Hamiltonian Truncation Effective Theory

Durham University

IP3

Rachel Houtz
HEFT 2022
Granada
June, 2022

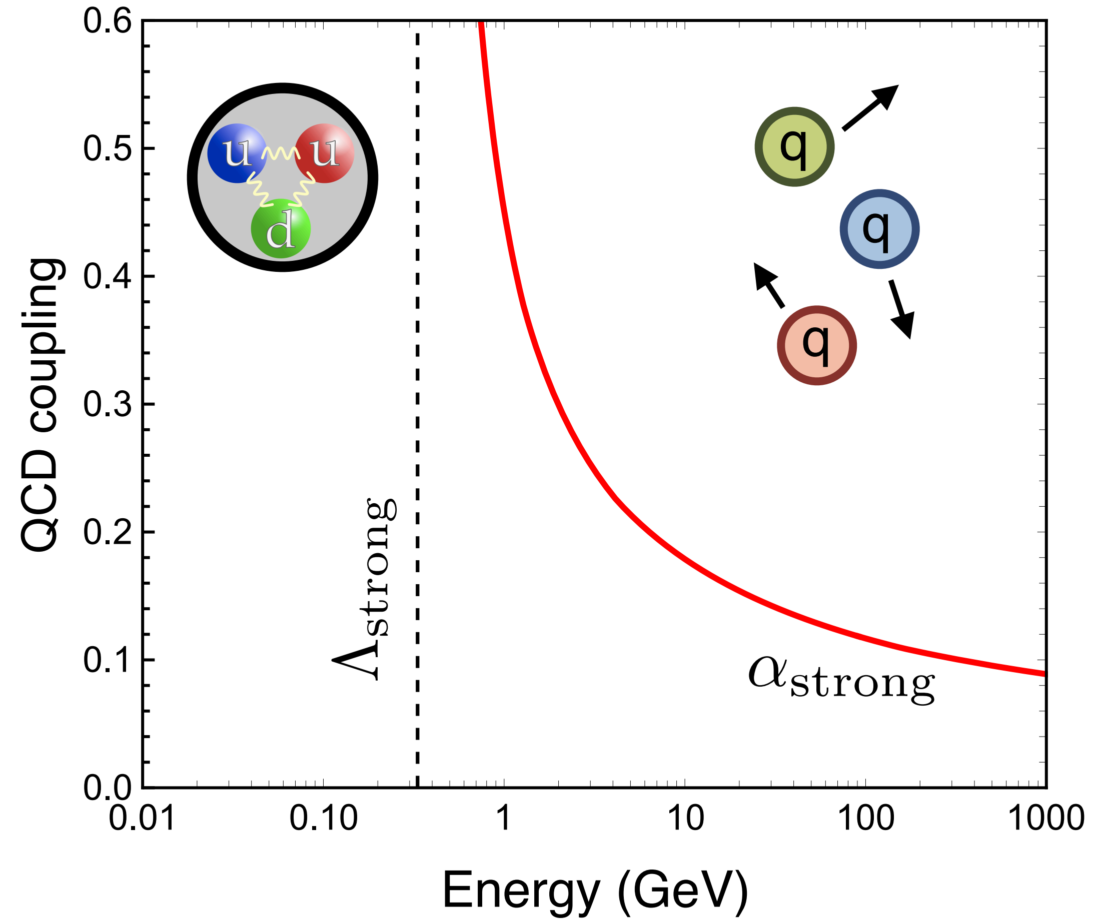
In Collaboration with T. Cohen (U. of Oregon), K. Farnsworth (Case Western), M. Luty (UC Davis), arXiv: 2110.08273

Hamiltonian Truncation Effective Theory (HTET)

- Not about the Higgs

- It **is** about effective field theory methodology

- A nonperturbative strategy for probing strong coupling



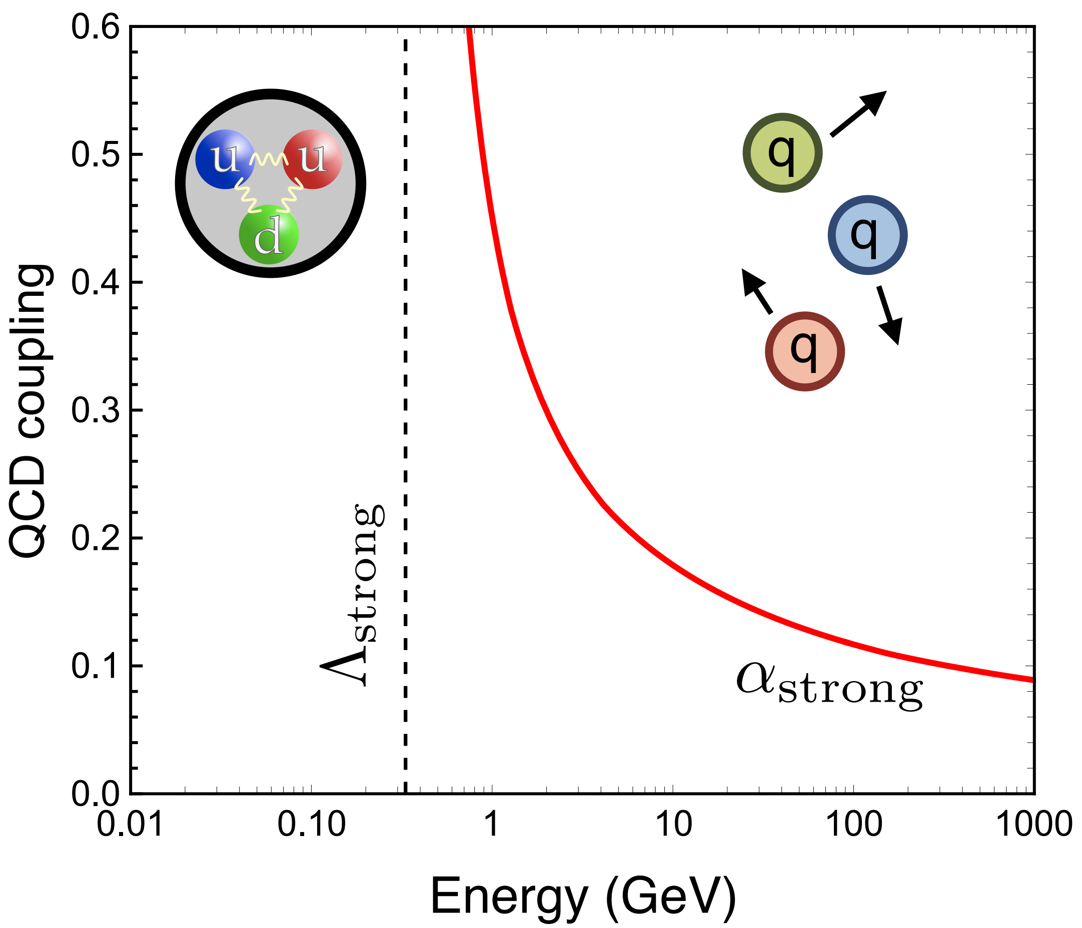
- An attempt to impose some EFT-inspired order-by-order control over nonperturbative methods to probe strongly coupled theories

Hamiltonian Truncation Effective Theory (HTET)

- Not about the Higgs

- It **is** about effective field theory methodology

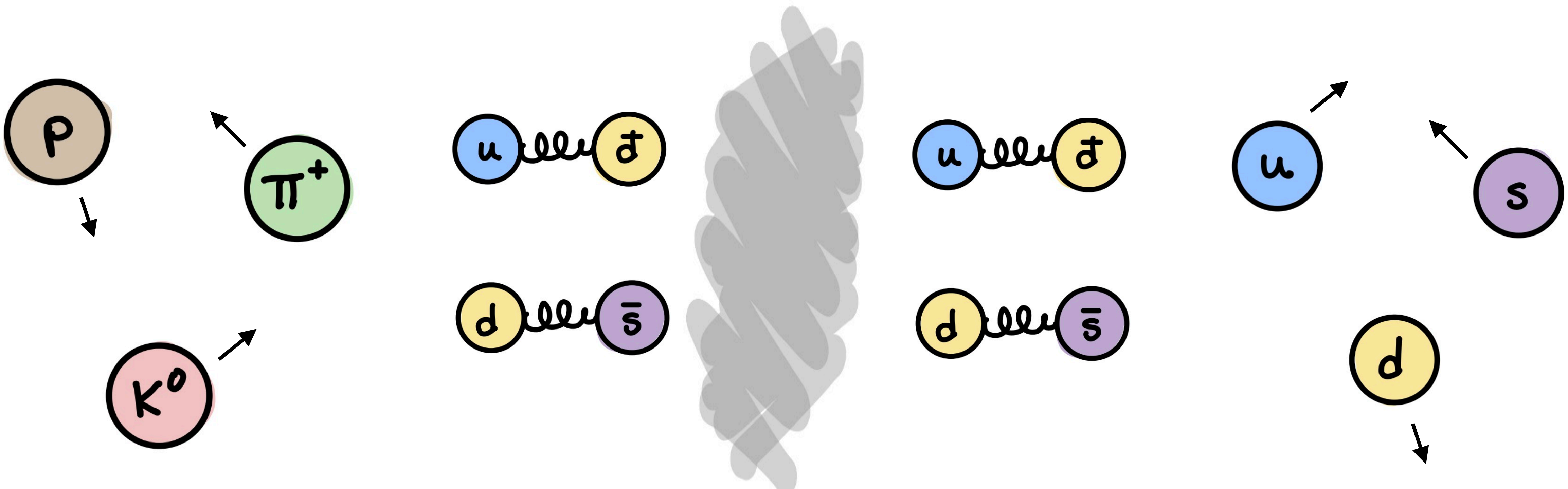
- A nonperturbative strategy for probing strong coupling



- An attempt to impose some EFT-inspired order-by-order control over nonperturbative methods to probe strongly coupled theories

Why Hamiltonian Truncation?

- We want to be able to understand particle physics at strong coupling



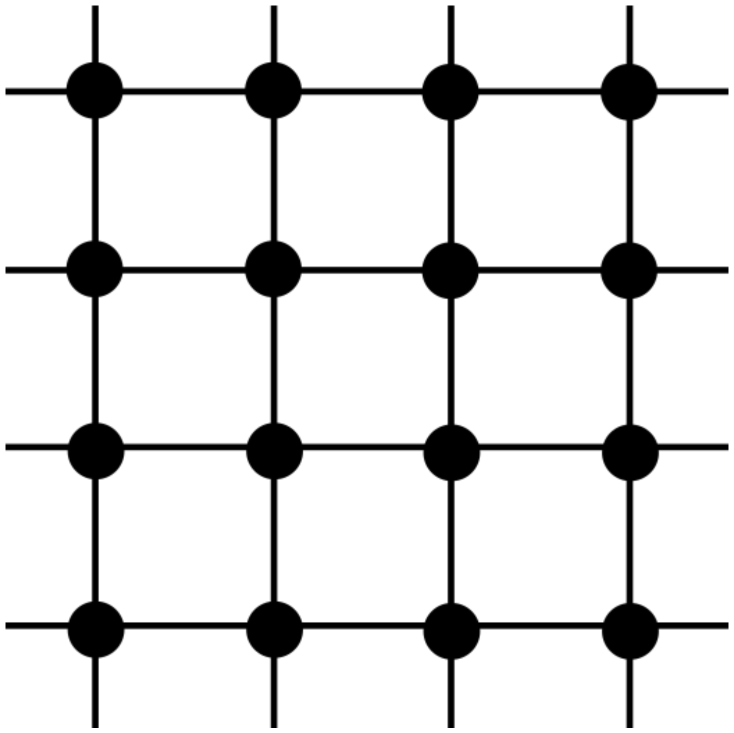
- Perturbation theory breaks down at confinement
- Calls for nonperturbative QFT methods, like lattice QCD

Alternatives to the Lattice?

- Lattice field theory is so far the only way to probe strong coupling with errors that are quantifiable and under control
- Lattice methods have some shortcomings
  - Tricky to model chiral fermions

    Karsten, Smit (1981)   Kaplan hep-lat/920601

  - Difficult to obtain dynamical quantities
  - Explicitly breaks continuous rotational and translational invariance
  - Requires ever increasing computational resources



Alternatives to the Lattice?

- Truncated Conformal Space Approach: $H = H_{\text{known}} + \lambda V_{\text{pert}}$

  Yurov, Zamolodchikov (1990), (1991)

  Integrable or conformal

- Truncate the basis:

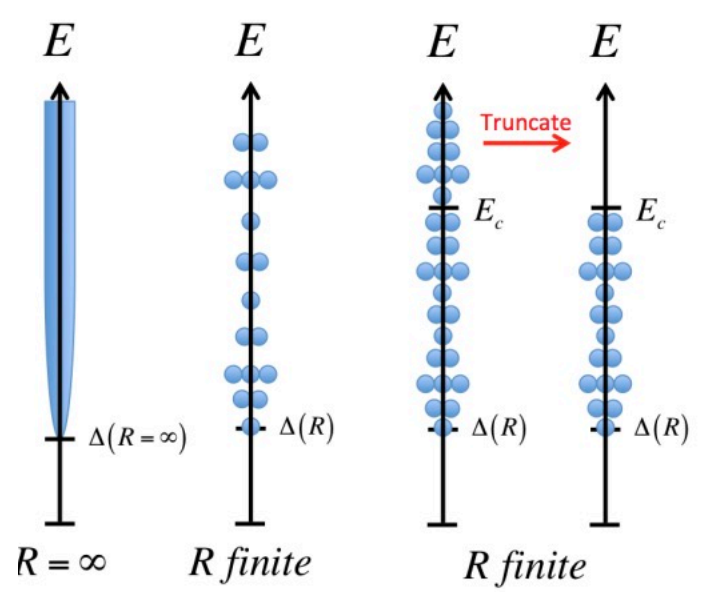


Figure lifted from James, Konik, Lecheminant, Robinson, Tsvelik, arXiv:1703.08421

- Simply compute

$$H_{ij}^{\text{trunc}} = E_i \delta_{ij} + \lambda \langle E_i | V_{\text{pert}} | E_j \rangle$$

- Results from diagonalizing H_{trunc} are surprisingly effective due to the *relevancy* of the perturbing operator

Hamiltonian Truncation

- **Hamiltonian Truncation** uses the free UV CFT as the starting point for the Truncated Conformal Space Approach

A Cheap Alternative to the Lattice?

