Mathematical Reasoning in University Physics

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Abstract. Physics and mathematics are deeply connected. However, there is ample research that mathematical reasoning in physics is not at all trivial for students. We present four contributions that discuss different aspects of mathematics and mathematical representations in university physics. The first contribution discusses student understanding of the Laplacian in the heat equation. In the second contribution, we report on students' interpretation of graphs representing non-constant acceleration motions. Study three deals with students' reasoning processes while constructing graphs. The last contribution reports on the translation and administration across different institutions in Europe and the U.S. of two instruments measuring quantitative reasoning.

The deep connection between mathematics and physics

Physics and mathematics are deeply connected. This holds true for all levels of physics, but at a more advanced level, the role of mathematics becomes even more important. As such, proficiency in mathematics is required to understand physical phenomena, and being able to combine the different fields is a prerequisite to become more proficient in physics.

Understanding an equation in physics is not just connecting the symbols to physical variables and being able to perform calculations and operations with that equation, it involves bridging the gap between a mathematical expression and its physical meaning and integrating the equation with its real-world implications [1]. This requires more than the sum of mathematics and physics and has proven to be challenging for students.

The relationship between mathematics and physics is an active research area in PER. Contributions span theoretical frameworks that describe the interplay [e.g., 2, 3], empirical research on students' challenges and views [e.g., 4, 5] and studies focusing on the pivotal role (mathematical) representations play in describing physical phenomena [e.g., 6, 7].

In this symposium, all presented studies are carried out with university students. We discuss several aspects of the intricate relationship between mathematics and physics using different theoretical lenses and methodologies. Each of the contributions starts from the idea that both disciplines are deeply connected. This implies that we do not see mathematics as 'a tool' for physics but rather see both in continuous mutual interaction and as such shaping each other.

Contributions in the symposium

The first contribution discusses students' reasoning on the Laplacian in the context of the heat equation. The authors start from APOS theory, a framework in mathematics education research describing mental constructions learners have to carry out to build an understanding of a mathematical concept, and combine it with physical concepts students should understand. By applying this framework, they designed a hypothetical trajectory for second year students majoring in Physics and Mathematics to learn and understand the Laplacian of a temperature distribution. In this trajectory, the coordination of 2nd partial derivative as rate of change of rate of change and the differential approach to the divergence of the gradient plays a pivotal role. The validity of the hypothesized trajectory was checked using task-based think aloud interviews.

Both the second and third contribution focus on graphical representations in physics. In the second study, student understanding of graphical representations of position-time, velocity-time and acceleration-time relations are studied in the context of motions with non-constant acceleration. It is well-known that analyzing kinematics graphs is hard for students and the challenges extend beyond linear motion. In the presented study, second year engineering students were asked to answer open-ended questions on motions with non-constant acceleration and their answers were analyzed using a phenomenographical approach focusing on the extraction, discrimination and interpretation of the graphs.

In the third contribution, we switch from students interpreting graphs to students constructing graphs: a series of questions relating to different hypothetical physics experiments were developed, where students were given a diagram and a piece of text describing an experiment and were then asked to complete a partially drawn graph of the predicted outcome. Along with drawing the graph, students were asked to explain why they had chosen that way to do so. Shapes of the drawings and explanations were analysed and will be discussed.

In the last presentation, physics quantitative reasoning (PQL) – the skills and habits of mind for the sophisticated use of algebraic mathematics to describe the world – is central. Two instruments to measure PQL, the PIQL for calculus based physics and the GERQN for algebra-based physics, were translated in Dutch and administered in English and Dutch to several student populations both in the U.S. and Europe. Conclusions both from the translation process and the administration will be discussed.

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Students' understanding of the Laplacian in the heat equation

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Abstract. The 2D heat equation describes how the temperature at any point on a plate evolves due to heat conduction. Our goal is to develop a research-validated learning path that supports students in developing a conceptual understanding of the 2D heat equation. The Laplacian of the temperature is a key concept in the 2D heat equation. Our previous research revealed that students struggle with understanding the Laplacian in mathematics and physics contexts. We have utilized the APOS framework and proposed a genetic decomposition that comprises both mathematics and physics principles. More specifically for the Laplacian, we hypothesize that coordinating second partial derivatives with the differential approach to divergence of gradient is important. An interview study validated this hypothetical genetic decomposition and revealed that only few students adopted a different approach to comprehend the Laplacian.

