Uncovering the Structure of the ε Expansion

Andreas Stergiou

King's College London

Based on past work with Hugh Osborn and Slava Rychkov,
and ongoing work with William Pannell

(arXiv:1707.06165, 1810.10541, 2010.15915)

ε expansion

Ideas of the renormalization group are unsurprisingly best understood when we can use perturbation theory.

Unfortunately, we typically don't have a small parameter with which to construct perturbative series for physical observables of interest.

In some cases, however, we can make one up!

The most notable example is the ε expansion, pioneered by Wilson and Fisher more than 50 years ago, in 1971.

The main pursuit since then has been to access the physics of fixed points in $d = 3$ dimensions using the following logic:

1. start in $d = 4 - \varepsilon$,
2. compute physical observables as series in ε,
3. resum and send $\varepsilon \to 1$ in the end.

ε expansion — Simplest example

$$\int d^{4-\varepsilon}x \left(\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \tfrac{1}{4!}\lambda\varphi^4 \right)$$

For $\lambda = 0$ the operator φ^4 is relevant:

$$\Delta_{\varphi^4} = 4\frac{d-2}{2} = 4 - 2\varepsilon < 4 - \varepsilon\,.$$

Thus, when the free theory is deformed by the operator φ^4, a renormalization-group flow is triggered.

The flow ends at another, interacting fixed point. The β function of λ is $\beta_\lambda = -\varepsilon\lambda + 3\lambda^2$, and has a non-trivial zero at $\lambda = \varepsilon/3$. This is a fixed point with $\mathbb{Z}_2$ global symmetry obtained in the Wilson–Fisher prescription.

It is infrared-attractive, for the operator φ^4 is irrelevant there:

$$\Delta_{\varphi^4} = d + \partial_\lambda\beta_\lambda|_{\lambda=\varepsilon/3} = 4 > 4 - \varepsilon\,.$$

ε expansion — Ising model

Scaling dimensions of operators are the main observables.

With regular Feynman diagrams or analytic bootstrap methods we may compute

$$\Delta_\varphi = 1 - \tfrac{1}{2}\varepsilon + \tfrac{1}{108}\varepsilon^2 + O(\varepsilon^3)\,, \qquad \Delta_{\varphi^2} = 2 - \tfrac{2}{3}\varepsilon + \tfrac{19}{162}\varepsilon^2 + O(\varepsilon^3)\,.$$

It turns out that the $\mathbb{Z}_2$-invariant fixed point we just found (with $\varepsilon \to 1$) is in the same universality class as the 3D Ising lattice model, the critical point of water as well as the second-order phase transition in ferromagnets at the Curie temperature.

Many scalars

The strategy we just described has been applied to a wide variety of problems.

An obvious generalization is to consider the multi-scalar case,

$$\int d^{4-\varepsilon}x \left(\tfrac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i + \tfrac{1}{4!}\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l \right) , \qquad i = 1, \ldots, N .$$

Then,

$$\beta_{ijkl} = -\varepsilon\,\lambda_{ijkl} + \lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ilmn}\lambda_{jkmn} .$$

There are $\frac{1}{4!}N(N+1)(N+2)(N+3)$ independent couplings and β functions.

Imposing a global symmetry under which the action is invariant reduces the number of couplings and β functions.

Various symmetries

There are a few known classes of fixed points with various global symmetry groups.

- $O(N)$: $(\varphi^2)^2$,
- ${\mathbb{Z}_2}^N \rtimes \mathcal{S}_N$ (hypercubic) : $(\varphi^2)^2$ and $\sum_{i=1}^N \varphi_i^4$,
- $\mathcal{S}_{N+1} \times \mathbb{Z}_2$ (hypertetrahedral) : $(\varphi^2)^2$ and $\sum_{a=1}^{N+1} (e_i^a \varphi_i)^4$,
- $O(m) \times O(n)/\mathbb{Z}_2$: $(\mathrm{tr}\,\varphi^2)^2$ and $\mathrm{tr}\,\varphi^4$,
- $O(m) \times O(n)$ (biconical) : $(\varphi^2)^2$, $(\chi^2)^2$ and $\varphi^2\chi^2$,
- …

These theories have been extensively analyzed due to their applications to critical phenomena, in many cases with results computed up to six loops.

Since the resulting series are asymptotic, resummation techniques are typically used to take the $\varepsilon \to 1$ limit.

This talk

We will be interested in a different set of questions that arise when one considers the overall structure of the ε expansion itself.

What are universal constraints that need to be satisfied by any theory obtained as a fixed point in the ε expansion?

Is there an organizing principle for fixed points in the ε expansion?

We will be interested in systems with scalar fields, scalars and fermions, and will also briefly consider line defects.

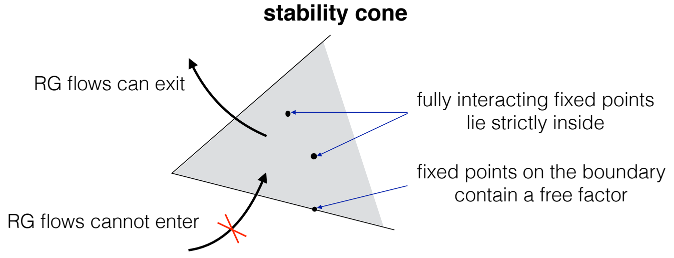
We want to assess how hard it might really be to "map the space of CFTs in 3D".