Matthijs Hogervorst,[1,2] Slava Rychkov,[2,1,3] and Balt C. van Rees[2]
[1]*Laboratoire de Physique Théorique de l'École normale supérieure, 75005 Paris, France*
[2]*CERN, Theory Division, 1211 Geneva, Switzerland*
[3]*Faculté de Physique, Université Pierre et Marie Curie, 75005 Paris, France*
(Received 22 September 2014; published 6 January 2015)

- Not cheaper, and in fact there is an exponential wall as the basis increases
- Still, intriguingly good results of the TCSA point to the promise of these methods
- While the truncation effects are small for wisely-chosen theories, we could do better if we *improve* the results of the truncation

Hogervorst, Rychkov, van Reese (2014), Rychkov, Vitale (2015), Katz, Marques Tavares, Xi (2014), Elias-Miro, Rychkov Vitale (2017), Elias-Mrio, Hardy (2020)

The Hamiltonian

- Split the Hamiltonian up into the free and interacting pieces

$$H = H_{\text{free}} + V$$

Strong coupling in IR

- We want to diagonalize H, but its basis is infinite

- Look at a finite corner

$$H = \begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix}$$

- The "low" and "high" states in the basis are defined by an energy cutoff E_{max}

$$H_{\text{free}}|\text{state}_i\rangle = E_i|\text{state}_i\rangle$$

$$E_i \leq E_{\text{max}} \Rightarrow |\text{state}_i\rangle \in \mathcal{H}_l \quad \text{(finite)}$$

$$E_i > E_{\text{max}} \Rightarrow |\text{state}_i\rangle \in \mathcal{H}_h \quad \text{(infinite)}$$

Truncating the Hamiltonian

- The easiest thing to do is just truncate the Hamiltonian (basically TCSA)

$$H = \begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix} \quad \Rightarrow H_{\text{eff}} = (H_{ll})$$

- The main source of error is mixing coming from H_{lh}

- We want to develop an improvement scheme to incorporate the effects of H_{lh} , **order by order**

- Our strategy will be to define **an operator** in the fundamental theory and **match** it to the effective theory to determine corrections to H_{eff}

Hamiltonian Truncation Effective Theory Strategy

- Fundamental theory:

$$H = H_{\text{free}} + V$$

Free theory defines basis:

$$H_{\text{free}}|\text{state}_i\rangle = E_i|\text{state}_i\rangle$$

- The energy eigenstates are: $H|i\rangle = \mathcal{E}_i|i\rangle$
  and can be written as

$$\mathcal{E}_i = E_i + \mathcal{E}_{1i} + \mathcal{E}_{2i} + \ldots \qquad \mathcal{E}_{ni} = \mathcal{O}(V^n)$$

- We want to define a calculable, finite-dimensional H_{eff} such that:

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \ldots \qquad H_n = \mathcal{O}(V^n)$$

Matching the Transition Matrix

- Time evolution: $U_{\mathrm{IP}}(t_f, t_i) = T\exp\left\{-i\int_{t_i}^{t_f} dt V_{\mathrm{IP}}(t)\right\}$ $V_{\mathrm{IP}}(t) = e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t}$

- Define an operator to match: $\langle f|\Sigma|i\rangle \equiv \lim_{t_f\to\infty} \langle f|U_{IP}(t_f, 0)|i\rangle$

- Implicit relation can be evaluated iteratively order-by-order

$$U_{\mathrm{IP}}(t_f, t_i) = \mathbb{1} - i\int_{t_i}^{t_f} dt U_{\mathrm{IP}}(t_f, t) V_{\mathrm{IP}}(t)$$

- This gives an expansion:

$$\langle f|\Sigma|i\rangle = \delta_{fi} + \frac{\langle f|V|i\rangle}{E_f - E_i} + \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{(E_f - E_i)\,(E_f - E_\alpha)} + \mathcal{O}(V^3)$$

Matching the Transition Matrix

- More convenient to define: $\langle f|\Sigma|i\rangle = \delta_{fi} + \frac{\langle f|T|i\rangle}{E_f - E_i}$

$$\langle f|T|i\rangle_{\text{fund}} = \langle f|V|i\rangle + \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha} + \sum_{\alpha,\beta} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle f|V|i\rangle}{(E_f - E_\alpha)(E_f - E_\beta)} + \mathcal{O}(V^4)$$

$$\langle f|T|i\rangle_{\text{eff}} = \langle f|V_{\text{eff}}|i\rangle + \sum_{\alpha}^{<} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|i\rangle}{E_f - E_\alpha} + \sum_{\alpha,\beta}^{<} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|\beta\rangle\langle\beta|V_{\text{eff}}|i\rangle}{(E_f - E_\alpha)\,(E_f - E_\beta)} + \mathcal{O}(V^4)$$

- Matching is accomplished by setting the T matrix equal in both theories and evaluating H_i

$$H_{\text{eff}} = H_0 + V_{\text{eff}} \;= H_0 + H_1 + H_2 + \ldots$$

Matching the Transition Matrix

$$\langle f|T|i\rangle_{\text{fund}} = \langle f|V|i\rangle + \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha} + \quad \ldots$$

$$\langle f|T|i\rangle_{\text{fund}} = \ \langle f|T|i\rangle_{\text{eff}} = \langle f|V_{\text{eff}}|i\rangle + \sum_{\alpha}^{<} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|i\rangle}{E_f - E_\alpha} + \ldots$$

- The matching defines H_{eff}, analogous to normal QM perturbation theory:

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^{>}\sum_{\beta}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_\alpha)(E_f - E_\beta)} + \sum_{\alpha}^{<}\sum_{\beta}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_\beta)(E_\alpha - E_\beta)}$$

$$\ldots$$

2D ϕ^4 Theory

- Test this method out on a specific theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad [m] = 1 \qquad [\lambda] = 2$$

- Relevant couplings: weakly coupled in UV, strongly coupled in IR
- Quantized on a circle of radius R: $H_0 = \sum_k \omega_k a_k^\dagger a_k \qquad \omega_k = \sqrt{(k/R)^2 + m_Q^2}$
- The goal is to calculate H_{eff}, which will be an expansion in V:

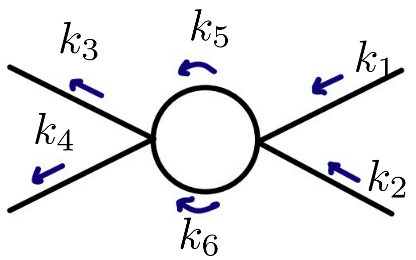
$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_\alpha^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha} \quad \text{uses} \quad V = \int dx \left[\frac{1}{2}m_V^2 :\phi^2: + \frac{\lambda}{4!} :\phi^4:\right]$$

- Develop Feynman rules to more easily evaluate these sums

Diagrammatic Calculation of the Transition Matrix

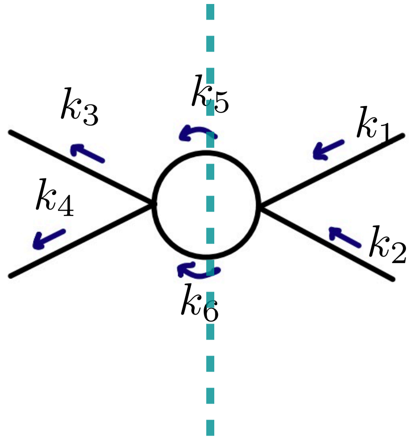
- One part of the transition matrix in the fundamental theory can be found using:

$$\langle f|T|i\rangle_{\text{fund}} \ni \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha}$$



$$= \frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{k_1,..k_6} \delta_{k_1+k_2,k_5+k_6} \langle f|\phi_4^- \phi_3^- \phi_2^+ \phi_1^+|i\rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{1}{\omega_3+\omega_4-\omega_5-\omega_6+i\epsilon}$$

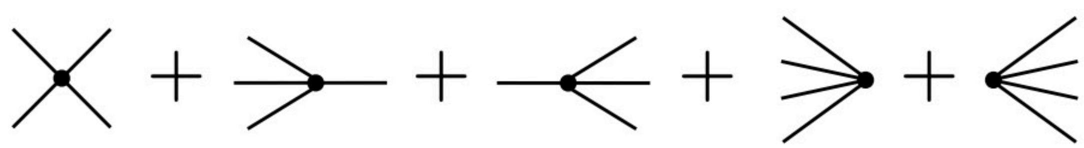
- In the effective theory, we instead find:



$$= \frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{k_1,..k_6} \delta_{k_1+k_2,k_5+k_6} \langle f|\phi_4^- \phi_3^- \phi_2^+ \phi_1^+|i\rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{\Theta(E_{\max}-E_f+\omega_3+\omega_4-\omega_5-\omega_6)}{\omega_3+\omega_4-\omega_5-\omega_6+i\epsilon}$$

Diagrammatic Representation of Matching $\mathcal{O}(V)$

- Matching at first order: $\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle$



$$= \frac{\lambda}{4!}\int \mathrm{d}x\,\langle f|:\phi^4:|i\rangle$$

$$= \frac{1}{2}m_V^2\int \mathrm{d}x\,\langle f|:\phi^2:|i\rangle.$$

- Same diagrams in both effective theory and fundamental theory at tree level

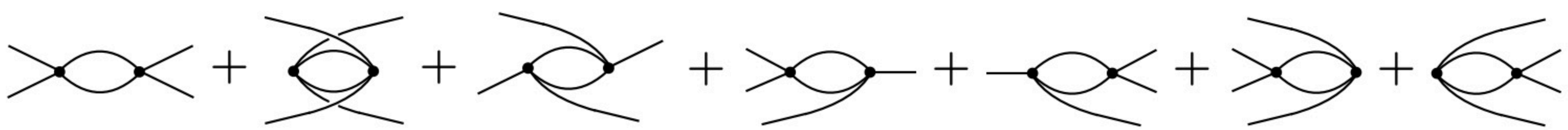
- Trivially gives: $$H_{\text{eff}} = H_0 + \int dx\left[\frac{1}{2}m_V^2 :\phi^2: + \frac{\lambda}{4!}:\phi^4:\right] + \ldots$$

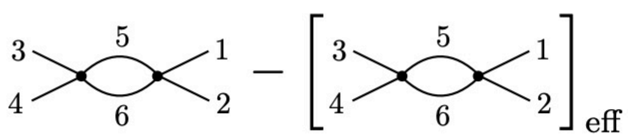
Diagrammatic Representation of Matching $\mathcal{O}(V^2)$

- Matching at second order: $\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_\alpha}$

- Vertex correction

$\langle f|T_{2,4}|i\rangle =$

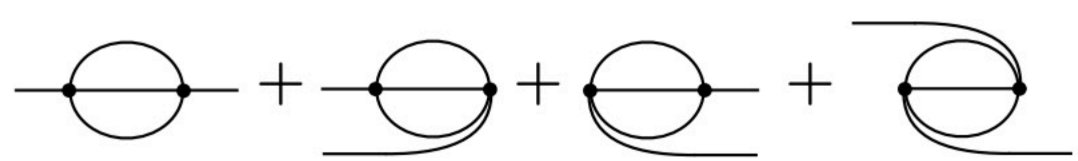


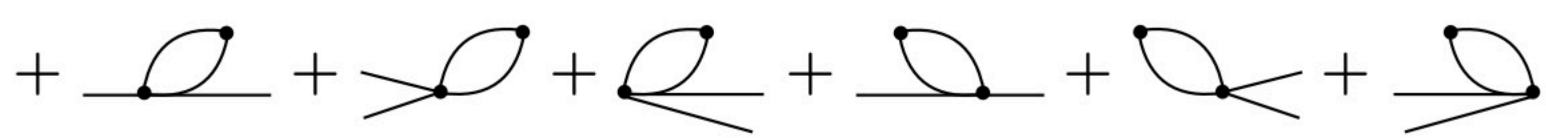


$$= \frac{\lambda^2}{128\pi^2 R^2}\sum_{1,\ldots,4}\delta_{12,34}\langle f|\phi_4^{(-)}\phi_3^{(-)}\phi_2^{(+)}\phi_1^{(+)}|i\rangle \sum_{5,6}\delta_{56,34}\frac{\Theta(\omega_5+\omega_6-\omega_3-\omega_4+E_f-E_{\max})}{\omega_5\omega_6(\omega_3+\omega_4-\omega_5-\omega_6)}$$

- Mass correction

$\langle f|T_{2,2}|i\rangle =$





Matching Results and Implementation

- Final answer at this order:

$$H_{\text{eff}} = H_0 + \int dx \left[\frac{1}{2} \left(m_V^2 + m_{V_2}^2 \right) :\phi^2: + \frac{\lambda^2 + \lambda_2}{4!} :\phi^4: \right]$$

