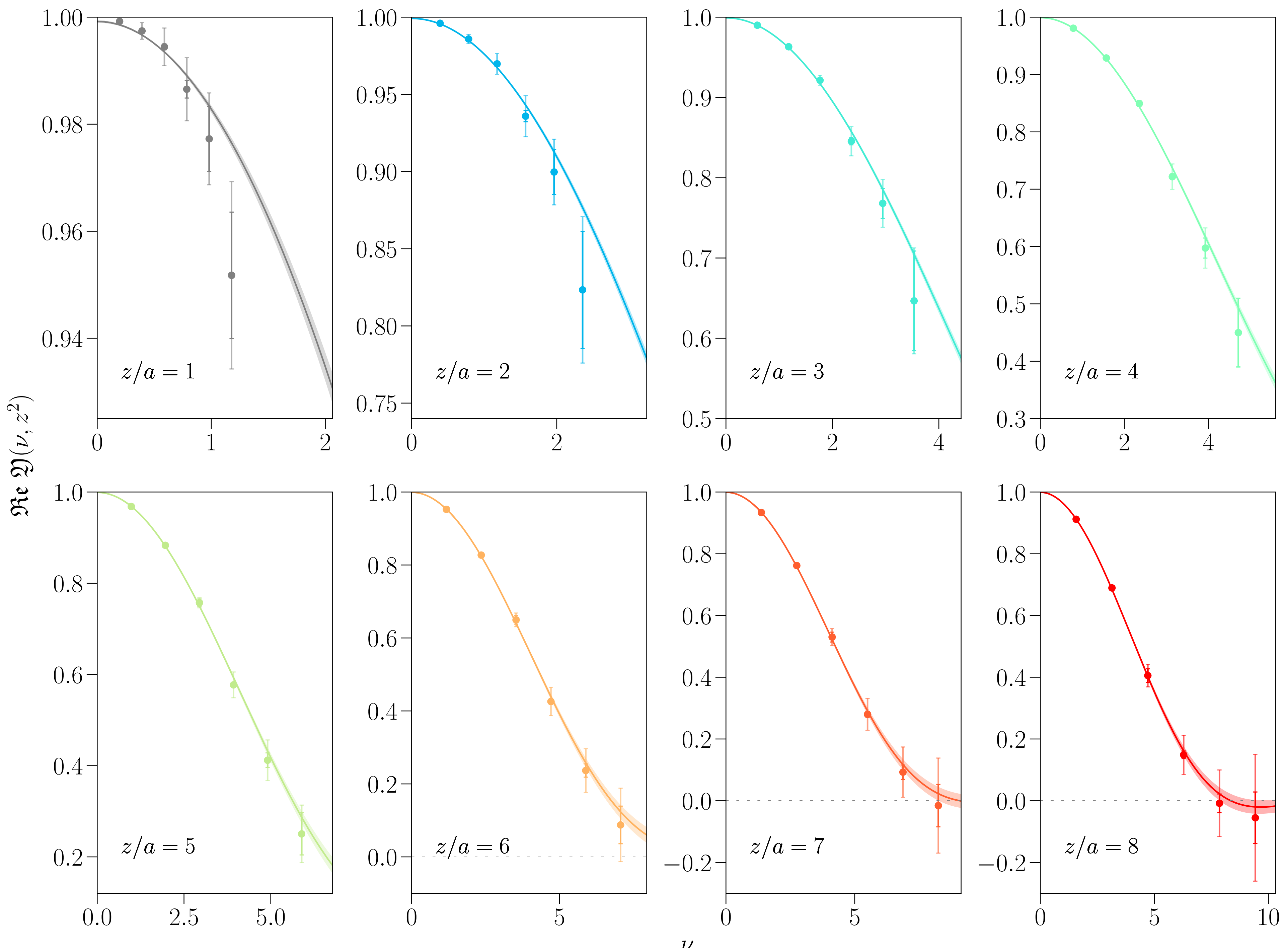
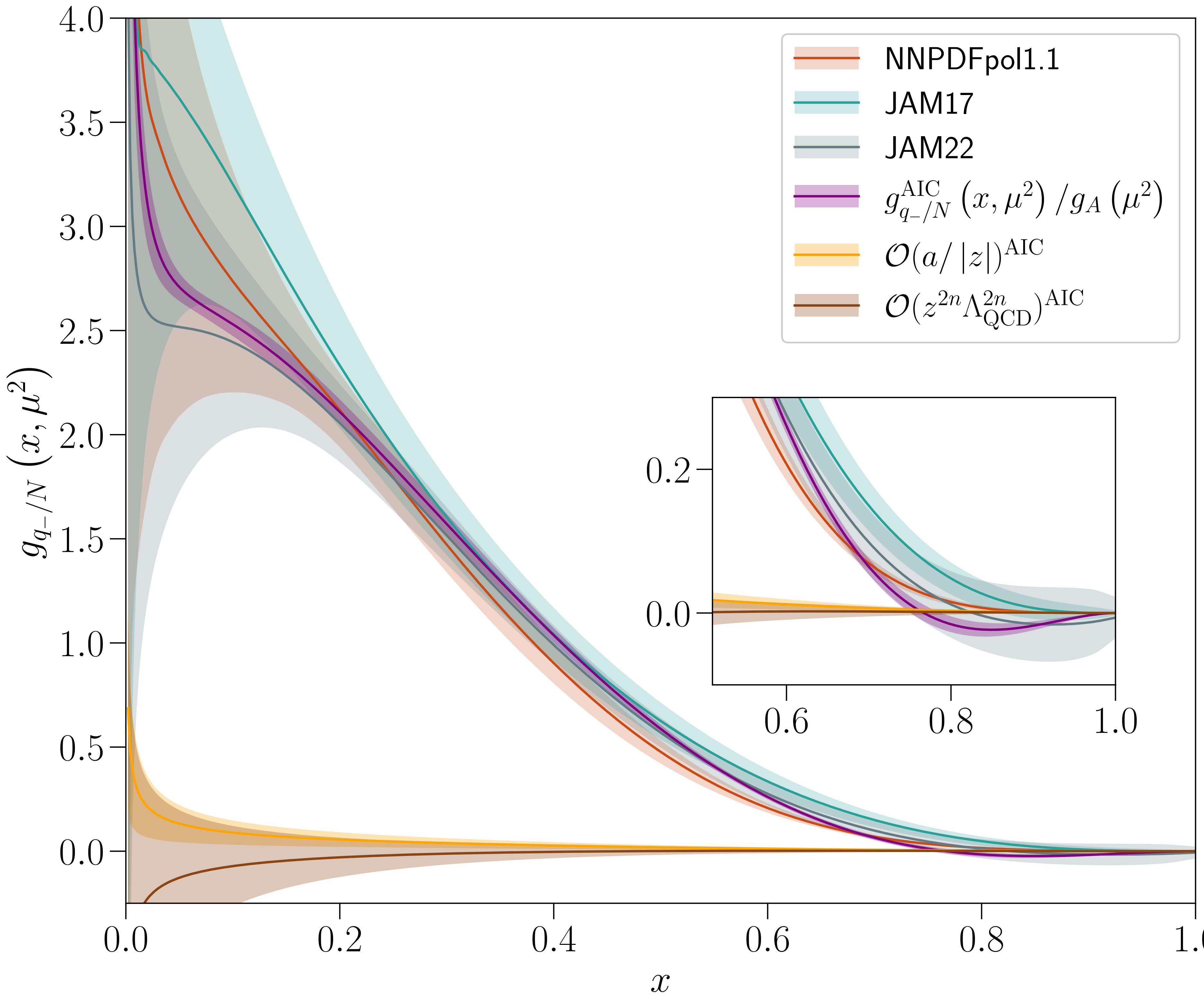
Helicity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion

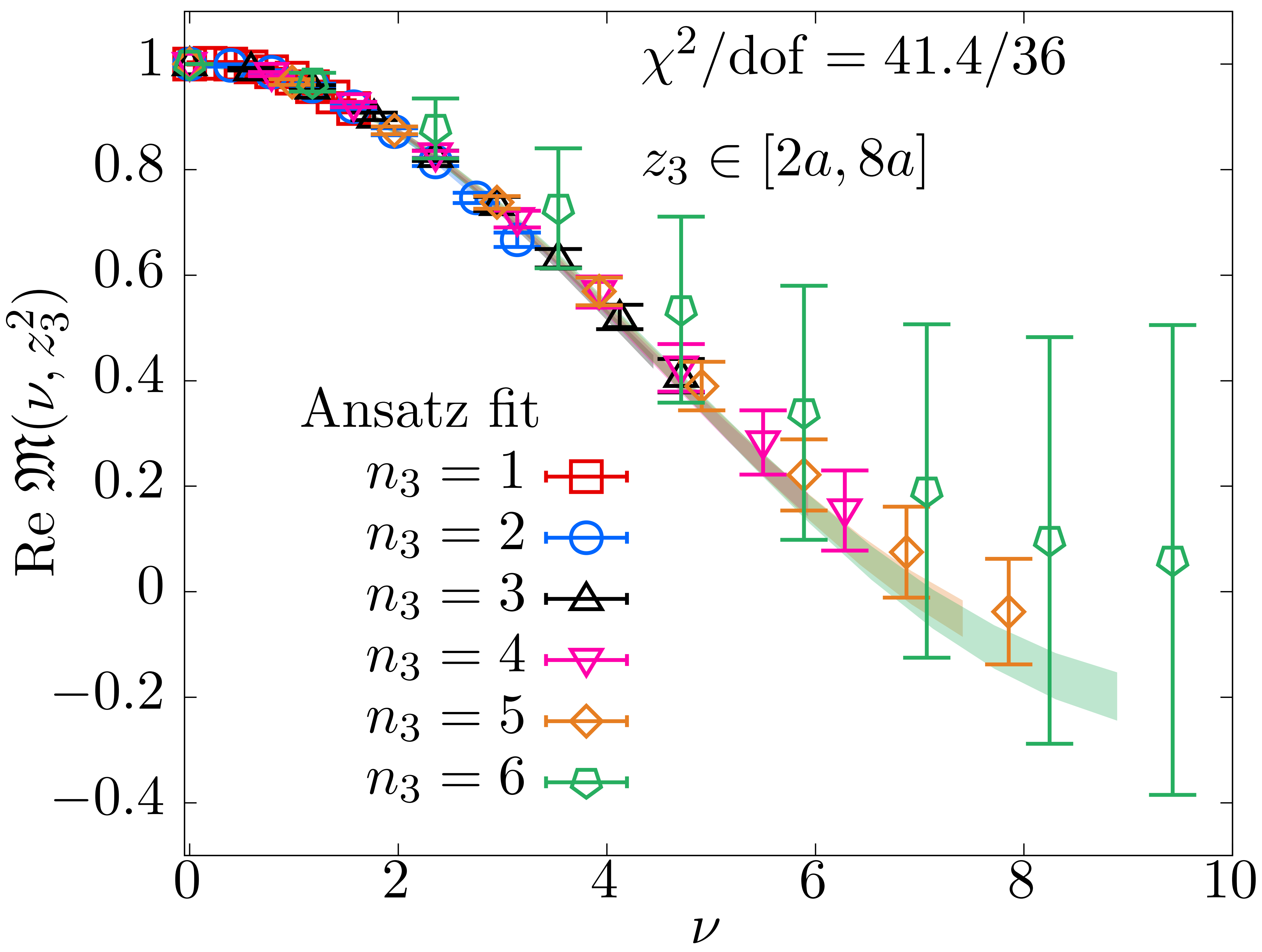


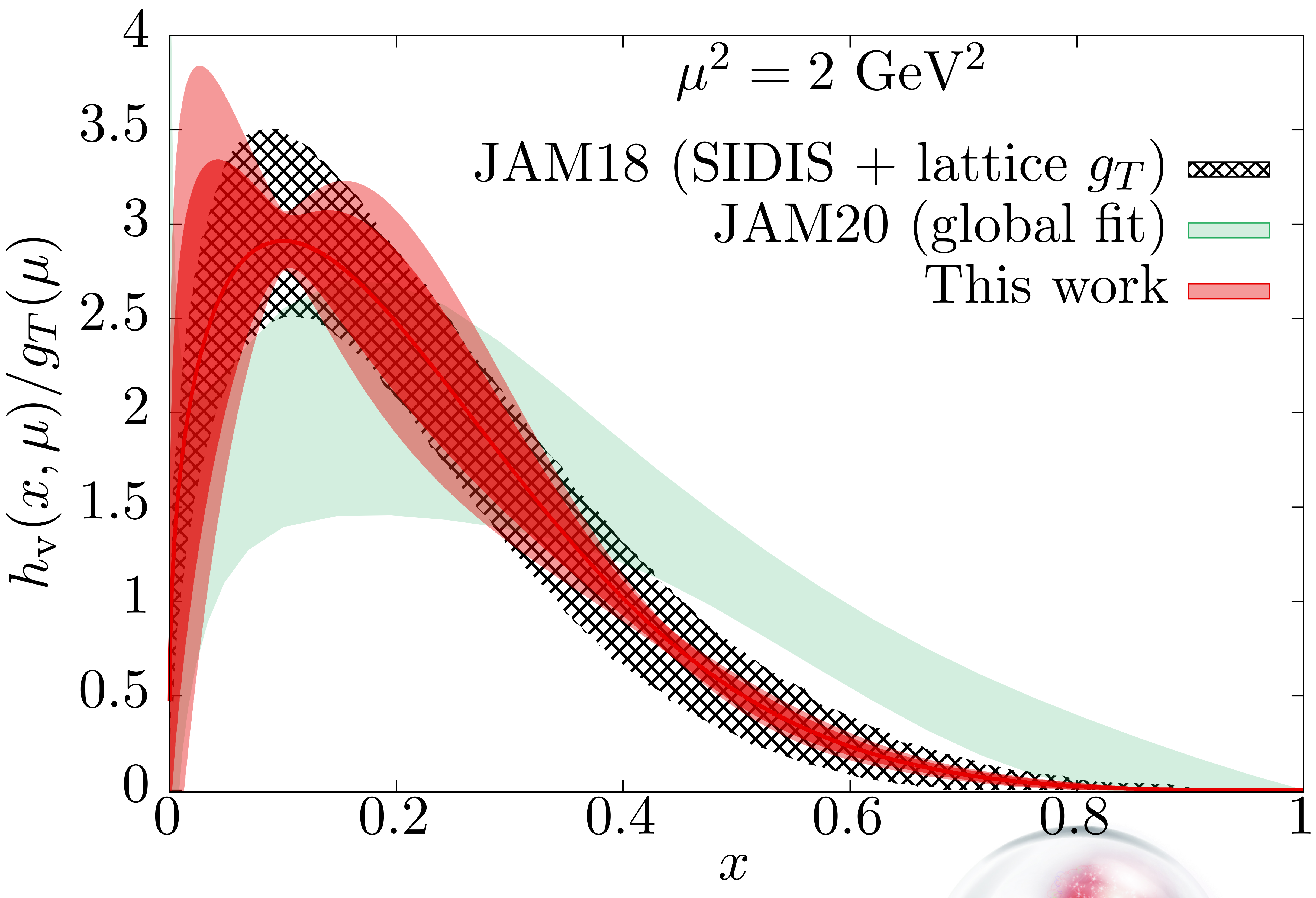


arXiv:2211.04434 **[hep-lat] C. Egerer** *et. al.*

Transversity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





[arXiv:2111.01808](https://arxiv.org/abs/2111.01808) **[hep-lat] C. Egerer** *et. al.*

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Conclusions

Outlook

- The understanding hadronic structure is a major goal in nuclear physics
  - Large experimental effort: JLab 12 GeV and future EIC
- Lattice QCD calculations can in principle compute (Generalized) Parton distribution functions from first principles
- Controlling all systematics of the calculation is important and that complicates the solution of the inverse problem at hand
  - Both lattice spacing and higher twist effects need to be controlled
- New ideas are needed for pushing to higher momentum and improved sampling of the Ioffe time
  - The range of Ioffe time is essential for obtaining the x-dependence of distribution functions
- The synergy between lattice and experiment may be proven essential in providing precision estimates of (Generalized) Parton distribution functions