For the rest of this talk we will mostly discuss results at leading order in ε.

Some general results for scalar fixed points

At any fixed point, the scalar potential is bounded below:

$$\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l = 3\,\lambda_{ijmn}\lambda_{klmn}\varphi_i\varphi_j\varphi_k\varphi_j = 3\,(\lambda_{ijmn}\varphi_i\varphi_j)(\lambda_{klmn}\varphi_k\varphi_l) \geqslant 0\,.$$



Given a set of quartic terms in the scalar potential, if a stable fixed point exists, then it is unique.(Michel; 1984)

A bound for scalar theories

The symmetric coupling tensor λ_{ijkl} can be decomposed into irreducible representations of $O(N)$ as

$$\lambda_{ijkl} = d_0(\delta_{ij}\delta_{kl} + \ldots) + (\delta_{ij} d_{2,kl} + \ldots) + d_{4,ijkl} \, ,$$

where d_2 and d_4 are symmetric and traceless.

Schematically, this is the decomposition

$$\text{rank-4 symmetric tensor} = \text{spin-0} \oplus \text{spin-2} \oplus \text{spin-4} \, .$$

Let us now define the $O(N)$ invariants

$$a_0 = \lambda_{iijj} \, , \qquad a_1 = \lambda_{ijkk}\lambda_{ijll} \, ,$$

which are the only invariants up to quadratic order beyond S.

A bound for scalar theories

We will work with the quantities(Hogervorst & Toldo; 2020. Osborn & AS; 2020)

$$a_0 = N(N+2)d_0\,, \qquad a_2 = (N+4)^2||d_2||^2 = a_1 - \tfrac{1}{N}a_0^2\,,$$
$$a_4 = ||d_4||^2 = S - \tfrac{6}{N+4}a_2 - \tfrac{3}{N(N+2)}a_0^2\,.$$

If $a_2 \neq 0$, there exists a non-trivial $d_{2,ij}$ tensor and there are then more than one quadratic invariants.

From the β-function equation,

$$\lambda_{iijj} = \lambda_{iimn}\lambda_{jjmn} + 2\lambda_{ijmn}\lambda_{ijmn} \quad \Rightarrow \quad a_0 = a_2 + \tfrac{1}{N}a_0^2 + 2S\,,$$

which can be brought to the form

$$S + \tfrac{1}{2}a_2 = \tfrac{1}{8}N - \tfrac{1}{2N}(a_0 - \tfrac{1}{2}N)^2 \leqslant \tfrac{1}{8}N\,.$$

Bound saturation

For $N \geqslant 4$ there are some known cases where the bound is saturated, all of them with $a_2 = 0$.

- $N = 4$: $O(4)$,
- $N = 5$: hypertetrahedral ($\mathcal{S}_6 \times \mathbb{Z}_2$),
- $N = m_i n_i$, with $(m_{i+1}, n_{i+1}) = (10m_i - n_i + 4, m_i)$, $m_1 = 7, n_1 = 1$: $O(m_i) \times O(n_i)/\mathbb{Z}_2$,
- $N = 2m_i n_i$, with $(m_{i+1}, n_{i+1}) = (10m_i - n_i, m_i)$, $m_1 = 5, n_1 = 1$: $U(m_i) \times U(n_i)/U(1)$.

Allowing factorized fixed points, the bound can be saturated for all N except for $N = 2, 3, 6, 7, 11$ (based on our current knowledge).

One can show that whenever the bound is saturated with $a_2 = 0$, there is a marginal operator in the theory.

Known fixed points for low N

We will be interested in fully-interacting fixed points only.

For $N = 1$ the only fixed point is Ising.

For $N = 2$ the only fixed point is the $O(2)$ fixed point. It does not saturate the bound, so the bound cannot be saturated for $N = 2$.(Osborn & AS; 2017)

For $N = 3$ the only fixed points were recently shown to be $O(3)$, cubic and biconical.

$N = 3$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
C_3	$\frac{10}{27}$	$\frac{4}{3}$	0	$\frac{2}{135}$	$B_3 = \mathbb{Z}_2{}^3 \rtimes \mathcal{S}_3$	1(3)	1, 5
B_{I*O_2}	0.370451	1.33713	0.000255	0.01265	$\mathbb{Z}_2 \times O(2)$	2(2,1)	1, 2
O_3	$\frac{45}{121}$	$\frac{15}{11}$	0	0	$O(3)$	1(3)	0, 0

Known fixed points for low N

$N=4$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa<0,=0$
O_4	$\frac{1}{2}$	2	0	0	$O(4)$	1(4)	0, 25
T_{4-}	$\frac{220}{441}$	$\frac{40}{21}$	0	$\frac{20}{441}$	$\mathcal{S}_5\times\mathbb{Z}_2$	1(4)	15, 6

$N=5$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa<0,=0$
O_5	$\frac{105}{169}$	$\frac{35}{13}$	0	0	$O(5)$	1(5)	55, 0
C_5	$\frac{28}{45}$	$\frac{8}{3}$	0	$\frac{4}{315}$	B_5	1(5)	40, 14
$T_{5\pm}$	$\frac{5}{8}$	$\frac{5}{2}$	0	$\frac{5}{56}$	$\mathcal{S}_6\times\mathbb{Z}_2$	1(5)	39, 11