Computationally costly to build matrices

- Tuning the couplings, adding matrices together, and diagonalizing is cheap

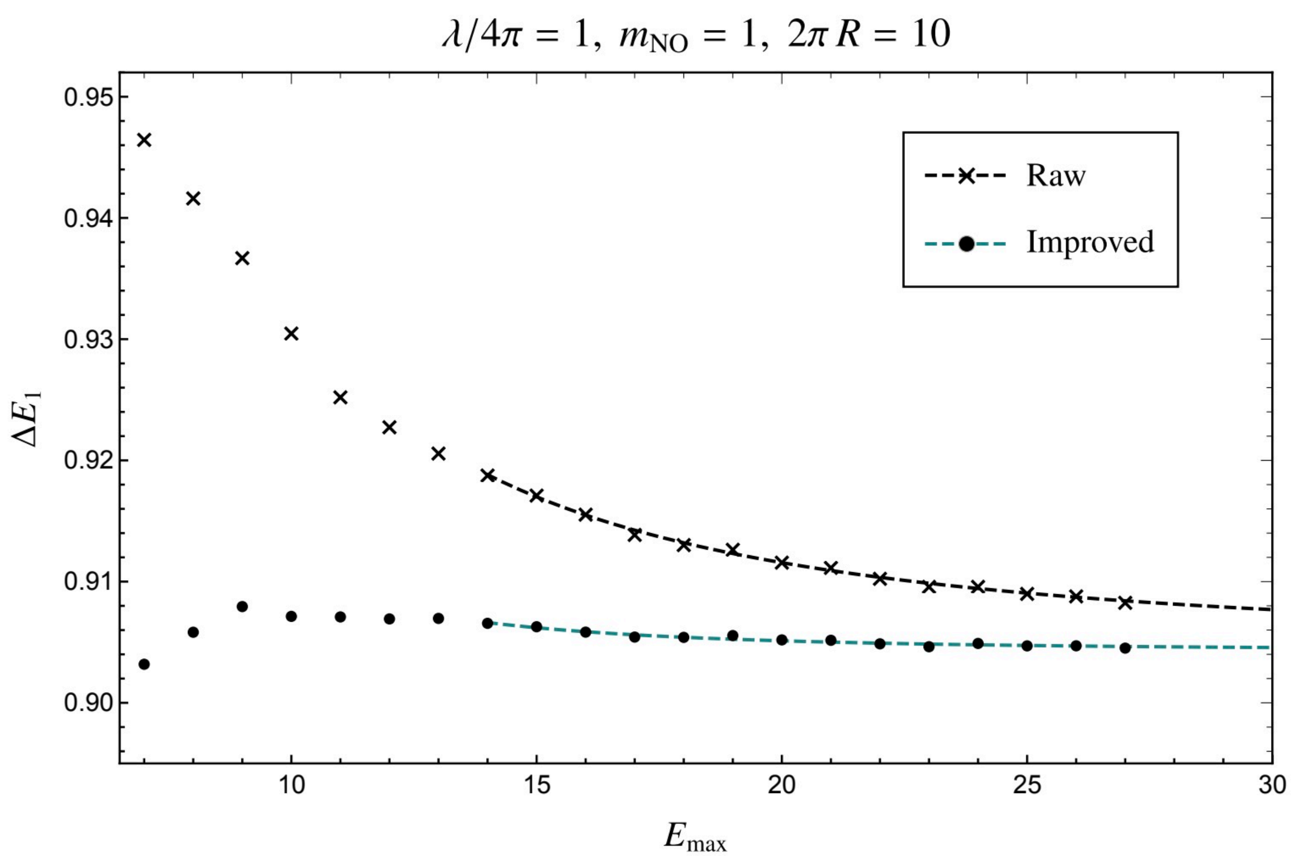
- Corrected couplings*

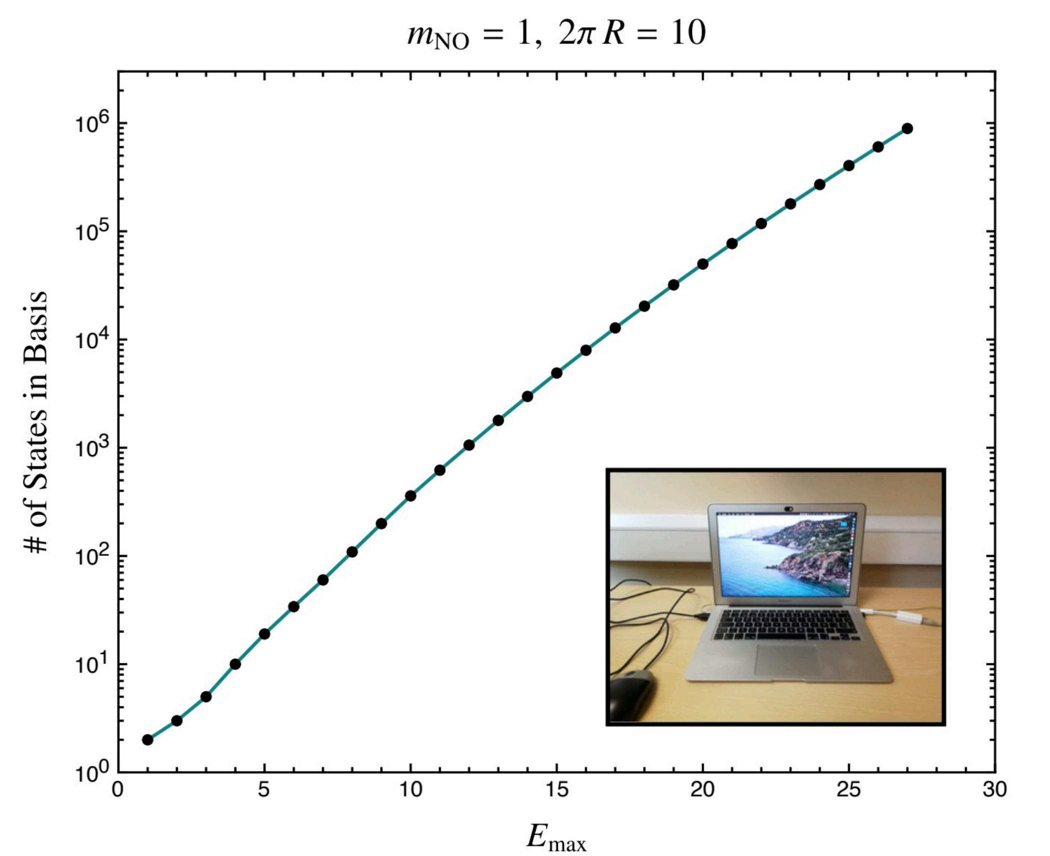
$$m_{V_2}^2 = \frac{\lambda}{16\pi R} \left[\frac{\lambda}{6\pi R} \sum_{345} \delta_{k_3+k_4+k_5,0} \frac{\Theta\left(\omega_3 + \omega_4 + \omega_5 - E_{\text{max}}\right)}{\omega_3\omega_4\omega_5\left(\omega_3 + \omega_4 + \omega_5\right)} - m_V^2 \sum_k \frac{\Theta\left(2\omega_k - E_{\text{max}}\right)}{\omega_k^3} \right]$$

$$\lambda_2 = -\frac{3\lambda^2}{16\pi R} \sum_k \frac{\Theta\left(2\omega_k - E_{\text{max}}\right)}{\omega_k^3}$$

*local approximation, neglecting external momenta $E_{i,f} \ll E_{\text{max}}$

Improved probe of ΔE_1





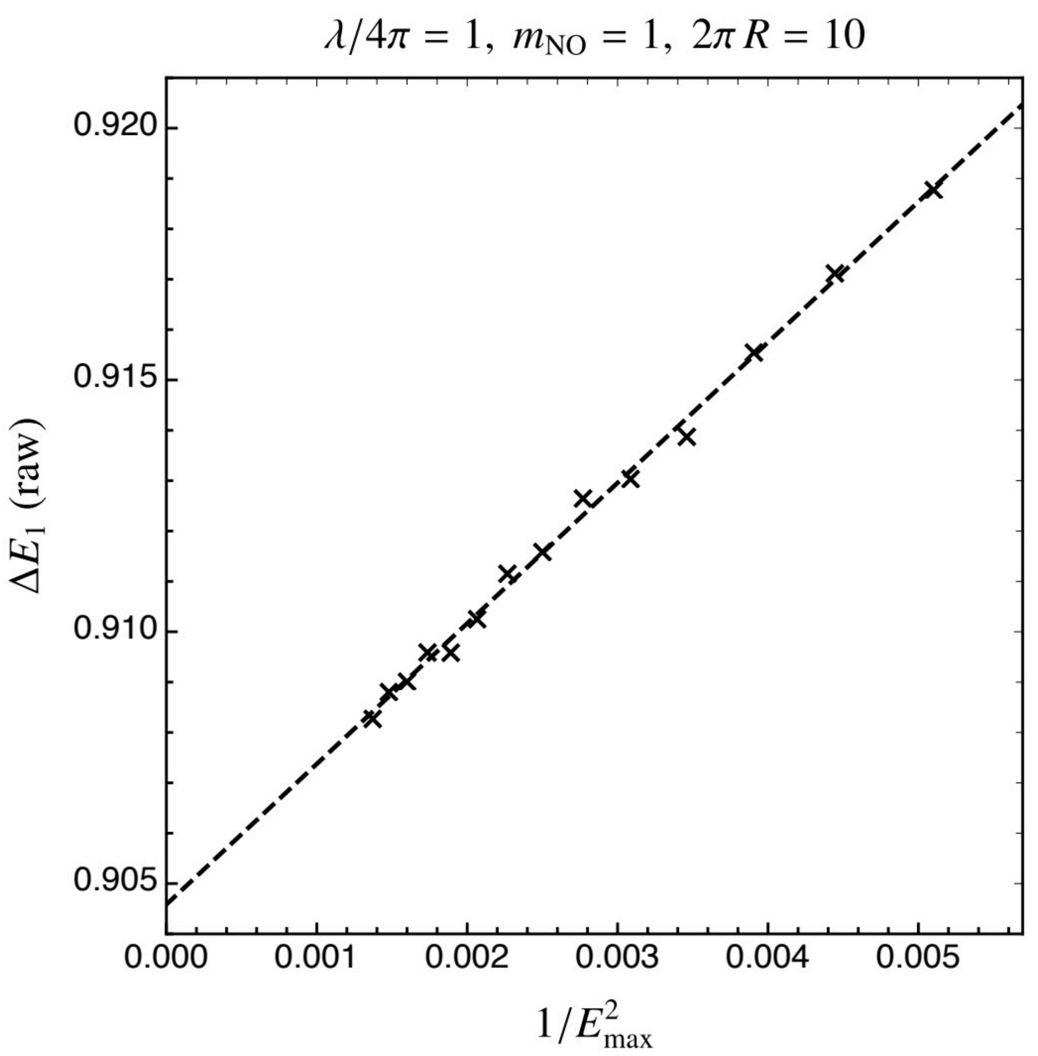
$$\Delta E_1 \equiv E_1 - E_0$$

- Very quick approach to the correct first excited energy level
- Results obtained on a laptop comparable to others requiring a grid

Elias-Miro, Rychkov Vitale (2017)

