Conformal Bootstrap universality between $c \geq 25$ and $c \leq 1$ two-dimensional CFTs

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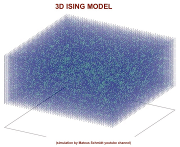
based on work to appear:
[2301.xxxxx], S. Ribault, I.T

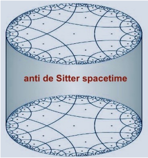
and also:
[1912.00222], S. Collier, A. Maloney, H. Maxfield, I.T.
[2011.09250], I.T.
[2202.01633], T. Numasawa, I.T.

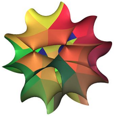
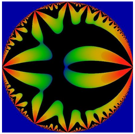
Introduction

Conformal Field Theories (CFTs)

- QFTs with conformal symmetry. Fixed points of RG flow. Universality for different systems at criticality. Quantum gravity in AdS, ...





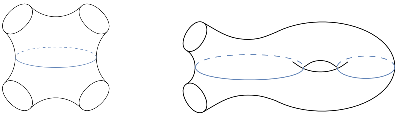
(from Wikipedia)

- Two dimensions are special! Infinite number of conserved charges. Ubiquitous in both physics and math: from 2d condensed matter systems and worldsheet string theory, to number theory, random matrix theory, quantum groups, $\cdots$

Introduction

Conformal Bootstrap Philosophy

We will be interested in Euclidean correlation functions of local (primary) operators on Riemann surfaces:



Using the power of the *Operator Product Expansion* (OPE), the basic **2d CFT data** at central charge c consist of:

- Dynamic: List of primary operators $\mathcal{O}_i$, along with scaling dimensions $\Delta_i = h_i + \overline{h}_i$ and spins $l_i = |h_i - \overline{h}_i|$, and their OPE coefficients C_{ijk}.
- Kinematic: *Conformal blocks*.

Question: How are these CFT data constrained from consistency conditions (e.g. associativity of OPE)? Are there any **universal** features that we can derive analytically?

Goal 1/2 of this talk

- Understand an important kinematic tool in 2d CFTs – the **crossing kernels** $\mathbb{K}^{(c)}_{P'P}$ – that implement change of basis transformations:

$$\mathcal{F}^{(\text{frame–1})}_{P} = \int_{\mathcal{C}} dP' \; \mathbb{K}^{(c)}_{P'P} \; \mathcal{F}^{(\text{frame–2})}_{P'},$$

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  For $c \in (-\infty, 1]$,we'll show:

$$\mathcal{K}^{(c)}_{P'P} = (\text{meromorphic function}) \times \mathbb{K}^{(26-c)}_{iP',iP}.$$

[S.Ribault, I.T., to appear]

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This includes and **generalizes** the already known expressions for e.g. Minimal Models.

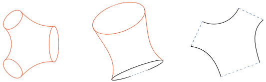
Goal 2/2 of this talk

Apply these formulas to:

- Analytically prove **crossing symmetry for time-like Liouville** theory, given that space-like Liouville theory is crossing symmetric.
- In general, provide **universal bootstrap connections** in universal kinematic regimes for $c \leq 1$ theories, given the analogous expressions for $c \geq 25$ theories.
(in the spirit of [S. Collier, A. Maloney, H. Maxfield, I.T., '19])
- Glimpse into the structure of conformal blocks, connection between large positive/negative central charge, $\cdots$

Outline

- 2D CFT primer
- Crossing Symmetry and Modular Covariance in 2d
- Crossing Kernels
- (*Goal 1*) Analyticity and Crossing Kernels for $c \leq 1$
- (*Goal 2*) Conformal bootstrap applications
- Summary & Future Directions



2D CFT primer

2D CFT primer

General

- Contrary to $d > 2$, there is an infinite dimensional algebra of symmetries.
- Two copies of **Virasoro algebra**:

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\mathbb{1}, \\
&(\text{same for } L_n \to \overline{L}_n), \\
&[L_n, \overline{L}_m] = 0 \qquad\qquad m, n \in \mathbb{Z}
\end{aligned}
$$

c : central charge.