Introduction

The 2D heat equation describes how the temperature distribution in a 2D plate evolves over time due to heat conduction. Heat conduction is described by Fourier's Law which characterizes the relationship between heat flux density $\vec{q}$ and the gradient of the temperature $\vec{\nabla}T$:

$$\vec{q} = -k\vec{\nabla}T. \qquad (1)$$

Using the divergence theorem, Fourier's Law, and the rate of change of thermal energy over time, we obtain the heat equation in 2D:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T\ , \qquad (2)$$

where α is the thermal diffusivity. A correct interpretation of the Laplacian is crucial to conceptually understand the heat equation. The Laplacian may be calculated as the sum of second partial derivatives (2^{nd} PDs) with respect to spatial variables, e.g. *x* and *y*. This calculation relates the Laplacian to the spatial variation of temperature between a point and its surroundings. If $\nabla^2 T > 0$, the temperature at a point is lower than the average temperature of the surrounding area. The Laplacian is equal to the divergence of the temperature gradient. Physically, it is a quantity proportional to the net heat flux density at a point. In a previous study [1], we found that students struggle with interpreting the Laplacian. Our goal is to design a research-validated learning path aimed at addressing their challenges.

Theoretical Framework and methodology

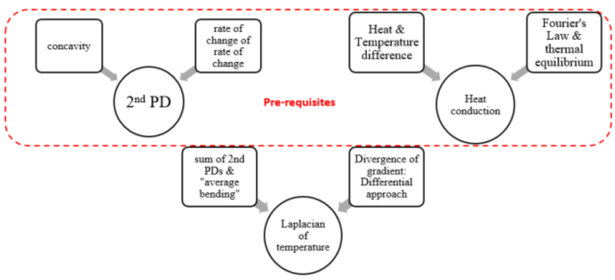

Fig. 1. Preliminary genetic decomposition of the Laplacian of temperature

To further study students' reasoning on the Laplacian in the context of the heat equation, we designed a hypothetical model of mental constructions that students need to carry out to learn and understand the Laplacian of a temperature distribution. We used the APOS framework from mathematics education research where such a hypothetical model is called a *genetic decomposition* (GD). We proposed a preliminary GD of the Laplacian (fig 1) comprising both mathematics and physics concepts [2] and emphasizing 2nd PD and heat conduction as pre-requisites. We do not delve into other pre-requisites in this paper. Our GD highlights the importance of understanding the concavity of a graph, interpreting rate of change of rate of change, comparing slopes of tangents in the vicinity of a point, discriminating between heat and temperature, and grasping Fourier's law. It suggests the need to conceptualize the summation of spatial 2nd PDs of temperature as the "average bending". Furthermore, it highlights the coordination between the differential approach to the divergence of a temperature gradient vector field and the rate of change of the rate of change. We hypothesize that understanding this coordination leads to interpreting the Laplacian of the temperature as a quantity proportional to the net heat flux through a closed boundary in 2D space. To test this hypothetical learning trajectory, we designed questions that probe these mental constructions. Eight students were chosen to take part in task-based think-aloud interviews.

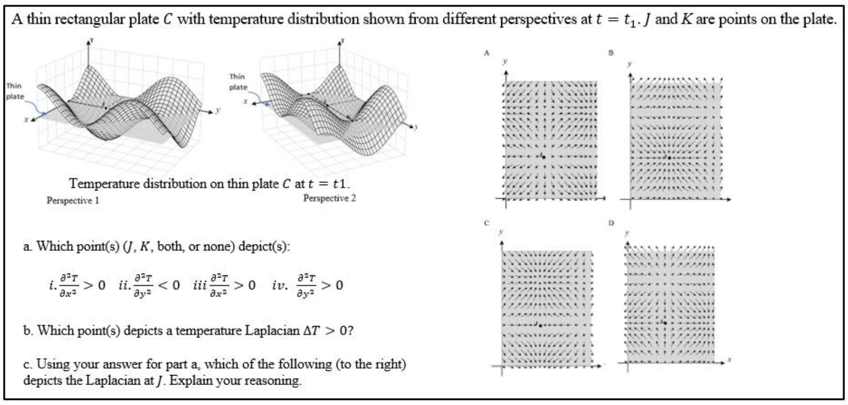

Fig. 2. Sample from questions designed to probe mental constructions of the GD.

Findings and conclusion

As hypothesized, the coordination of 2nd partial derivative as rate of change of rate of change and the differential approach to the divergence of the gradient, although challenging for students, proves to be important to develop an understanding of the temperature Laplacian. Only a few students used a different path to understanding originating directly from an understanding of Fourier's law. With these findings, we are closer to a more stable genetic decomposition of the Laplacian that accounts for students' reasoning and the difficulties associated with prerequisite concepts, and how they progress toward understanding the Laplacian. Details on student reasoning will be shown in the presentation.