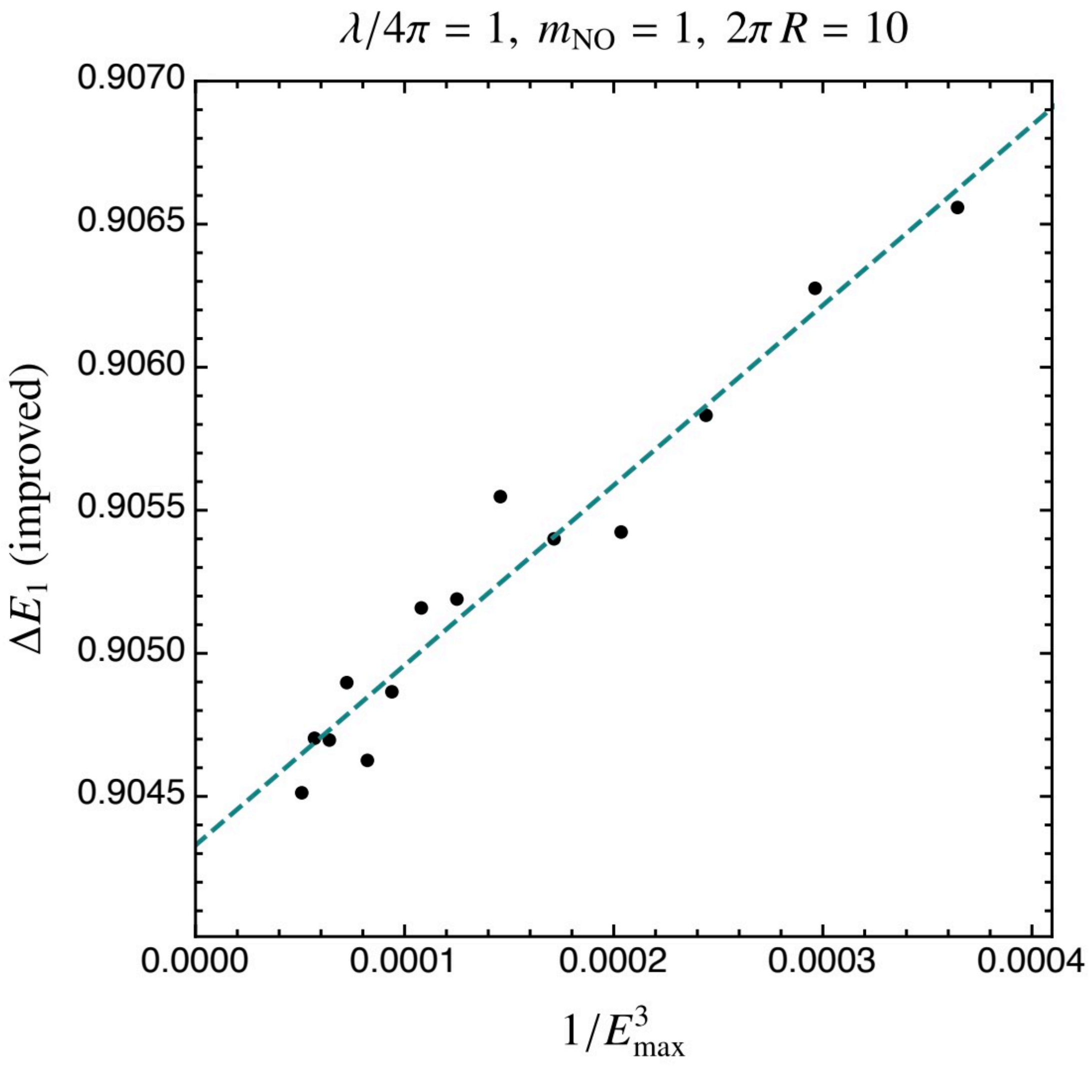
Error Scaling with $E_{\max}$

<u>Raw Truncation</u>



- $1/E_{\max}^2$ scaling

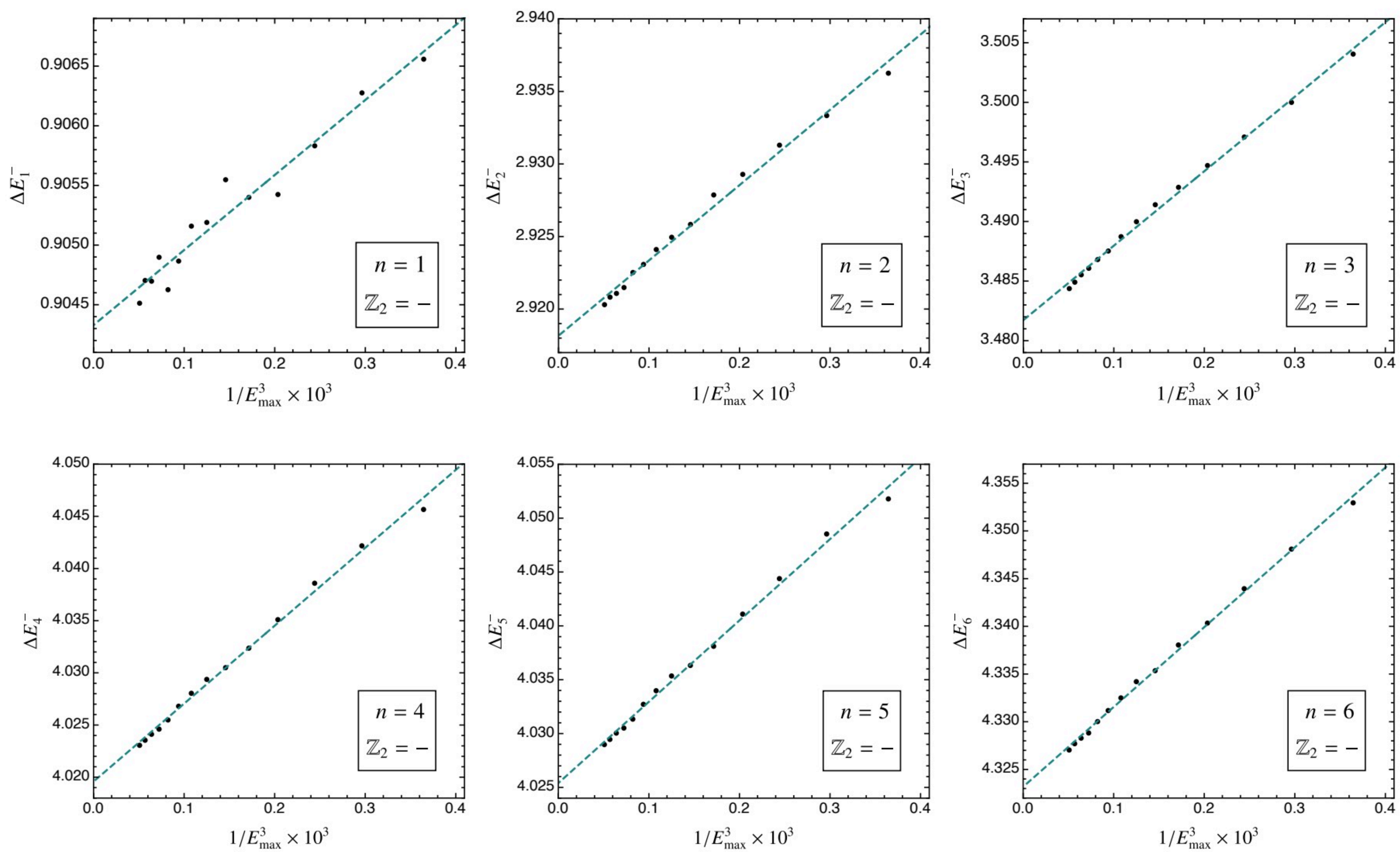
<u>Truncation + Improvement</u>



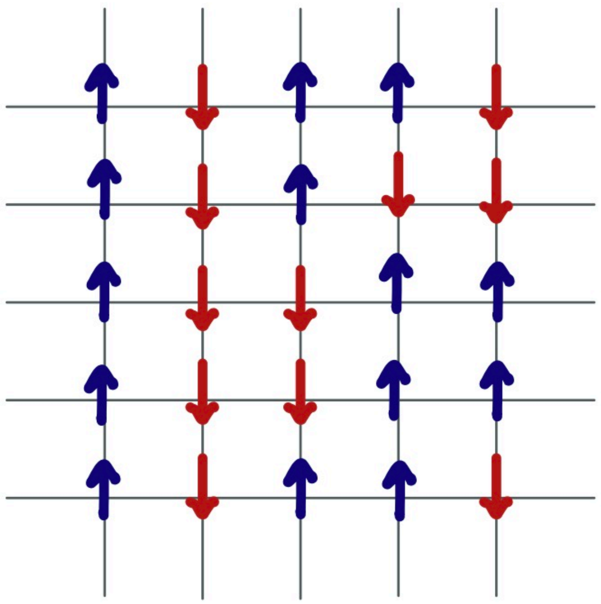
- $1/E_{\max}^3$ scaling

Excited States

- $1/E_{\max}^3$ scaling persists:



2D Ising Model Checks



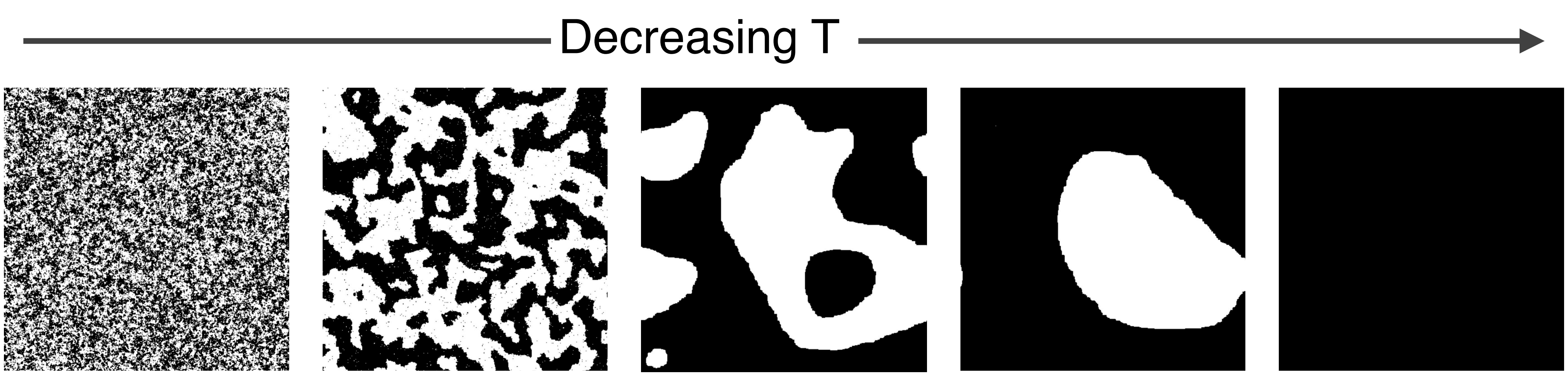
- 2D ϕ^4 is in the same universality class as the 2D Ising model

Onsager, Phys. Rev. 05, 117 (1944)

Anand, Genest, Katz, Khandker, Walters, arXiv:1704.04500

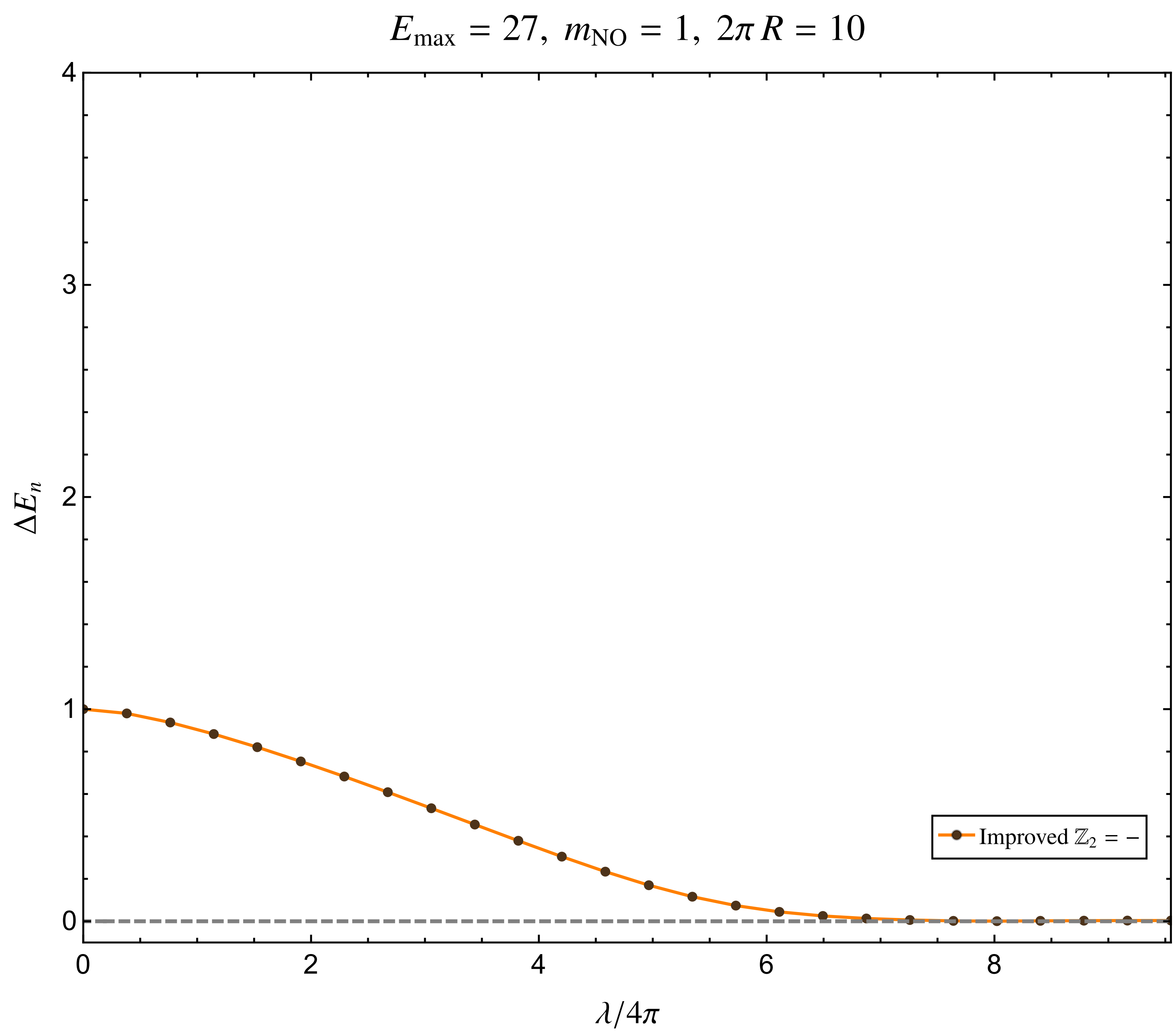
- Expect energy spectrum of 2D ϕ^4 to map onto operator dimensions of 2D Ising model

- 2D Ising Model phase transition:



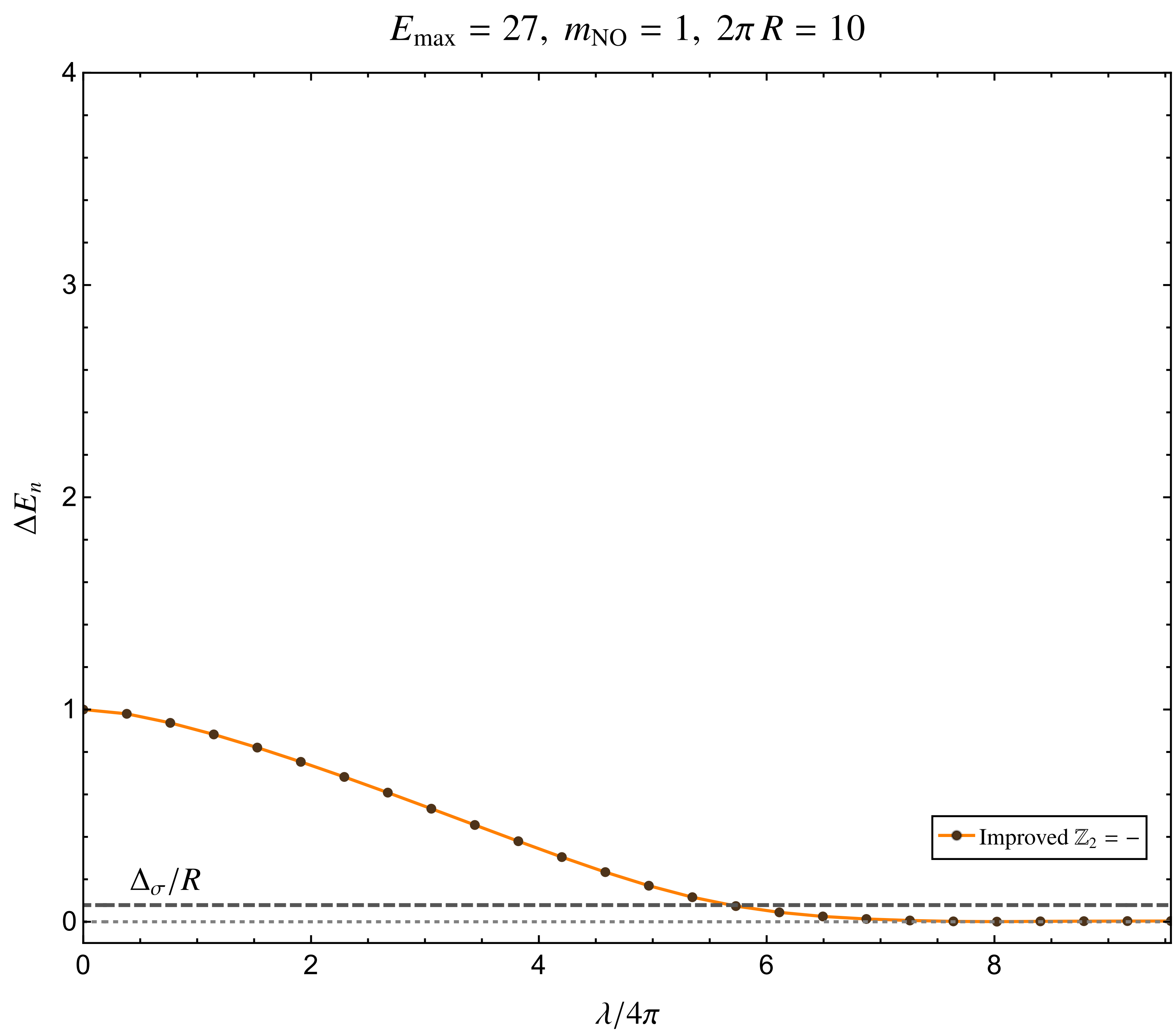
Demidov, www.ibiblio.org/e-notes/Perc/ising.htm

Probing the critical coupling



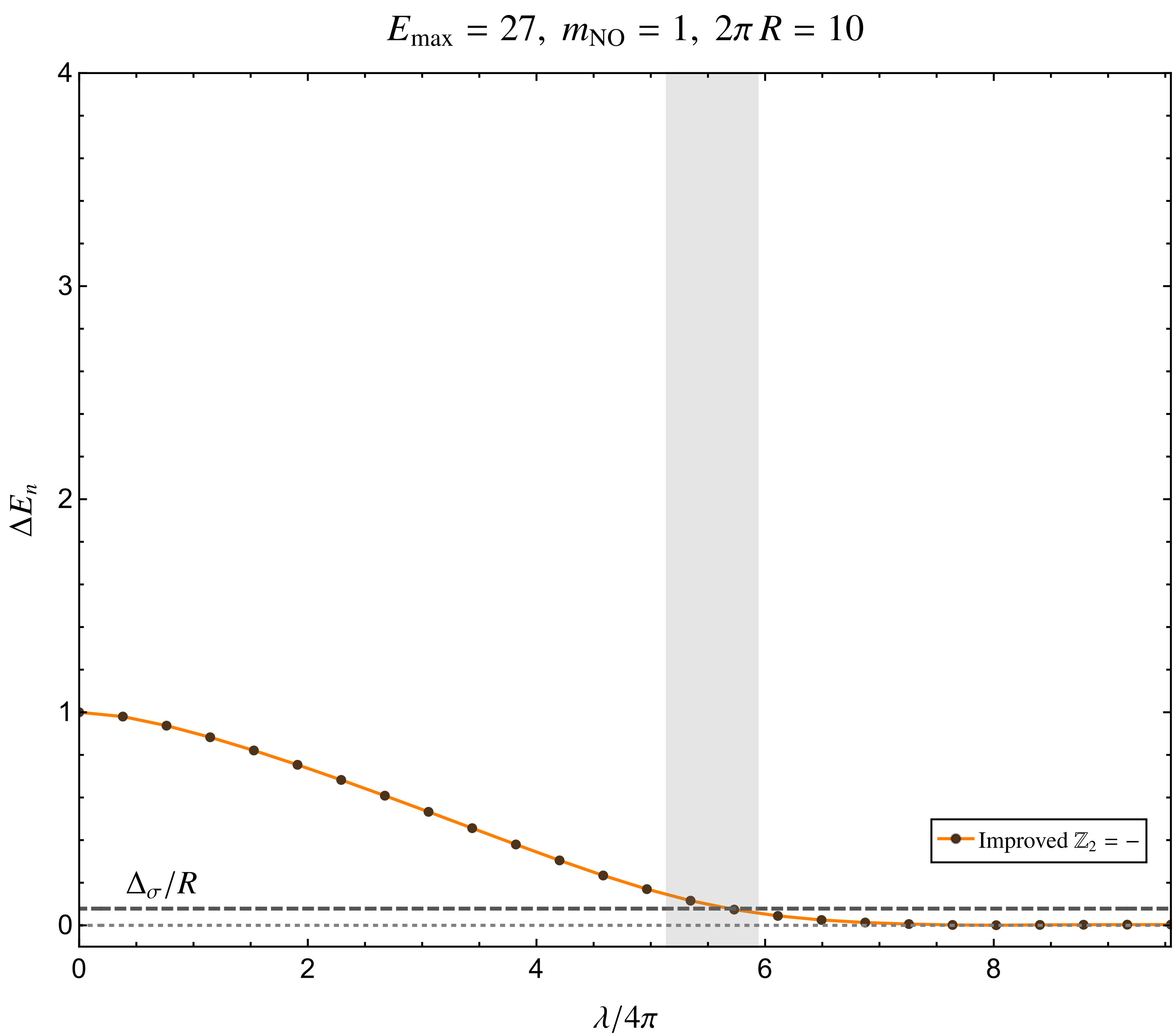
- What happens to the spectrum at larger coupling?
- Degenerate ground state emergence for strong coupling
- Indicates Z_2 symmetry is spontaneously broken

Probing the critical coupling



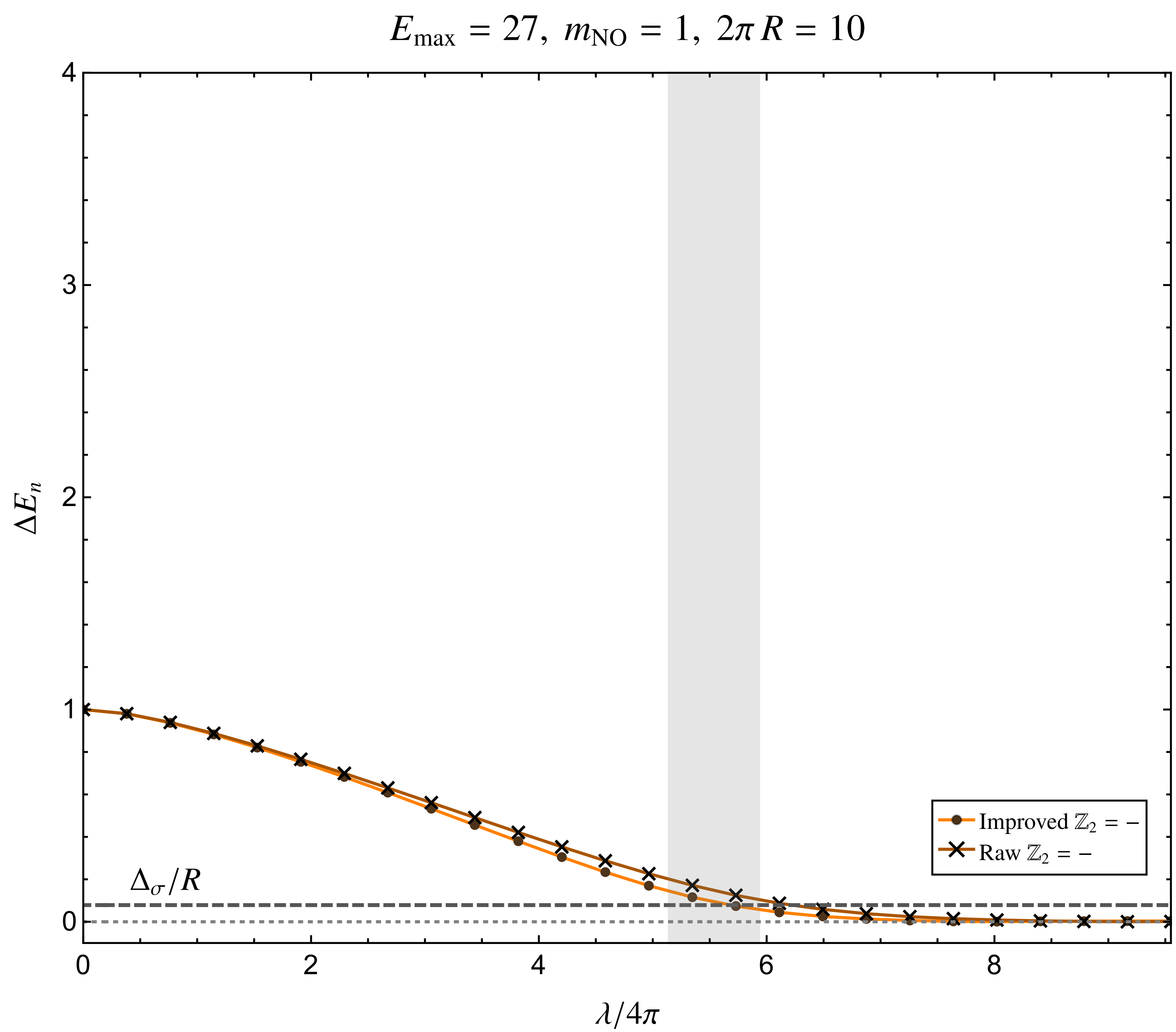
- What happens to the spectrum at larger coupling?
- 2D ϕ^4 is in same universality class as the 2D Ising model
- Should reproduce its spectrum near the critical coupling

Probing the critical coupling



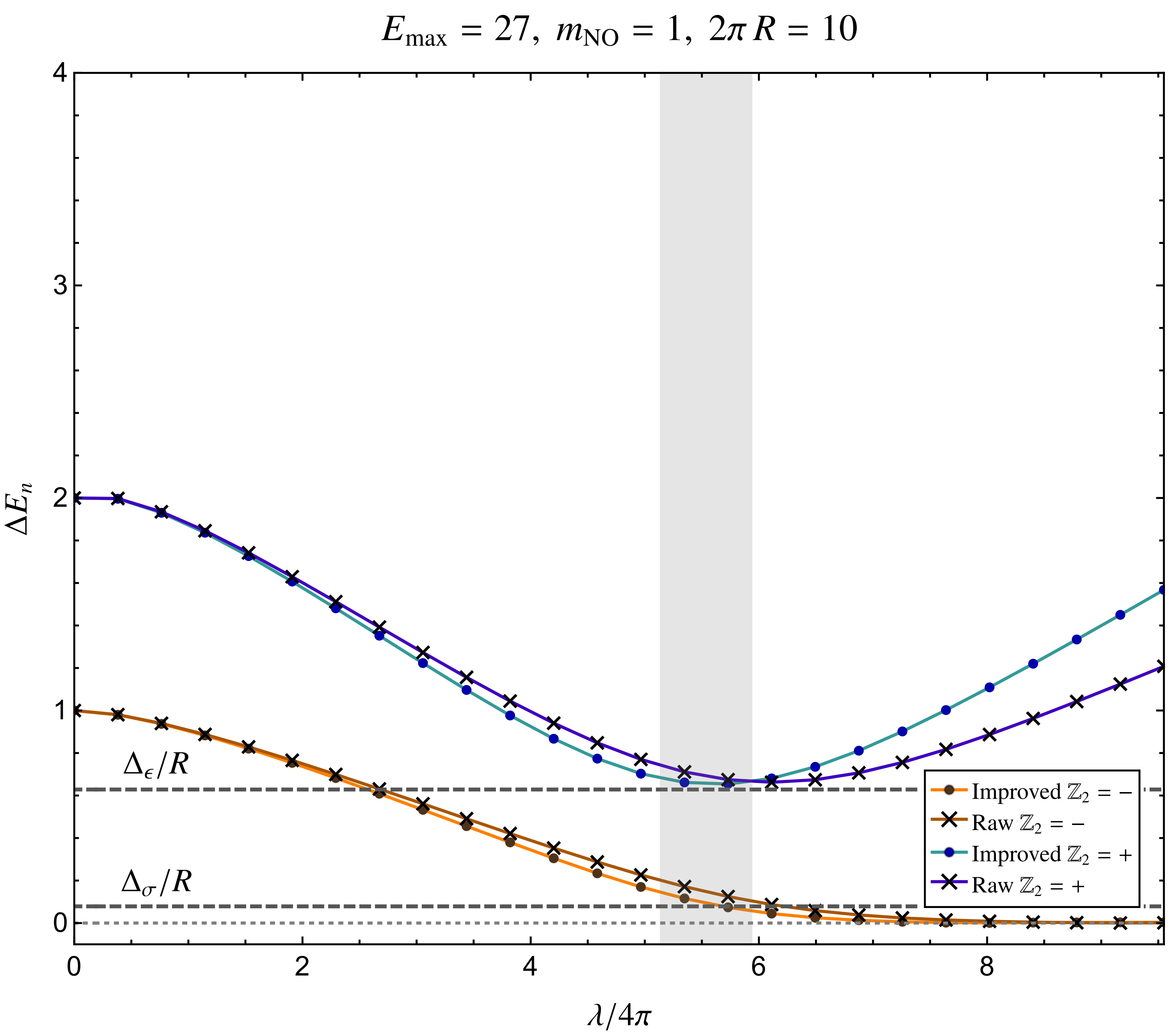
- What happens to the spectrum at larger coupling?
- 2D ϕ^4 is in same universality class as the 2D Ising model
- Shaded: Critical coupling region