2D CFT primer

General

- Irreps characterized by the conformal dimensions $(h, \overline{h})$ and associated highest weight/primary state $|h\rangle$

$$L_0 |h\rangle = h |h\rangle \quad , \quad \overline{L_0} |h\rangle = \bar{h} |h\rangle$$
$$L_n(\overline{L_n}) |h\rangle = 0 \quad , \quad \text{for} \quad n \in \mathbb{Z}_{>0}$$

- Heighest weight module $\mathcal{V}_h$

$$\mathcal{L}_{|N|} |h\rangle \equiv L_{-n_k}..L_{-n_2} L_{-n_1} |h\rangle \ , \quad n_1, n_2, .., n_k \in \mathbb{Z}_{>0} \ , \ N \equiv \sum_i n_i$$

$$L_0 \mathcal{L}_{|N|} |h\rangle = (h + N) \mathcal{L}_{|N|} |h\rangle \ ,$$

complete (albeit non-orthogonal) basis.

- Scaling dimension/Energy, Spin, Twist:

$$\Delta = h + \bar{h} \ , \quad l = |h - \bar{h}| \ , \quad \tau = \Delta - l = 2min(h, \bar{h}).$$

- Hilbert space of states:

$$\mathcal{H} = \oplus_{h,\bar{h}} \mathcal{V}_h \otimes \mathcal{V}_{\overline{h}}.$$

- State-Operator Correspondence:

$$\textit{primary state}\,|i\rangle\ \textit{on}\ S^1 \quad \leftrightarrow \quad \text{primary operator } \mathcal{O}_i(z,\bar{z})$$

- Algebraic product structure (OPE):

$$\mathcal{O}_i(z)\mathcal{O}_j(0) = \sum_k C_{ij}^k z^{h_k - h_i - h_j} \underbrace{\sum_N B_N(h_i, h_j; h_k|z)\mathcal{L}_{|N|}}_{descendants\ \ of\ \ \mathcal{O}_k, kinematic} \mathcal{O}_k(0)$$

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C_{ij}^k: OPE coefficients, *dynamic* data.

2D CFT primer

Correlation Functions on S^2

- Two-point function:

$$\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\rangle_{S^2} = \frac{\delta_{ij}}{z_{12}^{2h_i}} \qquad , \qquad z_{ij} \equiv z_i - z_j$$

- Three-point function:

$$\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\rangle_{S^2} = \frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k} z_{13}^{h_i+h_k-h_j} z_{23}^{h_j+h_k-h_i}} ,$$

$$\textit{with} \quad C_{ijk} = C_{ij}^{k} .$$

- Any higher point function can be readily constructed by successively using the OPE structure of operators!

Crossing Symmetry and Modular Covariance in 2d

Crossing Symmetry and Modular Covariance in 2d

Notation

We will be interested in kinematic quantities as functions of $c \in \mathbb{C}$. This includes both unitary ($c > 0$) and non-unitary ($c < 0$) theories .

"Natural" parametrization:

$$c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2$$

- For $b \in \mathbb{C}$ with $\mathrm{Re} b > 0 \quad \Rightarrow \quad c \in \mathbb{C} \backslash (-\infty, 1]$,
- For $b \in i\mathbb{R} \quad \Rightarrow \quad c \in (-\infty, 1]$.

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Conformal dimensions:

$$h = \alpha(Q - \alpha) = \frac{Q^2}{4} + P^2 \,, \qquad P \in \mathbb{C}$$

$$(\bar{h} = \bar{\alpha}(Q - \bar{\alpha}) = \frac{Q^2}{4} + \bar{P}^2)$$

Crossing Symmetry and Modular Covariance in 2d

Euclidean correlation functions out of elementary "legos"

Consider a general correlation function

$$G_{g,n_b} = \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_{n_b}(z_{n_b}) \rangle_{\Sigma_g}.$$

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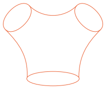
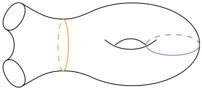
After successively using the OPE between operators, the amplitude is reduced to a product of elementary "legos" made out of the three-point structure constant C_{ij}^k:



e.g.



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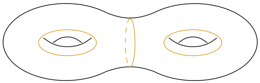

Crossing Symmetry and Modular Covariance in 2d

Conformal Bootstrap

"Sewing" the surface in different ways leads to *equivalent* descriptions of a correlation function. Manifestation of **locality**.