Known fixed points for low N

$N=6$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa<0, =0$
O_6	$\frac{36}{49}$	$\frac{24}{7}$	0	0	$O(6)$	1(6)	105, 0
C_6	$\frac{20}{27}$	$\frac{10}{3}$	0	$\frac{5}{108}$	B_6	1(6)	84, 20
$MN_{2,3}$	$\frac{90}{121}$	$\frac{36}{11}$	0	$\frac{9}{121}$	$O(2)^3 \rtimes \mathcal{S}_3$	1(6)	86, 12
$MN_{3,2}$	$\frac{216}{289}$	$\frac{54}{17}$	0	$\frac{135}{1156}$	$O(3)^2 \rtimes \mathbb{Z}_2$	1(6)	77, 9
T_{6+}	$\frac{110}{147}$	$\frac{20}{7}$	0	$\frac{5}{21}$	$\mathcal{S}_7 \times \mathbb{Z}_2$	1(6)	84, 15
T_{6-}	$\frac{182}{243}$	$\frac{28}{9}$	0	$\frac{35}{243}$	$\mathcal{S}_7 \times \mathbb{Z}_2$	1(6)	83, 15

$N=7$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa<0, =0$
O_7	$\frac{21}{25}$	$\frac{21}{5}$	0	0	$O(7)$	1(7)	182, 0
C_7	$\frac{6}{7}$	4	0	$\frac{2}{21}$	B_7	1(7)	154, 27
T_{7+}	$\frac{105}{121}$	$\frac{35}{11}$	0	$\frac{5}{21}$	$\mathcal{S}_8 \times \mathbb{Z}_2$	1(7)	154, 21
T_{7-}	$\frac{196}{225}$	$\frac{56}{15}$	0	$\frac{28}{135}$	$\mathcal{S}_8 \times \mathbb{Z}_2$	1(7)	153, 21

And now?

Is that really all there is?

There is a general perception that conformal field theories are rare.

But is this perception correct?

We are of course talking about unitary conformal field theories.

Our bound on S shows that fixed points in the ε expansion are indeed constrained. This could be seen as a hint suggesting their scarcity, but is there more we could say?

Do most fixed points in the ε expansion have rational S, a_0, a_2, a_4?

Numerical search for fixed points for low N

We numerically solved the β-function equations.

We made no assumptions about symmetries.

Somehow, this brute force approach had not been attempted before.

The algorithm we used is called IPOpt. It is an algorithm that can perform nonlinear constrained optimization.

We found that IPOpt performs very well for our problem for N as high as 9 (495 equations and couplings), but we will focus on $N \leqslant 7$. For $N = 7$ there are 210 equations and couplings.

Numerically-obtained fixed points for $N = 4$

$N = 4$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
O_4	$\frac{1}{2}$	2	0	0	$O(4)$	1(4)	0, 25
T_{4-}	$\frac{220}{441}$	$\frac{40}{21}$	0	$\frac{20}{441}$	$\mathcal{S}_5 \times \mathbb{Z}_2$	1(4)	15, 6
???	0.499115	1.92406	0.000328	0.036117		3(1,2,1)	14, 6
???	0.499144	1.92641	0.000359	0.034994		3(1,2,1)	13, 6
???	0.499606	1.95458	0.000273	0.021851		2(2,2)	12, 5

Quite surprising… three new fixed points.

This numerical method gives us numbers, but doesn't tell us anything about the nature of these fixed points, e.g. their global symmetries.

To uncover more information, the number of different eigenvalues of γ_{ij} and their degeneracies, as well as the number of zero modes of the stability matrix provide good hints.

“Double trace” perturbations

Take two known theories, add them up, and couple their quadratic invariants.

This follows the spirit of biconical theories:

$$V_{\text{biconical}} = \tfrac{1}{8}\lambda_1\,(\varphi^2)^2 + \tfrac{1}{8}\lambda_2\,(\chi^2)^2 + \tfrac{1}{4}h\,\varphi^2\chi^2\,.$$

It is by no means guaranteed that this procedure will yield new unitary fixed points.

It may just be that the only real solutions obtained are the ones where the coupling h of the quadratic invariants is set to zero.

However, if we apply this procedure with $V_{\mathcal{S}_3}(\varphi)$ and $V_{\text{Ising}}(\varphi)$, we find a new $N = 4$ fixed point with $S = 0.499115$, which is one of the numerically obtained solutions!

The other two $N = 4$ fixed points

$$V_2(\varphi) = \tfrac{1}{8}\,\lambda\,({\varphi_1}^2 + {\varphi_2}^2)^2 + \tfrac{1}{24}\,g\,({\varphi_1}^4 + {\varphi_2}^4) + \tfrac{1}{24}\,x_1\,{\varphi_3}^4 + \tfrac{1}{24}\,x_2\,{\varphi_4}^4 + \tfrac{1}{4}\,z\,{\varphi_3}^2{\varphi_4}^2 + \tfrac{1}{4}\,h_1({\varphi_1}^2 + {\varphi_2}^2){\varphi_3}^2 + \tfrac{1}{4}\,h_2({\varphi_1}^2 + {\varphi_2}^2){\varphi_4}^2 + h\,\varphi_1\varphi_2\varphi_3\varphi_4\,.$$