Probing the critical coupling



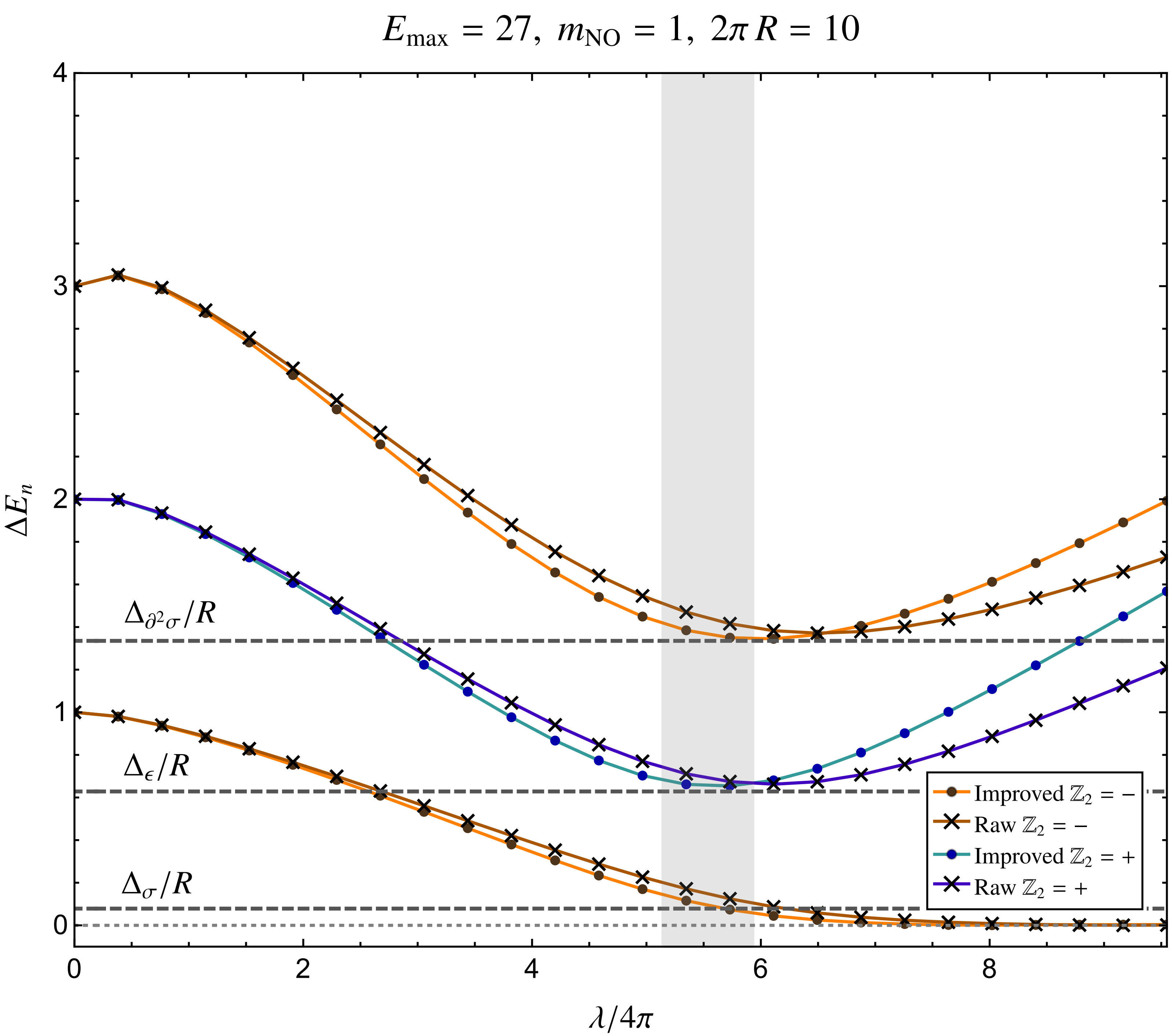
- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The **improved theory** shows better agreement when compared to the **raw truncation**

Probing the critical coupling



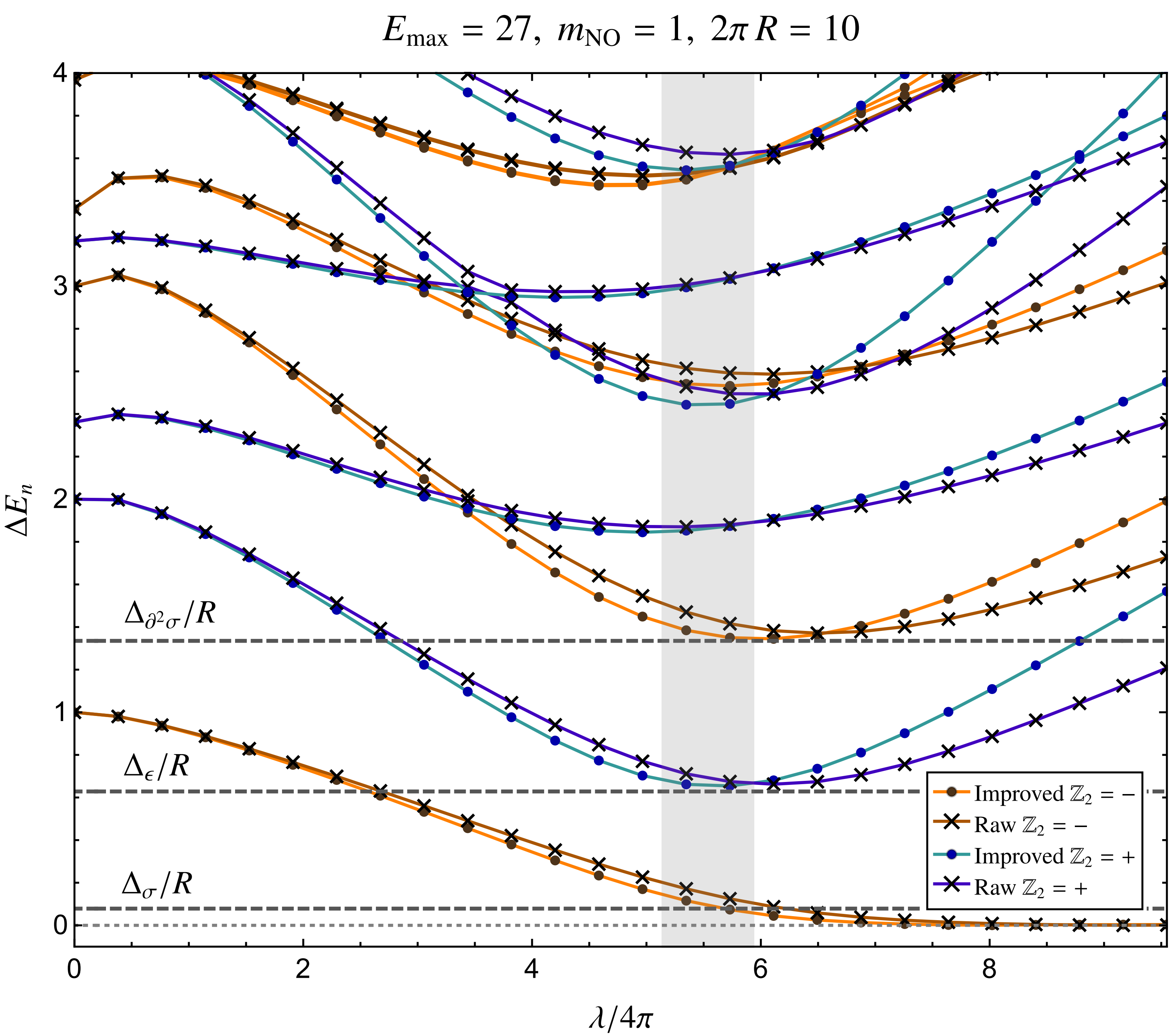
- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The second excited state also probes the 2D Ising model prediction

Probing the critical coupling



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The third excited state also probes the 2D Ising model prediction
- No Z_2 degeneracy for higher excited states, probably an artifact of finite R

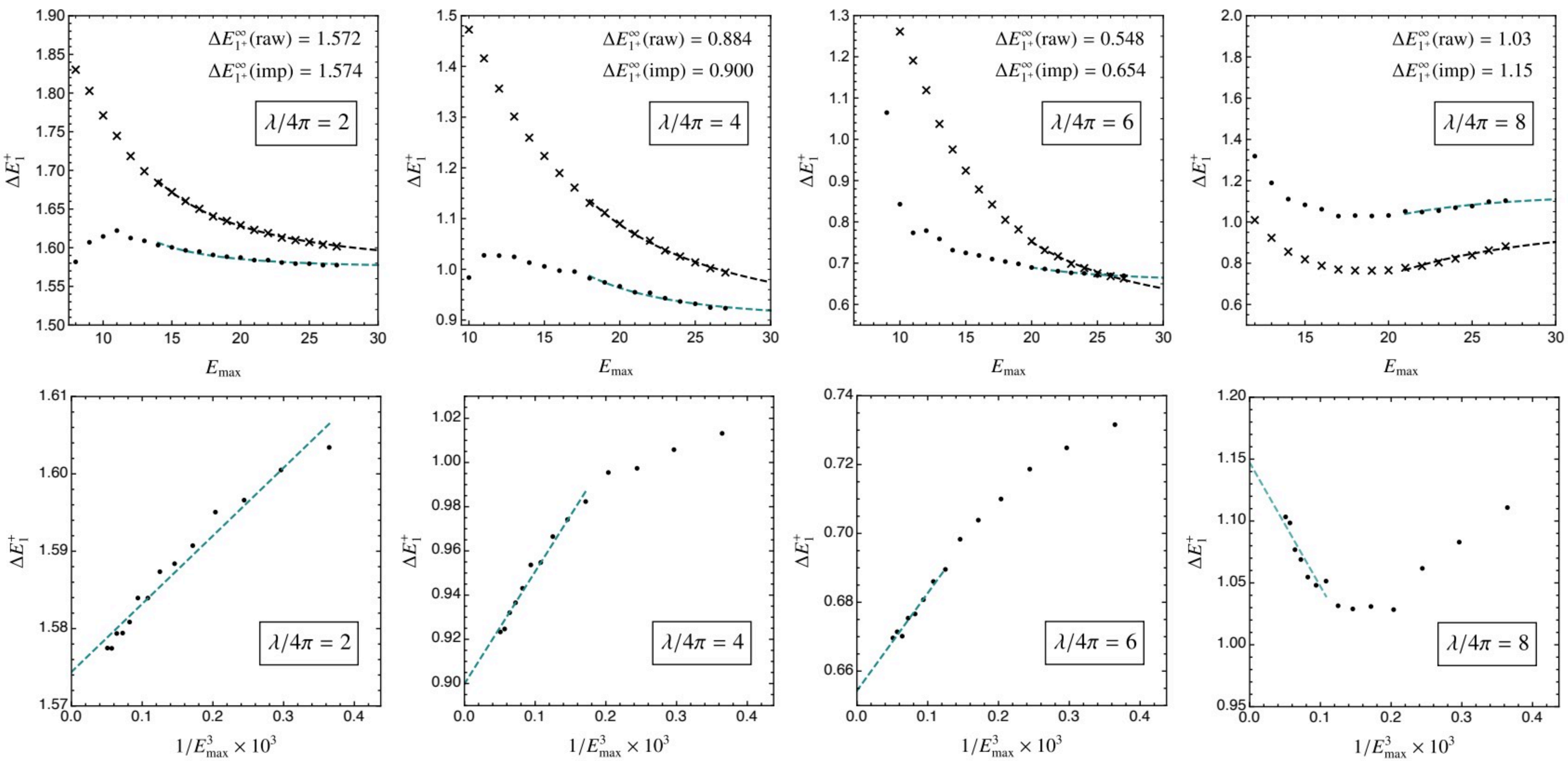
Probing the critical coupling



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- Odd/Even states
- The full spectrum
- No Z_2 degeneracy for higher excited states, probably an artifact of finite R