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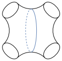
Two canonical examples:

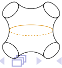
- Crossing symmetry of 4-pt functions on S^2

$$\Big\langle \underbrace{\mathcal{O}_1(0)\mathcal{O}_2(x,\bar{x})}_{OPE}\,\underbrace{\mathcal{O}_1(1)\mathcal{O}_2'(\infty)}_{OPE} \Big\rangle = \sum_{\alpha_s} C^2_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_s}\mathcal{F}(\alpha_s|x)\overline{\mathcal{F}}(\overline{\alpha_s}|\bar{x})$$



$$\Big\langle \underbrace{\mathcal{O}_1(0)\mathcal{O}_1(1)}_{OPE}\,\underbrace{\mathcal{O}_2(x,\bar{x})\mathcal{O}_2'(\infty)}_{OPE} \Big\rangle =$$

$$\sum_{\alpha_t} C_{\mathcal{O}_1\mathcal{O}_1\mathcal{O}_t}C_{\mathcal{O}_2\mathcal{O}_2\mathcal{O}_t}\mathcal{F}(\alpha_t|1-x)\overline{\mathcal{F}}(\overline{\alpha_t}|1-\bar{x})$$

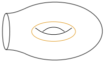


Crossing Symmetry and Modular Covariance in 2d

Conformal Bootstrap

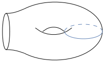
- Modular covariance of 1-pt functions on T^2

$$\langle \mathcal{O}_0 \rangle_{T^2} = \sum_{\alpha_{\mathcal{O}}} C_{\mathcal{O}\mathcal{O}\mathcal{O}_0} \mathcal{F}^{\mathcal{O}_0}(\alpha_{\mathcal{O}}|q) \mathcal{F}^{\overline{\mathcal{O}_0}}(\overline{\alpha_{\mathcal{O}}}|\overline{q})$$



Under $S : \tau \to -\frac{1}{\tau}$ the primary operator transforms non-trivially by definition.

$$\langle \mathcal{O}_0 \rangle_{T^2} (-1/\tau, -1/\overline{\tau}) = \tau^{\alpha_0(Q-\alpha_0)} \bar{\tau}^{\bar{\alpha}_0(Q-\bar{\alpha}_0)} \langle \mathcal{O}_0 \rangle_{T^2} (\tau, \bar{\tau})$$



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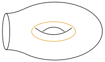

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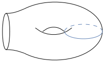

$\Rightarrow$ In the case $\mathcal{O}_0 = \mathbb{1}$ $(\alpha_0 = 0, C_{\mathcal{O}\mathcal{O}\mathbb{1}} = 1)$,

$$\sum_{\alpha,\overline{\alpha}} d_{\alpha,\overline{\alpha}} \chi_\alpha(\tau) \chi_{\overline{\alpha}}(\overline{\tau}) = \sum_{\alpha,\overline{\alpha}} d_{\alpha,\overline{\alpha}} \chi_\alpha(-1/\tau) \chi_{\overline{\alpha}}(-1/\overline{\tau})$$

Modular Invariance of the Partition Function.

Crossing Symmetry and Modular Covariance in 2d

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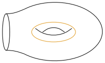

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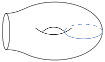


- ☐ <u>Powerful result</u> [G.W. Moore, N. Seiberg, '88]: (i) Crossing symmetry of 4-pt functions + (ii) Modular covariance of torus 1-pt functions are **sufficient** to imply *higher point* crossing symmetry and *higher genus* modular covariance.

Crossing Kernels

Kernel as Fundamental

- We saw the statement of crossing symmetry:

$$\sum_{\alpha_s} C^2_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_s}\mathcal{F}(\alpha_s|x)\overline{\mathcal{F}}(\overline{\alpha_s}|\overline{x}) =$$

$$= \sum_{\alpha_t} C_{\mathcal{O}_1\mathcal{O}_1\mathcal{O}_t}C_{\mathcal{O}_2\mathcal{O}_2\mathcal{O}_t}\mathcal{F}(\alpha_t|1-x)\overline{\mathcal{F}}(\overline{\alpha_t}|1-\overline{x}).$$

- Trivial fact of life[Exercise in Moore-Seiberg;'89]:
  If $\{f_i\},\{g_i\},\{h_i\},\{k_i\}$ sets of linearly independent analytic functions s.t.