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Non-constant acceleration kinematics. University students' difficulties related to graphical representation systems

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Abstract. The aim of this work is to analyse the difficulties of Chilean students related to the treatment of graphical representation (position, velocity, acceleration vs. time) for the subject of kinematics of non-constant acceleration. This research has been carried out by means of a peer-validated open-ended questionnaire. In this study, we present three of the 12 questions that make up the complete questionnaire. These are the three questions that ask students to extract, discriminate and interpret data from graphs so that by treating them they can base their answers using graphs and explanations. Phenomenographic analysis allowed categorization of students' reasoning in their responses. Initial analyses show that despite the fact that about one third of the students answer the questions correctly, the reasoning reveals confusions between physical concepts and application of rote learning strategies without physical meaning. These findings would guide educators in future learning itinerary designs.

Introduction

Within Mechanics, analysing the physics of motion with variable acceleration is essential for university students to understand how objects vary in position, velocity, and acceleration over time. However, this understanding poses intrinsic challenges, especially when addressing the graphical and algebraic representations used to describe such phenomena[1]. Despite these representations being crucial for modelling and predicting the behaviour of systems under these conditions, their interpretation and elaboration can be enigmatic, partly due to the complex nature of this type of motion [2].

Graphical representations are widely used as visual aids to unravel aspects of motion. However, deciphering and analysing these graphs is not a straightforward task. Students are challenged to understand curves representing changes in velocity over extremely small-time segments, demanding a solid and detailed comprehension of the underlying physical principles. This challenge extends beyond linear motion and manifests in a variety of contexts and scenarios [3].

Therefore, the aim of this study is to examine the difficulties and reasoning of second-year engineering students in Chile in order to detect learning difficulties in the treatment of the information (extract, discriminate and interpret) into graphical representations of position, velocity, and acceleration versus time while learning concepts of kinematics with non-constant acceleration [4].

Experimental design and methodology

In this study, the participation of 120 students was analysed. The students were enrolled in a traditional second year of physics course at a private university in Chile. Non-constant acceleration motion was taught for four weeks and was assessed in an end of chapter problem based final exam. Data was collected after the four weeks of instruction and before the final exam during an in-person class. To address the aim of the research, a 12 questions open-ended questionnaire was developed, focusing in the learning objective regarding different representations (algebraic and graphical) and the treatment and conversion between them [4]. In this work we are presenting questions 8, 9 and 10 related to de graphical representation treatment focusing in the extraction, discrimination and interpretation of the data in graphical representations of position, velocity, and non-constant acceleration over time. The questionnaire, peer-validated by experts in Physics and Physical Education Research, facilitated enhancing its robustness and clarity.

Phenomenographic analysis was carried out [5] to define students' explanatory categories following the needed steps in the data analysis [6]. The process of definition, refinement and validation of categories was done by different researchers. The reliability of the analysis was assessed using Cohen's kappa coefficient, yielding an average value of 0.95.

Findings

The phenomenographic analysis of students' reasoning in response to the open-ended questionnaire identifies explicative categories in the treatment between initial and final graphical representation in the analysed three questions. The results are shown in table 1.

Explanatory category	% of answers		
	Q8	**Q9**	**Q10**
Extract, discriminate and interpret information from graphs	25	35	37
Uses algebraic tools (Derivative, slope,…) without physical meaning	21	23	18
Misunderstood physical concepts as trajectory, velocity, acceleration	37	32	29
No answer	17	10	16

Table 1. Percentages of answers in each category for the question Q8, Q9 and Q10.

About one third of the students extract, discriminate and interpret information correctly and justify their answer. However, about 20% of students use the concept of derivative or slope without any physical meaning. In addition, another third of students confuses basic concepts of kinematics such as trajectory, velocity or acceleration.

Conclusions

In university mechanics physics teaching, instructors should not assume that the conversion between representations is automatic, as students encounter difficulties in interpreting, discriminating, and extracting information from non-constant kinematics graphs. We conclude that the identified difficulties can guide the design of teaching learning sequences based on the learning demands that will be defined based on this evidence.

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Constructing Graphs in Physics

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Abstract. In this talk we explore the thought processes students engage in when asked to draw graphs representing a qualitative physics scenario. We presented first year undergraduate students with a hypothetical physics experiment and asked them to sketch a graph of what they predicted the results would look like. We also asked them to give a written explanation as to why they drew the graph that way. We have analysed their responses under different theoretical frameworks based on the knowledge-in-pieces framework.