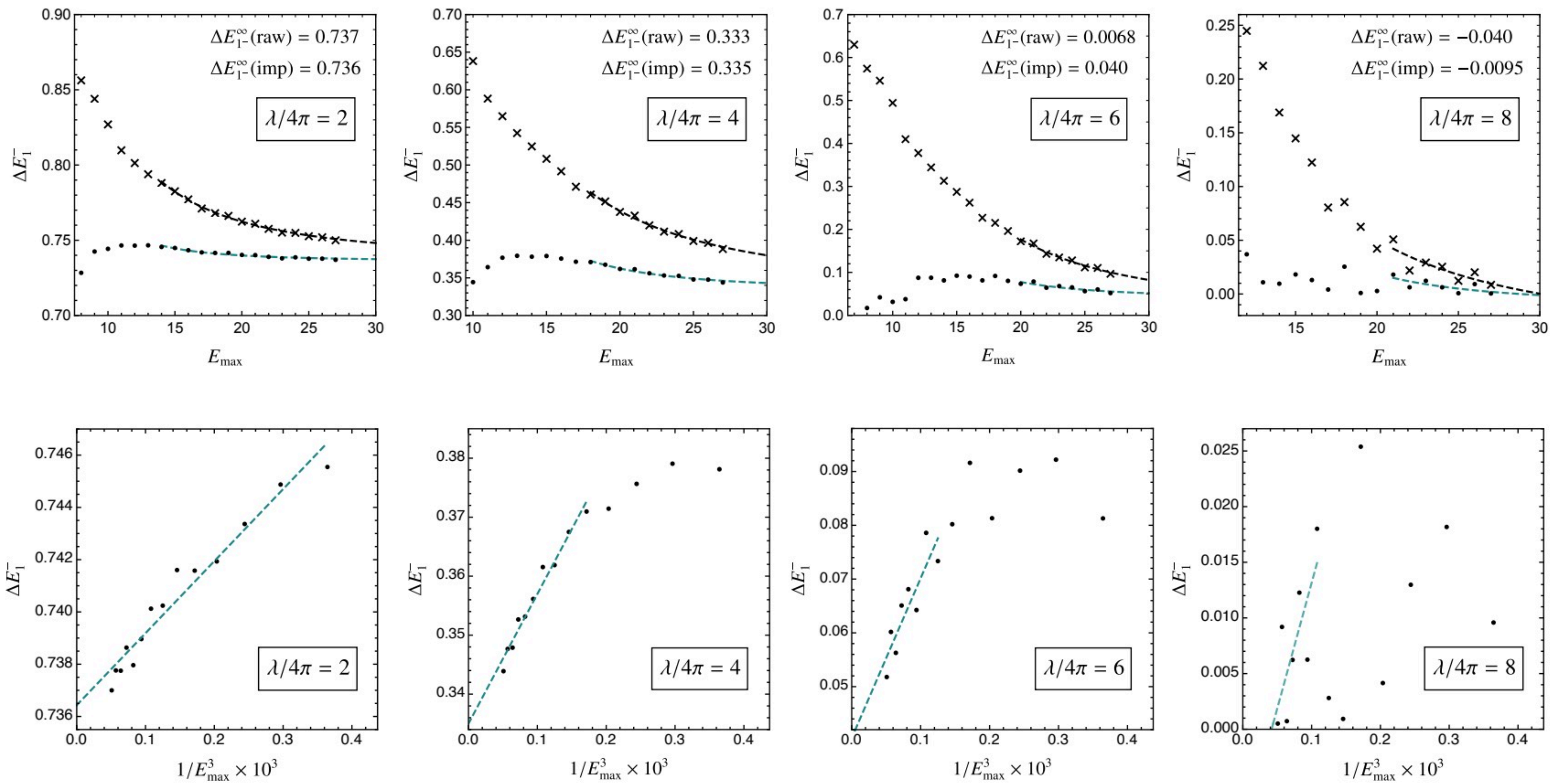
Stronger Couplings (Z_2 even)

- Need to increase truncation scale to see power-law scaling



Stronger Couplings (Z_2 odd)

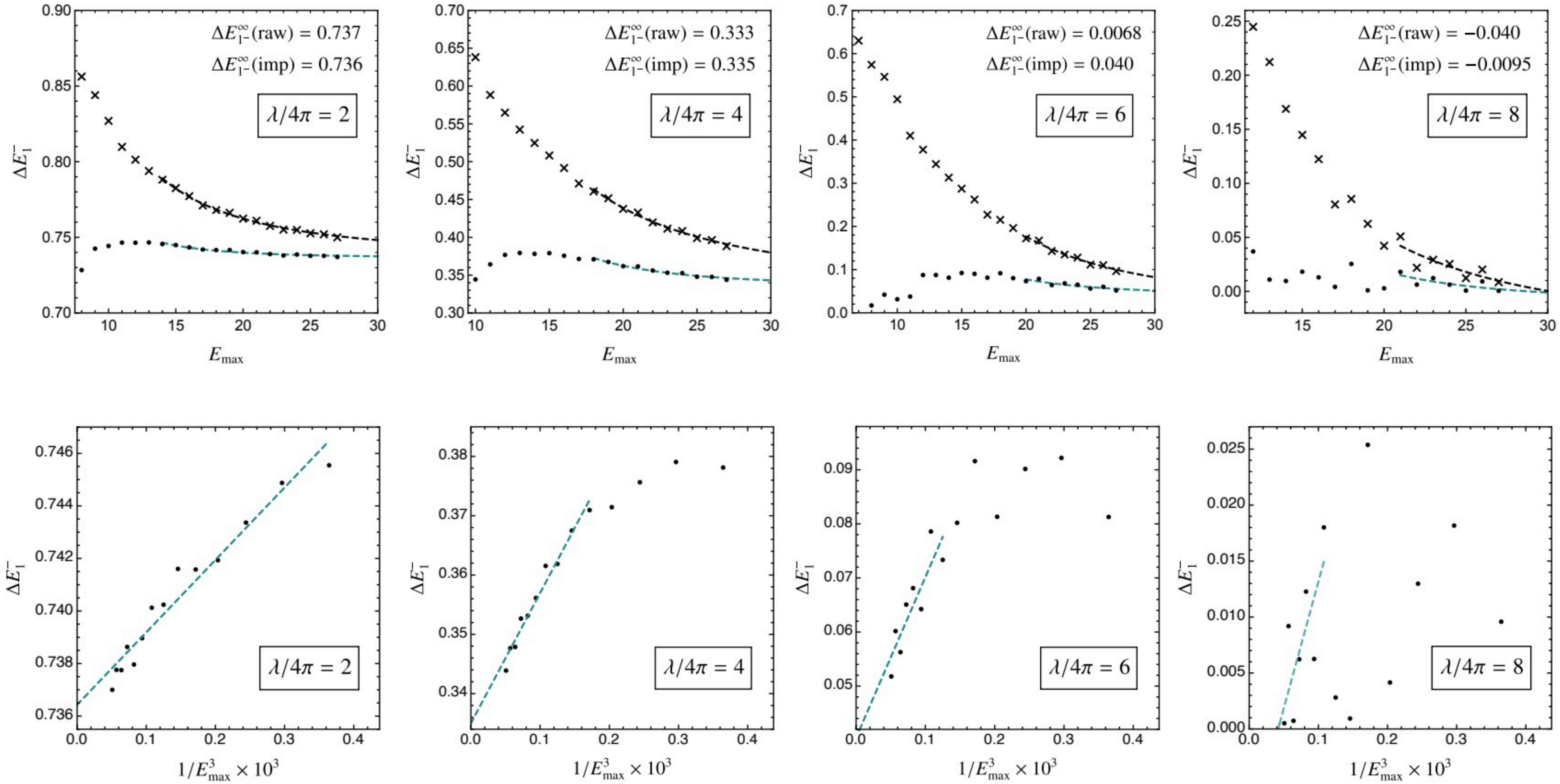
- Need to increase truncation scale to see power-law scaling



Stronger Couplings (Z_2 odd)

- Need to increase truncation scale to see power-law scaling

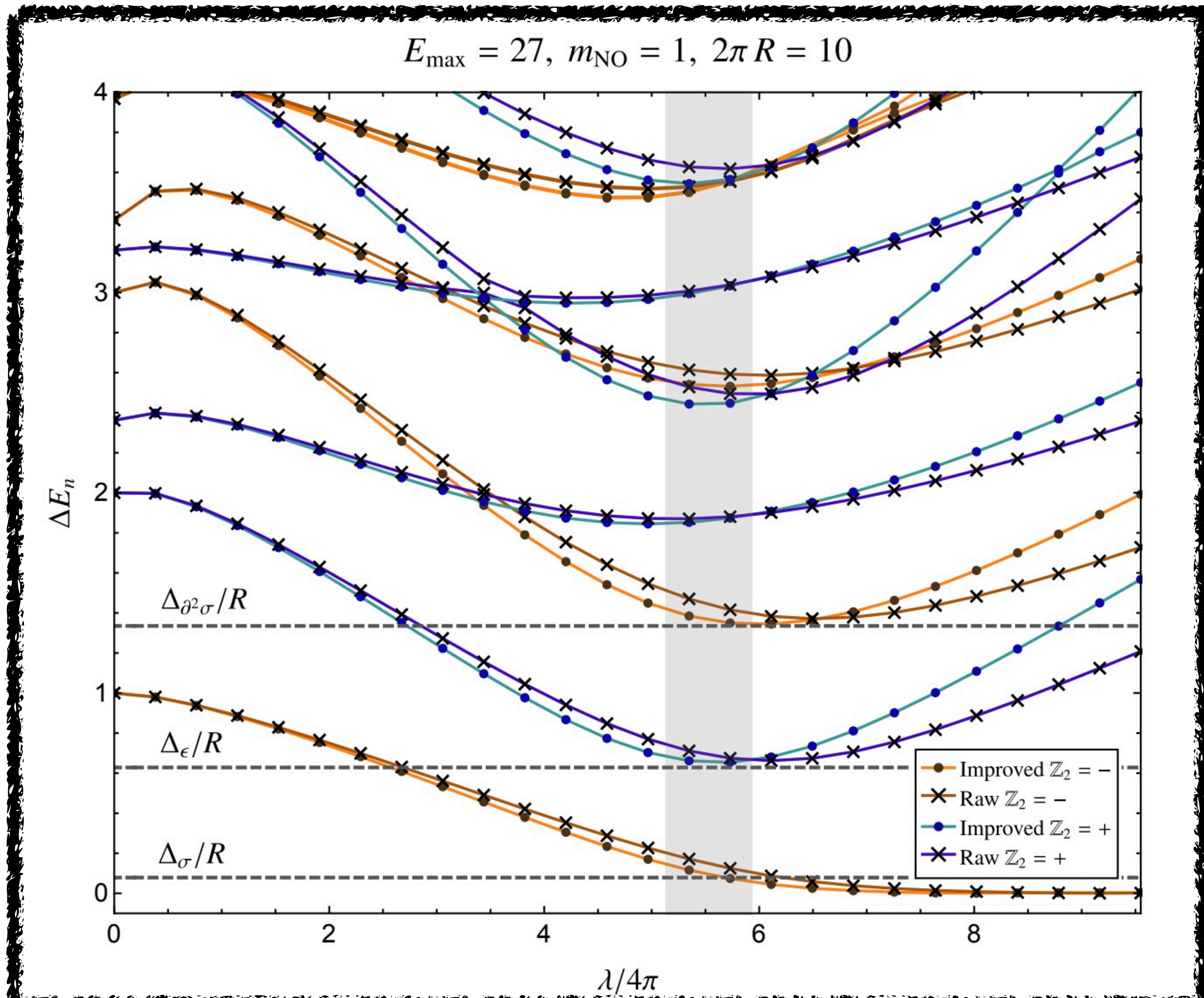
Numerical Noise from degenerate eigenvalues

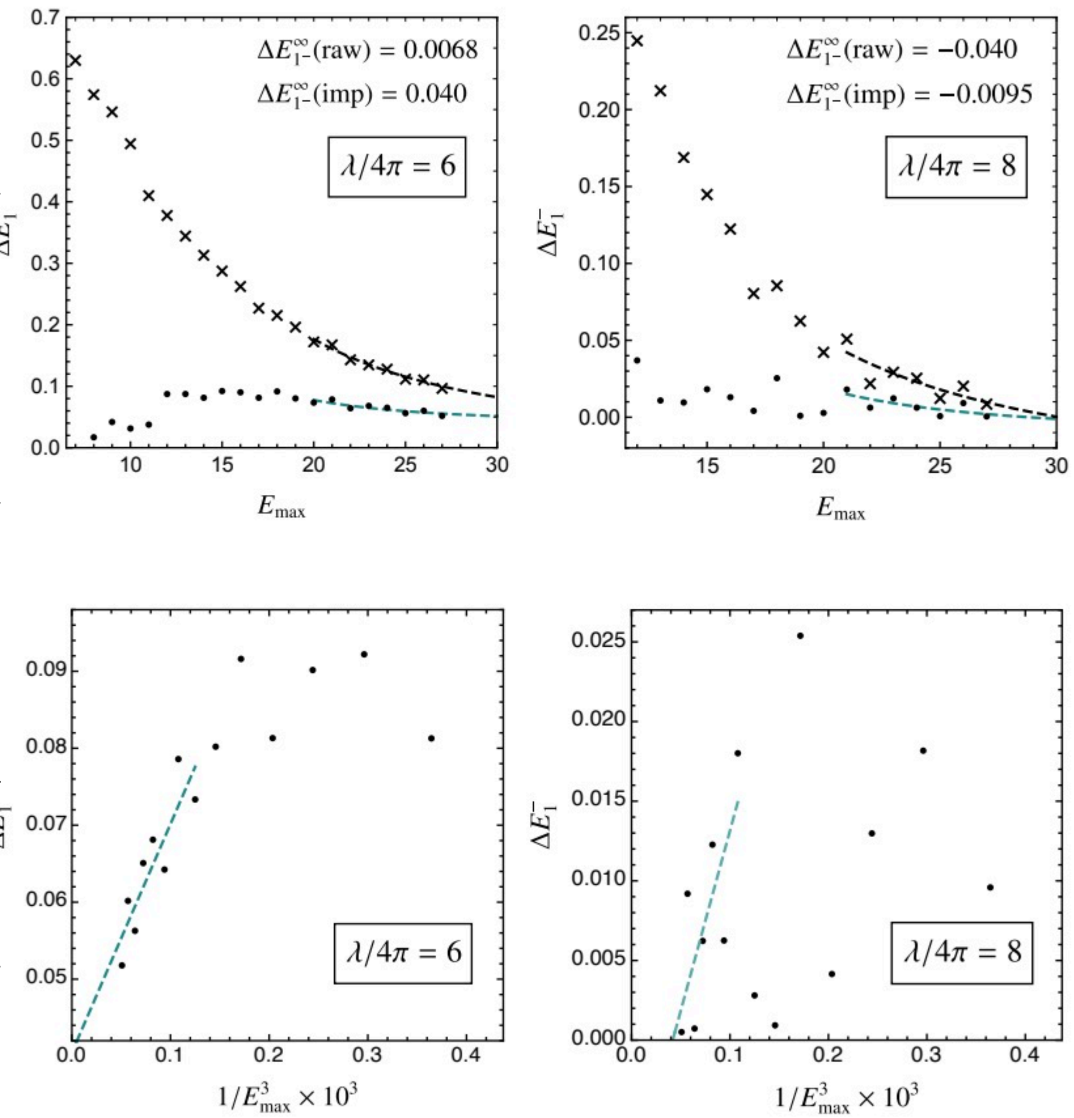


Stronger Couplings (Z_2 odd)