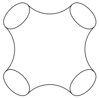
$$\sum_{i=1}^{N} f_i g_i^* = \sum_{i=1}^{M} h_i k_i^*$$

  Then, $N=M$, $\vec{f}=A\vec{h}$, $\vec{g}=(A^{-1})^\dagger\vec{k}$ for some invertible matrix A.

$\Rightarrow$ The blocks $\mathcal{F}(\alpha_s|x),\mathcal{F}(\alpha_t|1-x)$ should be related linearly on their common domain of analyticity via a *crossing kernel*!

Kernel as Fundamental

Schematically,



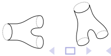
$$= \sum_i C_i \times (\text{Conformal Blocks})_i^s$$

$$= \sum_i \tilde{C}_i \times (\text{Conformal Blocks})_i^t$$

$$(\text{Conformal Blocks})_i^t = \sum_j \mathbb{F}_{ij} \times (\text{Conformal Blocks})_j^s$$

Lesson[Friedan,Shenker;'87,Moore-Seiberg;'88]:

- Conformal block expansion in a specific *channel* ⇒ expansion in a specific *basis*. Different basis should be related linearly.
- Sum over a complete set of states ⇒ "sewing" pair of pants and hence, consistency conditions ⇒ equivalence of different *independent* ways to "sew" the same Riemann surface.



The Elementary Crossing Kernels

- What are those kernels in 2d CFTs?

The Elementary Crossing Kernels

- What are those kernels in 2d CFTs?
- Amazingly, for $c \in \mathbb{C} \backslash (-\infty, 1]$ there is a closed-form expression for the **torus one-point** kernel $\mathbb{S}$ and the **sphere four-point** kernel $\mathbb{F}$ due to [B.Ponsot, J.Teschner,'99,'01,B.Ponsot;03'].
- Torus 1-point kernel $\mathbb{S}_{\alpha\alpha'}$:

$$G_{1,1}(\tau, \bar{\tau}) \equiv \langle O_0(0) \rangle_{T^2} = \sum_{\alpha} C_{\mathcal{O}\mathcal{O}\mathcal{O}_0} \mathcal{F}^{\mathcal{O}_0}(\alpha|\tau) \mathcal{F}^{\overline{\mathcal{O}_0}}(\overline{\alpha}|\overline{\tau})$$

$$\tau^{h_{\mathcal{O}_0}} \mathcal{F}^{\mathcal{O}_0}(\alpha'| - 1/\tau) = \int_{\mathcal{C}_{\mathbb{S}}} \frac{d\alpha}{2i} \, \mathbb{S}_{\alpha\alpha'}[O_0] \, \mathcal{F}^{\mathcal{O}_0}(\alpha|\tau)$$

- Sphere 4-point kernel $\mathbb{F}_{\alpha\alpha'}$:

$$G_{0,4}(z, \bar{z}) \equiv \sum_{\alpha} C_{12\mathcal{O}} C_{\mathcal{O}34} \mathcal{F}_S(\alpha|z) \overline{\mathcal{F}}_S(\bar{\alpha}|\bar{z})$$

$$\mathcal{F}_T(\alpha'|1 - z) = \int_{\mathcal{C}_{\mathbb{F}}} \frac{d\alpha}{2i} \, \mathbb{F}_{\alpha\alpha'} \begin{bmatrix} \alpha_2 & \alpha_1 \\ \alpha_3 & \alpha_4 \end{bmatrix} \, \mathcal{F}_S(\alpha|z)$$

The Elementary Crossing kernels

Torus 1-point kernel $\mathbb{S}_{\alpha\alpha'}$

$$\mathbb{S}_{\alpha\alpha'}[\mu] = \frac{\sqrt{2}S_b(2\alpha)}{S_b(2\alpha - Q)S_b(\mu)} \frac{\Gamma_b(2\alpha')\Gamma_b(2\alpha-\mu)(\times\text{reflections})}{\Gamma_b(2\alpha)\Gamma_b(2\alpha'-\mu)}$$
$$\times \int_{\mathcal{C}'_{\mathbb{S}}} \frac{d\xi}{i} e^{4\pi i(\frac{Q}{2}-\alpha')\xi} \frac{S_b(\alpha - \frac{Q-\mu}{2} + \xi)S_b(\alpha - \frac{Q-\mu}{2} - \xi)}{S_b(\alpha + \frac{Q-\mu}{2} + \xi)S_b(\alpha + \frac{Q-\mu}{2} - \xi)}$$

[B.Ponsot;03']

where

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$$

$\Gamma_b(x)$: b-defomred Γ function.
Can be thought of as a generalization of the usual Γ function with simple poles at $x = -mb - nb^{-1}$, $n, m \in \mathbb{Z}_{\geq 0}$ and no zeroes.