Symmetry: $D_4 \times \mathbb{Z}_2$

$$V_3(\varphi) = \tfrac{1}{8}\,\lambda_1({\varphi_1}^2 + {\varphi_2}^2)^2 + \tfrac{1}{8}\,\lambda_2({\varphi_3}^2 + {\varphi_4}^2)^2 + \tfrac{1}{4}\,h({\varphi_1}^2 + {\varphi_2}^2)({\varphi_3}^2 + {\varphi_4}^2) + \tfrac{1}{6}\,\hat{h}\left({\varphi_1}^3 - 3\,\varphi_1{\varphi_2}^2,\ {\varphi_2}^3 - 3\,{\varphi_1}^2\varphi_2\right)\cdot\left(\varphi_3,\ \varphi_4\right).$$

Symmetry: $O(2)$

These new $N = 4$ fixed points were independently discovered recently, but their global symmetry groups were not identified correctly.(Codello, Safari, Vacca & Zanusso; 2020)

Numerically-obtained fixed points for $N = 4$

This is (very likely) the complete table of $N = 4$ fixed points:

$N = 4$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
O_4	$\frac{1}{2}$	2	0	0	$O(4)$	1(4)	0, 25
T_{4-}	$\frac{220}{441}$	$\frac{40}{21}$	0	$\frac{20}{441}$	$\mathcal{S}_5 \times \mathbb{Z}_2$	1(4)	15, 6
$B_{\mathcal{S}_3 * I}$	0.499115	1.92406	0.000328	0.036117	$\mathcal{S}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	3(1,2,1)	14, 6
$\hat{B}_{O_2 * I * I}$	0.499144	1.92641	0.000359	0.034994	$D_4 \times \mathbb{Z}_2$	3(1,2,1)	13, 6
$O_2 \circ O_2$	0.499606	1.95458	0.000273	0.021851	$O(2)$	2(2,2)	12, 5

Fixed points for $N = 5$

$N = 5$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
O_5	$\frac{105}{169}$	$\frac{35}{13}$	0	0	$O(5)$	1(5)	55, 0
C_5	$\frac{28}{45}$	$\frac{8}{3}$	0	$\frac{4}{315}$	B_5	1(5)	40, 14
$T_{5\pm}$	$\frac{5}{8}$	$\frac{5}{2}$	0	$\frac{5}{56}$	$\mathcal{S}_6 \times \mathbb{Z}_2$	1(5)	39, 11
B_{I*O_4}	0.621937	2.67255	0.000170	0.009605	$\mathbb{Z}_2 \times O(4)$	2(4,1)	50, 4
$B_{C_2*O_3}$	0.622163	2.66667	0.000118	0.012561	$B_2 \times O(3)$	2(3,2)	46, 7
$B_{C_3*O_2}$	0.622230	2.66560	0.000056	0.013157	$B_3 \times O(2)$	2(2,3)	41, 9
$B_{I*O_2*O_2}$	0.623037	2.63897	0.000064	0.026068	$\mathbb{Z}_2 \times O(2) \times O(2)$	3(2,1,2)	40, 8
$B_{C_3*O_2}$	0.623040	2.63881	0.000066	0.026139	$B_3 \times O(2)$	2(3,2)	38, 9
$B_{O_2*O_3}$	0.623053	2.63808	0.000082	0.026474	$O(2) \times O(3)$	2(3,2)	37, 6

Irrational fixed points for $N = 6$

$N = 6$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
B_{I*O_5}	0.738216	3.35878	0.002115	0.031859	$\mathbb{Z}_2 \times O(5)$	2(5,1)	99, 5
$B_{C_2*O_4}$	0.739865	3.33333	0.001752	0.044369	$B_2 \times O(4)$	2(3,3)	94, 9
$B_{C_3*O_3}$	0.740572	3.32649	0.001091	0.048323	$B_3 \times O(3)$	2(3,3)	90, 12
$B_{C_4*O_2}$	0.740798	3.32758	0.000520	0.048438	$B_4 \times O(2)$	2(2,4)	85, 14
$B_{O_2*O_4}$	0.744334	3.32362	0.002037	0.088569	$O(2) \times O(4)$	2(4,2)	94, 8
$B_{I*O_2*O_3}$	0.744373	3.23709	0.001886	0.088318	$\mathbb{Z}_2 \times O(2) \times O(3)$	3(3,1,2)	90, 11
$B_{C_4*O_2}$	0.7443770	3.23720	0.001868	0.088288	$B_4 \times O(2)$	2(4,2)	87, 14
$B_{\mathcal{S}_4*O_2}$	0.7443773	3.23721	0.001867	0.088286	$\mathcal{S}_4 \times \mathbb{Z}_2 \times O(2)$	3(3,1,2)	86, 14
$B_{C_2*O_2*O_2}$	0.744379	3.23726	0.001860	0.088272	$B_2 \times O(2) \times O(2)$	3(2,2,2)	85, 13
$B_{O_2*O_2*O_2}$	0.744437	3.23901	0.001605	0.087776	$(O(2)^2 \rtimes \mathbb{Z}_2) \times O(2)$	2(4,2)	85, 12
$B_{I*I*O_2*O_2}$	0.746610	3.19983	0.000125	0.106603	$(\mathbb{Z}_2 \times O(2))^2 \rtimes \mathbb{Z}_2$	2(4,2)	83, 13
$B_{\mathcal{S}_4*O_2}$	0.746638	3.19991	0.000063	0.106637	$\mathcal{S}_4 \times \mathbb{Z}_2 \times O(2)$	3(2,3,1)	81, 14
$B_{I*O_2*O_3}$	0.746962	3.18917	0.000112	0.111220	$\mathbb{Z}_2 \times O(2) \times O(3)$	3(2,3,1)	80, 11
$B_{C_3*O_3}$	0.746991	3.18955	0.000030	0.111147	$B_3 \times O(3)$	2(3,3)	78, 12