- Need to increase truncation scale to see power-law scaling

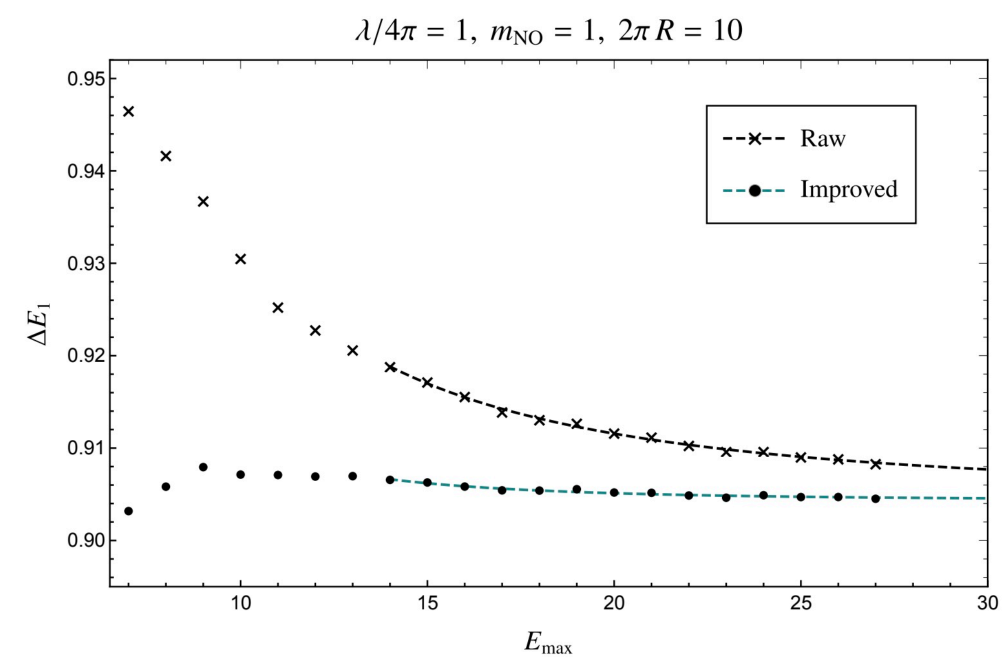
Numerical Noise from degenerate eigenvalues





Conclusions

- Introduced a scheme for systematic, order-by-order improvement for Hamiltonian truncation: HTET



- Fast results obtained on a laptop

- Demonstrated improvement from $1/E_{\mathrm{max}}^2$ scaling for raw truncation to $1/E_{\mathrm{max}}^3$ scaling for improved theory

- Future directions
  - 3D ϕ^4 theory with nontrivial UV divergences
  - 2D ϕ^4 theory at next order ($1/E_{\mathrm{max}}^4$ scaling)
  - Next order may require state-dependent counterterms

Thank you!

Back-up Slides

Matching an Operator

- We would prefer to match an operator order by order
- Start by turning off interactions adiabatically $V \to V e^{-\epsilon t}$
- Evolving in time, we find an ill-defined phase

$$\lim_{t_f \to \infty} \langle f | e^{-iH_{\text{eff}} t_f} | i \rangle = \lim_{\epsilon \to 0^+} \langle f | T \exp \left\{ -i \int_0^\infty dt H_0 + V e^{-it} \right\} | i \rangle$$

- Can be factored out in the interaction picture

$$|\Psi(t)\rangle_{\text{IP}} = e^{iH_0 t} |\Psi(t)\rangle_{\text{SP}} \qquad \mathcal{O}_{\text{IP}} = e^{iH_0 t} \mathcal{O}_{\text{SP}} e^{-iH_0 t}$$

- Time evolution: $U_{\text{IP}}(t_f, t_i) = T \exp \left\{ -i \int_{t_i}^{t_f} dt V_{\text{IP}}(t) \right\}$

Expansion of the Transition Matrix

- Time evolution:

$$U_{\rm IP}(t_f,t_i) = T\exp\left\{-i\int_{t_i}^{t_f} dt V_{\rm IP}(t)\right\} \qquad V_{\rm IP}(t) = e^{iH_0t}Ve^{-\epsilon t}e^{-iH_0t}$$

$$\lim_{t_f\to\infty}\langle f|U_{\rm IP}(t_f,0)|i\rangle \ni \langle f|i\rangle$$

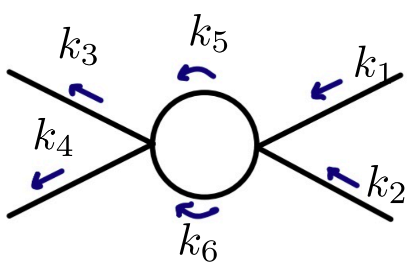
$$\lim_{t_f\to\infty}\langle f|U_{\rm IP}(t_f,0)|i\rangle \ni -i\int_0^\infty dt\langle f|e^{iH_0t}Ve^{-\epsilon t}e^{-iH_0t}|i\rangle = \langle f|V|i\rangle\frac{1}{E_{fi}+i\epsilon}$$

$$\lim_{t_f\to\infty}\langle f|U_{\rm IP}(t_f,0)|i\rangle \ni -\int_0^\infty dt\int_\infty^t dt'\langle f|e^{iH_0t'}Ve^{-\epsilon t'}e^{-iH_0t'}e^{iH_0t}Ve^{-\epsilon t}e^{-iH_0t}|i\rangle$$

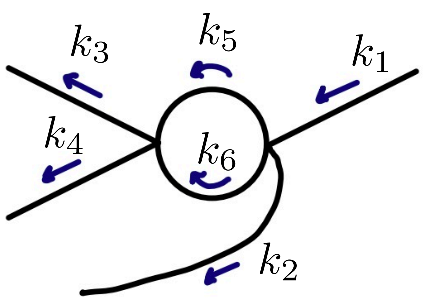
$$= \sum_\alpha\int_0^\infty dt\langle f|V\frac{1}{iE_{f\alpha}-\epsilon}|\alpha\rangle\langle\alpha|e^{iE_{f\alpha}t+iE_{\alpha i}t-2\epsilon t}V|i\rangle$$

$$= \sum_\alpha\frac{\langle f|V|\alpha\rangle}{E_{f\alpha}+i\epsilon}\frac{\langle\alpha|V|i\rangle}{E_{fi}+2i\epsilon}$$

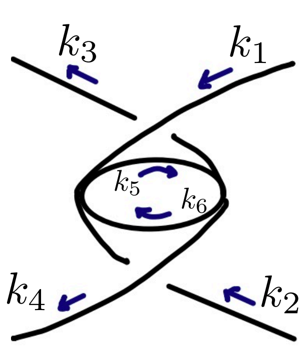
Diagrammatic Representation of Matching



$$\frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{k_1,..k_6} \delta_{k_1+k_2,k_5+k_6}\langle f|\phi_4^-\phi_3^-\phi_2^+\phi_1^+|i\rangle \frac{1}{2\omega_{k_5}}\frac{1}{2\omega_{k_6}}\frac{1}{\omega_3+\omega_4-\omega_5-\omega_6+i\epsilon}$$

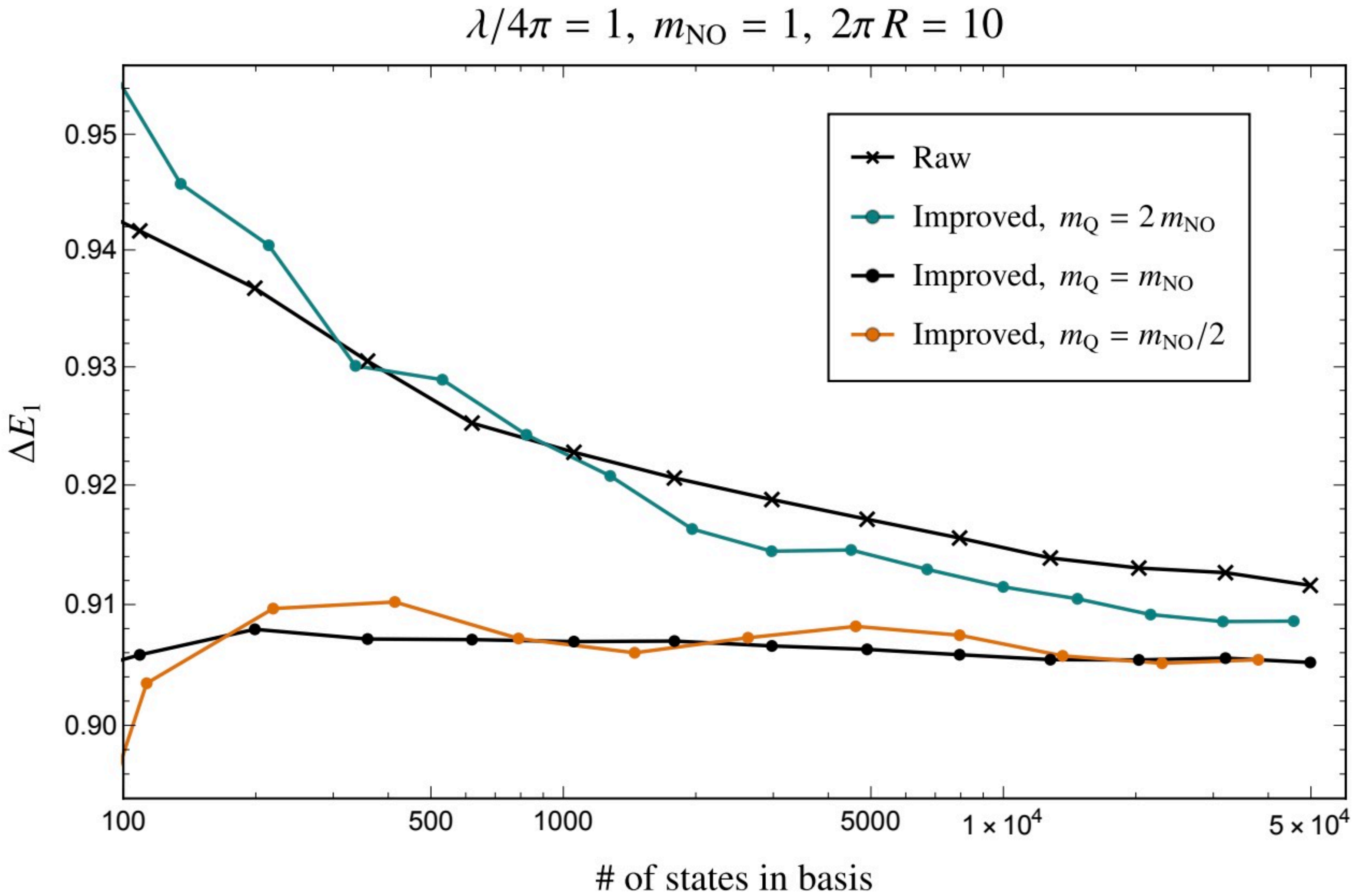


$$\frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{k_1,..k_6} \delta_{k_1,k_2+k_5+k_6}\langle f|\phi_4^-\phi_3^-\phi_2^-\phi_1^+|i\rangle \frac{1}{2\omega_{k_5}}\frac{1}{2\omega_{k_6}}\frac{1}{\omega_3+\omega_4-\omega_5-\omega_6+i\epsilon}$$



$$\frac{1}{8}\left(\frac{\lambda}{2\pi R}\right)^2 \sum_{k_1,..k_6} \delta_{k_1+k_2,k_5+k_6}\langle f|\phi_4^-\phi_3^-\phi_2^+\phi_1^+|i\rangle \frac{1}{2\omega_{k_5}}\frac{1}{2\omega_{k_6}}\frac{1}{-\omega_1-\omega_2-\omega_5-\omega_6+i\epsilon}$$

Varying mQ



$\lambda/4\pi = 1,\ m_{NO} = 1,\ 2\pi R = 10$
0.95
0.94
0.93
0.92
0.91
0.90
ΔE_1
100
500
1000
5000
1×10^4
5×10^4
# of states in basis
Raw
Improved, $m_Q = 2\,m_{NO}$
Improved, $m_Q = m_{NO}$
Improved, $m_Q = m_{NO}/2$