The Elementary Crossing kernels

Sphere 4-point kernel $\mathbb{F}_{\alpha\alpha'}$

$$\mathbb{F}_{\alpha\alpha'} \begin{bmatrix} \alpha_2 & \alpha_1 \\ \alpha_3 & \alpha_4 \end{bmatrix} = P_b(\alpha_i; \alpha, \alpha') \int_{\mathcal{C}'_{\mathbb{F}}} \frac{ds}{i} \prod_{k=1}^{4} \frac{S_b(s + U_k(\alpha_i))}{S_b(s + V_k(\alpha_i))}$$

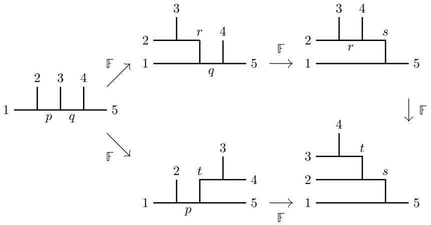
[B.Ponsot, J.Teschner;'99,'01]

where P_b is again made out of Γ_b functions.

- It is remarkable that these expressions are known explicitly for $c \in \mathbb{C} \backslash (-\infty, 1]$, whereas Virasoro conformal blocks are not!
- Once $\mathbb{S}$, $\mathbb{F}$ are known, all higher genus + higher point kernels are appropriate convolutions of these two kernels.
- In great generality, these kernels satisfy fundamental consistency conditions known as e.g. "pentagon identities".

The Elementary Crossing kernels

Pentagon identities



- This leads to

$$\sum_r \mathbb{F}_{rp}\begin{bmatrix} 1 & q \\ 2 & 3 \end{bmatrix} \mathbb{F}_{sq}\begin{bmatrix} 1 & 5 \\ r & 4 \end{bmatrix} \mathbb{F}_{tr}\begin{bmatrix} 2 & s \\ 3 & 4 \end{bmatrix} = \mathbb{F}_{tq}\begin{bmatrix} p & 5 \\ 3 & 4 \end{bmatrix} \mathbb{F}_{sp}\begin{bmatrix} 1 & 5 \\ 2 & t \end{bmatrix}$$

- Similarly, from the torus two-point function one gets

$$\mathbb{F}_{\mathcal{O}\mathbb{1}}\begin{bmatrix} t & t \\ t & t \end{bmatrix} \mathbb{S}_{st}[\mathcal{O}] = \mathbb{S}_{s\mathbb{1}}[\mathbb{1}] \sum_u e^{2\pi i(h_s+h_t-h_u-h_{\mathcal{O}}/2)} \mathbb{F}_{u\mathbb{1}}\begin{bmatrix} s & t \\ s & t \end{bmatrix} \mathbb{F}_{\mathcal{O}u}\begin{bmatrix} t & t \\ s & s \end{bmatrix}.$$

Analyticity and Crossing Kernels for $c \leq 1$

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- On the other hand, there are Minimal Models (MM) in that regime. For particular values of $\beta = \beta_{(MM)}$ we know some of these kernels: they are *finite dimensional matrices* [yellow book]
- But the "pentagon identities" – once specified into degenerate conformal dimensions – provide *difference equations* for the kernels that are *analytic* for any $b \in \mathbb{C}$. The Ponsot-Teschner expressions are just a special class of solutions of those equations.

Analyticity and Crossing Kernels for $c \leq 1$

Solutions of the pentagon identities

In [S.Ribault, I.T., to appear], we **solve explicitly** the pentagon identities in the regime $b \in i\mathbb{R}$!