Irrational fixed points for $N = 7$

$N = 7$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
B_{I*O_6}	0.848454	4.05973	0.008335	0.059079	$\mathbb{Z}_2 \times O(6)$	2(6,1)	175, 6
$B_{C_3*O_4}$	0.855735	3.97989	0.005630	0.098402	$B_3 \times O(4)$	2(4,3)	164, 15
$B_{C_3*C_4}$	0.857146	3.99516	0.000681	0.098711	$B_3 \times B_4$	2(3,4)	156, 21
$B_{C_5*O_2}$	0.857297	3.98590	0.001676	0.099839	$O(2) \times B_5$	2(2,5)	155, 20
$B_{O_2*O_5}$	0.862416	3.82034	0.010508	0.161683	$O(2) \times O(5)$	2(5,2)	169, 10
$B_{I*O_2*O_4}$	0.863351	3.82328	0.008369	0.162715	$\mathbb{Z}_2 \times O(2) \times O(4)$	3(2,4,1)	164, 14
$B_{C_2*O_2*O_3}$	0.863688	3.82583	0.007459	0.162621	$B_2 \times O(2) \times O(3)$	3(3,2,2)	160, 17
$B_{C_5*O_2}$	0.863748	3.82704	0.007224	0.162369	$O(2) \times B_5$	2(5,2)	158, 20
$B_{C_2*C_3*O_2}$	0.863750	3.82693	0.007230	0.162405	$O(2) \times B_2 \times B_3$	3(3,2,2)	156, 20
$B_{O_2*O_2*C_3}$	0.863776	3.82689	0.007183	0.162473	$O(2) \times O(2) \times B_3$	3(2,3,2)	155, 19
$B_{I*O_2*O_2*O_2}$	0.865351	3.85371	0.001426	0.157379	$\mathbb{Z}_2 \times O(2) \times (O(2)^2 \rtimes \mathbb{Z}_2)$	3(2,1,4)	155, 18
$B_{I*O_2*O_2*O_2}$	0.865360	3.84698	0.002082	0.159497	$\mathbb{Z}_2 \times O(2) \times O(2) \times O(2)$	4(2,1,2,2)	154, 18
$B_{O_2*O_2*C_3}$	0.865363	3.85323	0.001450	0.157553	$B_3 \times (O(2)^2 \rtimes \mathbb{Z}_2)$	2(3,4)	153, 19
$B_{O_2*O_2*C_3}$	0.865370	3.84723	0.002036	0.159439	$O(2) \times O(2) \times B_3$	3(3,2,2)	152, 19
$B_{O_2*O_2*O_3}$	0.865427558	3.84923	0.001721	0.158937	$O(3) \times (O(2)^2 \rtimes \mathbb{Z}_2)$	2(3,4)	152, 16
$B_{O_2*O_2*O_3}$	0.865427563	3.84907	0.001738	0.158988	$O(2) \times O(2) \times O(3)$	3(3,2,2)	151, 16
$B_{I*C_2*O_4}$	0.8712962	3.68437	0.002552	0.223496	$\mathbb{Z}_2 \times B_2 \times O(4)$	3(4,2,1)	162, 15
$B_{I*C_2*C_4}$	0.87129773	3.684606	0.002536	0.223423	$\mathbb{Z}_2 \times B_2 \times O(4)$	3(4,2,1)	155, 21
	0.87129775	3.684611	0.002536	0.223421		4(2,2,2,1)	153, 20
$B_{C_3*O_4}$	0.8712983	3.68496	0.002516	0.223311	$B_3 \times O(4)$	2(4,3)	161, 15
	0.8712989	3.70402	0.001456	0.217183		3(4,2,1)	152, 21
	0.8712994	3.68487	0.002519	0.223342		3(4,2,1)	153, 19

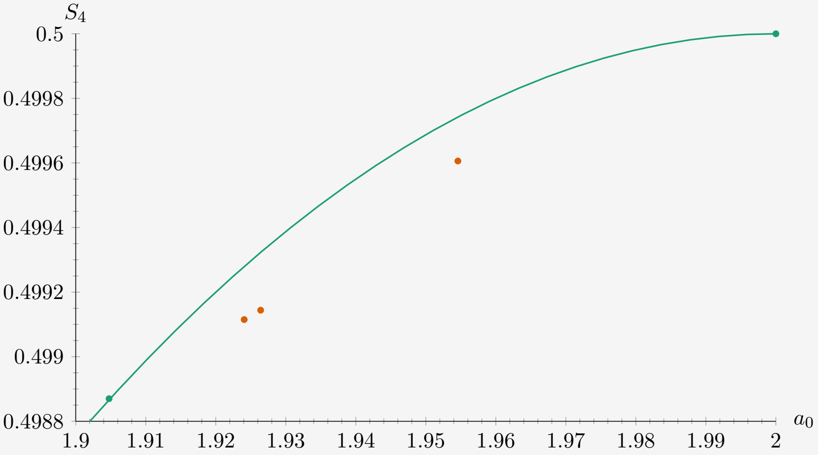
Irrational fixed points for $N = 7$ (cont'd)

$N = 7$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
$B_{I*O_3*C_3}$	0.8712996	3.68516	0.002503	0.223247	$\mathbb{Z}_2 \times B_3 \times O(3)$	3(3,1,3)	157, 18
$B_{C_3*C_4}$	0.871299832	3.6852003	0.00250046	0.2232359	$B_3 \times B_4$	2(4,3)	154, 21
$B_{I*C_3*C_3}$	0.871299833	3.6852004	0.00250045	0.2232358	$\mathbb{Z}_2 \times B_3 \times B_3$	3(3,1,3)	153, 21
$B_{O_2*C_2*C_3}$	0.87129986	3.68521	0.002500	0.223234	$O(2) \times B_2 \times B_3$	3(2,2,3)	152, 20
$B_{O_2*O_2*C_3}$	0.871301	3.68547	0.002483	0.223153	$B_3 \times (O(2)^2 \rtimes \mathbb{Z}_2)$	2(4,3)	152, 19
$B_{C_3*T_4}$	0.871304	3.70466	0.001409	0.216987	$B_3 \times \mathcal{S}_5 \times \mathbb{Z}_2$	2(4,3)	151, 21
	0.871305	3.70164	0.001581	0.217961		5(1,2,1,2,1)	151, 21
	0.87130606	3.70132	0.001598	0.218064		5(1,1,2,2,1)	150, 21
$B_{\mathcal{S}_5*O_2}$	0.871306	3.70264	0.00152144	0.217639	$\mathcal{S}_5 \times \mathbb{Z}_2 \times O(2)$	3(2,4,1)	151, 20
	0.871310	3.70227	0.001536	0.217767		4(1,2,1,3)	150, 21
	0.871311	3.70195	0.001553	0.217871		4(1,1,2,3)	149, 21
	0.871314	3.70006	0.001655	0.218486		5(1,2,2,1,1)	150, 20
	0.8713147	3.69972	0.0016724	0.218597		5(1,2,1,2,1)	149, 20
$B_{I*O_2*O_4}$	0.8713152	3.68073	0.002703	0.224709	$\mathbb{Z}_2 \times O(2) \times O(4)$	3(2,4,1)	161, 14
$B_{I*I*O_2*O_3}$	0.871316	3.68092	0.002691	0.224648	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times O(2) \times O(3)$	4(3,2,1,1)	157, 17
$B_{I*O_2*C_4}$	0.87131659	3.68096	0.002689	0.224637	$\mathbb{Z}_2 \times O(2) \times B_4$	3(2,4,1)	154, 20
	0.87131661	3.68097	0.002688	0.224636		4(2,2,2,1)	152, 19
$B_{I*O_2*O_2*O_2}$	0.871318	3.68121	0.002673	0.224559	$\mathbb{Z}_2 \times O(2) \times (O(2)^2 \times \mathbb{Z}_2)$	3(2,4,1)	152, 18
	0.8713206	3.68941	0.002233	0.221922		3(4,2,1)	151, 21
	0.87132074	3.68949	0.002229	0.221899		5(1,2,1,2,1)	150, 21
	0.87132076	3.6895	0.002228	0.221894		5(1,1,2,2,1)	149, 21
$B_{C_3*T_4}$	0.8713233	3.69025	0.002183	0.221659	$B_3 \times \mathcal{S}_5 \times \mathbb{Z}_2$	2(4,3)	150, 21
	0.87132340	3.69033	0.002178	0.221632		4(1,2,1,3)	149, 21
	0.87132342	3.69035	0.002177	0.221627		4(1,1,2,3)	148, 21

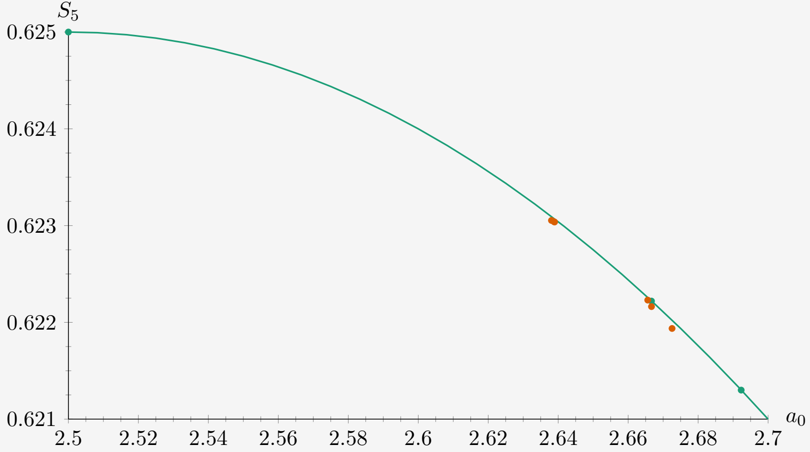
Irrational fixed points for $N = 7$ (cont'd)