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We find the *unique** solutions:

$$\Sigma_{P'P}[P_0] = \left[\frac{P}{P'}\frac{B_{tL}(P)C_{tDOZZ}(P_0,P',P')}{B_{tL}(P')C_{tDOZZ}(P_0,P,P)}\right] \times \mathbb{S}^{(ib)}_{iP',iP}[iP_0],$$

$$\Phi_{P'P}\begin{bmatrix} P_2 & P_1 \\ P_3 & P_4 \end{bmatrix} = \left[\frac{P}{P'}\frac{B_{tL}(P)C_{tDOZZ}(P_1,P_2,P')C_{tDOZZ}(P_3,P_4,P')}{B_{tL}(P')C_{tDOZZ}(P_1,P_2,P)C_{tDOZZ}(P_3,P_4,P)}\right]$$

$$\times \mathbb{F}^{(ib)}_{iP',iP}\begin{bmatrix} iP_2 & iP_1 \\ iP_3 & iP_4 \end{bmatrix}, \qquad b = -i\beta, \beta \in \mathbb{R}.$$

where B_{tL}, C_{tDOZZ} are repsectively the two and three-point structure constants of "time-like" Liouville theory.

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Solutions of the pentagon identities

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Some comments:

- These kernels reproduce all the known MM expressions + provide the **natural analytic continuation** for any $c \in (-\infty, 1]$!
- Note that $b \to ib$ means $c \to 26 - c$. These formulas seem to realize an explicit large/small central charge connection in 2d CFTs at the level of kinematic quantities.
- The appearance of the time-like Liouville theory quantities is not an accident...

Conformal Bootstrap applications

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Proof of crossing symmetry for time-like Liouville theory

For a theory with only scalar primaries, the modular covariance equation for the torus 1-pt function can be recast as:

$$\frac{C_{P,P,P_0}}{B(P)}\,\mathbb{S}_{PP'}[P_0] = \frac{C_{P,'P',P_0}}{B(P')}\,\mathbb{S}^{-1}_{P'P}[P_0] \qquad (*)$$

- For **space-like Liouville theory** (formally defined for $c \in \mathbb{C}\backslash(-\infty, 1]$) $\Rightarrow C_{P,P,P_0} \equiv C_{DOZZ}(P,P,P_0)$ and $B(P) \equiv B_L(P)$. Equation $(*)$ is satisfied non-trivially as a particular instance of the pentagon identity.

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- What about **"time-like" Liouville theory**? Defined for $c \leq 1$.If we substitute $C_{P,P,P_0} \equiv C_{tDOZZ}(P, P, P_0)$, $B(P) \equiv B_{tL}(P)$, and also use our kernel $\mathbb{\Sigma}$, is the equation:

$$\frac{C_{tDOZZ}(P, P, P_0)}{B_{tL}(P)}\ \mathbb{\Sigma}_{PP'}[P_0] = \frac{C_{tDOZZ}(P', P', P_0)}{B_{tL}(P')}\ \mathbb{\Sigma}^{-1}_{P'P}[P_0]$$

true?!

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Yes!

[S.Ribault, I.T., to appear]

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Yes! [S.Ribault, I.T., to appear]

- The proof starts from modular covariance of the space-like theory, and uses the explicit relation between the kernels $\mathbb{S}$ and Σ, as well as the non-trivial relations between the structure constants of the two theories:

$$B_{tL}(P) = -\frac{1}{4P^2}\frac{1}{B_L^{(ib)}(iP)}\quad,\ C_{tDOZZ}(P_1,P_2,P_3) = \frac{1}{C_{DOZZ}^{(ib)}(iP_1,iP_2,iP_3)}.$$

- Similarly, we prove crossing symmetry of the four-point functions on the sphere by using the relations between the kernels $\mathbb{F}$ and Φ.

Summary & Future Directions

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- We studied the *crossing kernels* for torus 1-pt functions and sphere 4-pt functions in 2d CFTs. We completed the study of [Ponsot,Teschner] by providing solutions in the 'missing' regime $c \in (-\infty, 1]$:

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[S.Ribault, I.T., to appear]

- Novel application is an *analytic* proof of crossing symmetry for time-like Liouville theory, which was so far elusive.
- Intricate relation between $c \leftrightarrow 26 - c$ (or $b \leftrightarrow ib$). Rigid structure of Analytic Conformal Bootstrap as a function of the central charge. Could we explore this further in some universal kinematic regimes of the bootstrap (e.g. lightcone bootstrap)?
- Reverse logic: could the existence of Minimal Models for $c \leq 1$ teach us something about 'irrational' $c \geq 25$ CFTs? Correspondence between large positive/large negative c?

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Thank you!