$N = 7$	S	a_0	a_2	a_4	Symmetry	# different γ and degeneracies	$\#\kappa < 0, = 0$
$B_{\mathcal{S}_5 * O_2}$	0.871337	3.68539	0.00241684	0.223251	$\mathcal{S}_5 \times \mathbb{Z}_2 \times O(2)$	3(2,4,1)	150, 20
	0.87133668	3.68544	0.0024141	0.223236		5(1,2,2,1,1)	149, 20
	0.87133669	3.68545	0.0024136	0.223233		5(1,2,1,2,1)	148, 20
$B_{T_4 * O_3}$	0.872241	3.68634	0.000557	0.224839	$O(3) \times \mathcal{S}_5 \times \mathbb{Z}_2$	2(4,3)	150, 18
	0.872269	3.68187	0.000737	0.226337		4(1,2,3,1)	149, 18
	0.872273	3.68132	0.000758	0.226521		4(1,1,2,3)	148, 18
$\hat{B}_{(O_2 \circ O_2) * O_3}$	0.872388	3.66736	0.001223	0.231267	$O(2) \times O(3)$	3(2,2,3)	147, 17
$B_{O_3 * O_4}$	0.8724124	3.65263	0.001847	0.236084	$O(3) \times O(4)$	2(4,3)	160, 12
$B_{I * O_3 * O_3}$	0.8724128	3.65273	0.001842	0.236054	$\mathbb{Z}_2 \times O(3) \times O(3)$	3(3,1,3)	156, 15
$B_{C_4 * O_3}$	0.8724129	3.65275	0.001841	0.236049	$O(3) \times B_4$	2(4,3)	153, 18
$B_{O_2 * O_2 * O_3}$	0.872413	3.65286	0.001835	0.236012	$O(3) \times (O(2)^2 \rtimes \mathbb{Z}_2)$	2(4,3)	151, 16
$B_{T_4 * O_3}$	0.872418318	3.654206	0.0017663	0.235587	$O(3) \times \mathcal{S}_5 \times \mathbb{Z}_2$	2(4,3)	149, 18
	0.872418321	3.654208	0.0017662	0.235586		4(1,1,2,3)	147, 18
$\hat{B}_{(O_2 \circ O_2) * O_3}$	0.872419	3.65456	0.001749	0.235474	$O(2) \times O(3)$	3(2,2,3)	146, 17

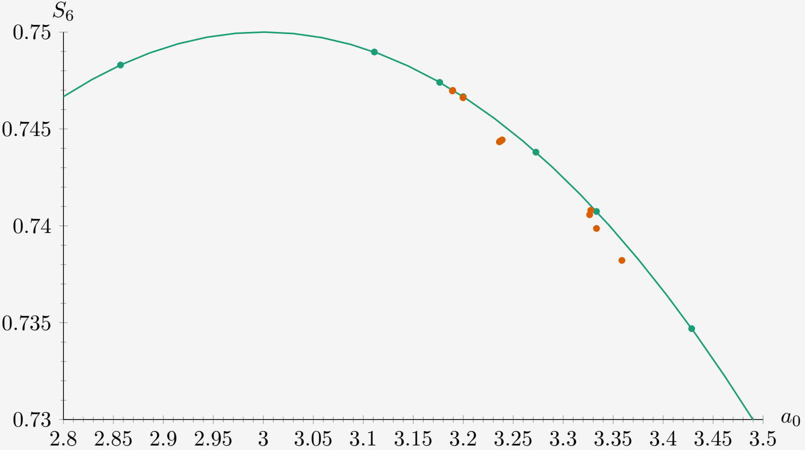
Fixed points for $N = 4$



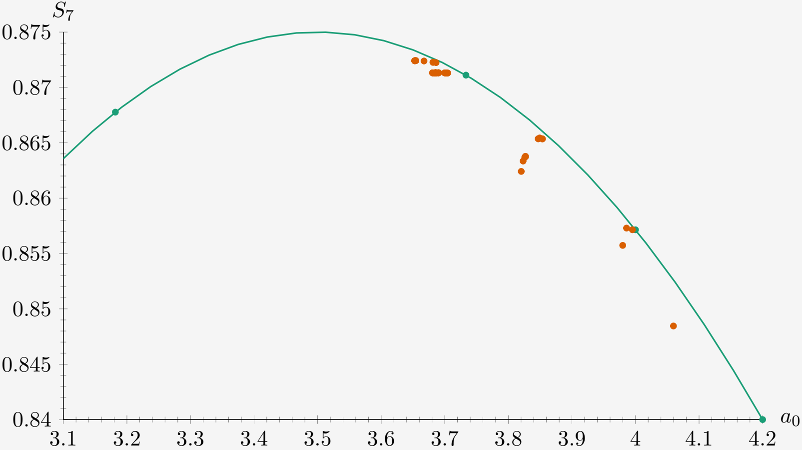
Fixed points for $N = 5$



Fixed points for $N = 6$



Fixed points for $N = 7$



Fixed points in scalar-fermion systems

Consider

$$\int d^{4-\varepsilon}x\,\big(\tfrac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_j + i\bar{\psi}_a\bar{\sigma}^\mu\partial_\mu\psi_a + \tfrac{1}{4!}\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l + \big(\tfrac{1}{2}y_{iab}\varphi_i\psi_a\psi_b + \text{h.c.}\big)\big)\,.$$

Now we have the Yukawa β functions too and the Yukawa contributions to the quartic coupling β functions:

$$\beta_{iab} = -\tfrac{1}{2}\varepsilon\, y_{iab} + \text{“}(y^3)_{iab}\text{”}\,,$$
$$\beta_{ijkl} = -\varepsilon\,\lambda_{ijkl} + \text{“}(\lambda^2)_{ijkl}\text{”} + \text{“}(\lambda\bar{y}y)_{ijkl}\text{”} + \text{“}(\bar{y}^2y^2)_{ijkl}\text{”}\,.$$

Well-known models of this type include the Gross–Neveu–Yukawa model and the Nambu–Jona-Lasinio–Yukawa model.

There are suggestions for emergent supersymmetry in $d=3$ in these models.(Fei, Giombi, Klebanov & Tarnopolsky; 2016)

Some general results for scalar-fermion fixed points

Since the Yukawa β function does not depend on λ, we may set it to zero, obtain the Yukawa solutions and then feed them one by one into the quartic β-function equation.

Each Yukawa solution will be characterized by a set of invariants.

For each distinct Yukawa solution we will get a few distinct solutions of the quartic β-function equation.

It turns out that if one of them corresponds to a stable fixed point, then it is unique. This generalizes Michel's theorem.

Unfortunately, scalar potentials are not guaranteed to be bounded below if the Yukawa solutions are non-trivial.

A bound for scalar-fermion theories

Similarly to the scalar case, we can here define invariants that now involve the Yukawa coupling tensor too.

These invariants satisfy two bounds, one coming from the quartic and one from the Yukawa β function:

$$S + b_0 - \tfrac{1}{2}(b_1 - b_2) - 2b_3 \leqslant \tfrac{1}{8}N_s \,,$$

$$b_1 - 2b_3 + 6b_4 \leqslant \tfrac{1}{4}N_s \,.$$

These constraints are universal: they apply to any scalar-fermion fixed point obtained in the ε expansion at leading order in ε.

Line defects

In the ε expansion $\Delta_\varphi = 1 - \frac{1}{2}\varepsilon < 1$, and so one can consider

$$\mathscr{L}_{\text{CFT}} \to \mathscr{L}_{\text{CFT}} + h_i \int d\tau\, \varphi_i(\tau, \mathbf{0})\,.$$

$\mathscr{L}_{\text{CFT}}$ could involve only scalars, or scalars and fermions.

The question is if there exists an IR defect CFT, where the couplings h_i flow to a fixed point.

The β function of h_i for a multi-scalar bulk CFT is

$$\beta_i = -\tfrac{1}{2}\varepsilon\, h_i + \tfrac{1}{6}\lambda_{ijkl} h_j h_k h_l\,.$$

General results for line defects

Recently the so-called g-theorem was established non-perturbatively for line defects in any d.(Komargodski, Cuomo & Mezei; 2021)

The g-theorem says that there exists a monotonically-decreasing function for RG flows on a defect.

For any line defect in the ε expansion this quantity is

$$A = -\tfrac{1}{4}\varepsilon h^2 + \tfrac{1}{24}\lambda_{ijkl}h_i h_j h_k h_l \,.$$

Since $\beta_i h_i = 0 \Rightarrow \lambda_{ijkl}h_i h_j h_k h_l = 3\varepsilon h_i$, we find that, at a defect CFT,

$$A = -\tfrac{1}{8}\varepsilon h^2 \leqslant 0\,.$$

Among unitary dCFTs, only the trivial theory has $h^2 = 0$.

Line defect in $O(N)$ model

As an example take the $O(N)$ model in the bulk. Then

$$\beta_i = -\tfrac{1}{2}\varepsilon h_i(1 - \tfrac{1}{N+8}h^2)\,, \qquad h^2 = h_i h_i\,.$$

A non-trivial fixed point is found for

$$h^2 = N + 8\,.$$

Notice that we cannot fix the individual vector h_i but only its norm.

There is thus a manifold of equivalent theories. The manifold is S^{N-1} and it arises because the defect breaks the bulk symmetry from $O(N)$ to $O(N-1)$. S^{N-1} is of course the coset $O(N)/O(N-1)$.

Summary

We found novel constraints on fixed points in the ε expansion.

We found dozens of previously undiscovered fixed points in $d = 4 - \varepsilon$.

A similar analysis can be performed for multi-scalar models in $d = 3 - \varepsilon$. There are again many new fixed points.

The nature of these fixed points gives hints about the structure of the ε expansion ("double trace" perturbations).

These observations provide possible avenues to pursue to fully classify fixed points in the ε expansion.

There are interesting results and structures for line defects in the ε expansion.

Some open questions

How many of these fixed points survive in the $\varepsilon \to 1$ limit?

Are any of them relevant for non-zero temperature phase transitions of physical condensed-matter systems?

Can we fully classify fixed points in $d = 4 - \varepsilon$ and in $d = 3$?

Can we prove that there are no scalar fixed points with just $\mathbb{Z}_2$ symmetry in $d = 4 - \varepsilon$ besides the Ising model, or find other fixed points with just $\mathbb{Z}_2$ symmetry?

Since in $d = 4 - \varepsilon$ we have $\Delta_{\varphi^2} = 2 - \frac{2}{3}\varepsilon < 2$, what happens if we consider surface defects?