Introduction

Despite a long tradition in physics and mathematics education research on graphing, little is known about student construction of qualitative Cartesian graphs in physics contexts. We have started a systematic investigation of what students attend to when they draw graphs pertaining to a hypothetical experimental setup. Recently we analysed how the students represented equal distances on a track, beaker, or wire in their graphs, and found that students drew unequal intervals on the position axis to indicate unequal time or resistance intervals [1]. We extend this research to analyse what shapes students drew and why.

We initially posed a variety of questions involving different experimental setups, ranging from everyday scenarios to more abstract physics experiments. Whilst we discovered that context plays a huge role in whether or not students draw an appropriate graph, we saw that in all cases a large number of students drew straight line graphs where a curved graph would have been the correct response.

In an attempt to better understand what caused students to draw straight lines instead of curves, we posed the same hypothetical experiment to different groups of students but phrased the accompanying text differently to see what effect the wording had on the students. We asked students about hypothetical experimental set-ups in four different ways: (1) the original wording, in which the students are given a written and pictorial description of the experiment and are asked to complete a graph and explain their answer, (2) the original wording accompanied by a detailed narrative describing what happens throughout the experiment, (3) the original wording in a multiple choice version that allowed students to see a correct answer, and (4) a version where students were asked to explain what would happen before they drew the graph, in addition to the original wording. Interestingly, we saw that the posing of the question had little effect on students' ability to recognise and draw the correct graph.

Theoretical framework and methods

We administered the questions in midterm and end-of-semester assessments. The shapes of the graphs students drew could be categorised as curved, straight line, or other, with the former categories typically capturing 80% to 90% of the responses. One researcher coded the students' explanations in an emergent coding scheme and created a first version of a codebook.

The codebook that emerged is based on knowledge-in-pieces (Hammer, 2005). We note that recently analysis of students' reasoning with graphs in chemistry contexts has similarly utilised a resources framework (Rodriguez et al, 2021). Two other researchers coded a subset of the responses independently. Comparing coding helped us refine the codebook. A second iteration in which we coded a different subset of responses independently yielded high inter-rater reliability.

Findings and conclusions

We found that irrespective of how the questions were phrased, roughly equal fractions of students chose straight and curved graphs to represent the position of a ball rolling down an uphill or downhill track or the change in water level when a narrowing or widening object is placed inside a beaker when water is poured into it. We will report on some interesting correlations between elements of students' reasoning and the graph they drew.

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Assessing students' physics quantitative literacy in Ireland, Belgium, and the U.S.: towards insights for teacher preparation

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Abstract. Quantitative literacy – the ways in which one uses mathematics to describe the world around them – is an essential skill in physics. We have translated two inventories, one for calculus-based physics and one for algebra-based physics, into Flemish to compare Belgian, Irish, and U.S. students and pre-service teachers' quantitative literacy in physics contexts. Pilot instructional materials that may help to improve students' PQL are discussed.

Introduction

Quantitative reasoning is an essential skill in physics. *Quantitative literacy* is a term from mathematics education research which describes the nuanced ways that experts use foundational, algebraic mathematics to represent the world around them [1, 2]. In physics, these skills and habits of mind are referred to as *Physics Quantitative Literacy* (PQL) [3]. Prior work has identified three facets of PQL: reasoning about signs and signed quantities [4], proportional reasoning, which is defined as reasoning about linear and inverse relationships [5], and covariational reasoning, which is defined as reasoning about how small changes in one quantity affect changes in another quantity [6-8]. Recent research has produced an assessment of PQL called the Physics Inventory of Quantitative Literacy (PIQL) [8-9], and an algebra-based version of the assessment called the Generalized Equation-based Reasoning inventory of Quantity and Negativity (GERQN) [10]. Administration of these two assessments in the United States has shown that traditional physics instruction is unlikely to improve students' PQL on its own – direct instruction about graphical and symbolic representations, and their meaning in physics, is required to help students learn how to reason mathematically in physics contexts [8].

Historically, European school systems have incorporated higher levels of conceptual mathematical reasoning in pre-college instruction. To probe how this difference in cultural context may change students' experiences in college physics courses, we administered the PIQL to nearly identical populations in the United States and Belgium. Preliminary results suggest that Belgian students are likely to earn higher scores on the assessment. In particular, these students tend to have higher correct response rates for items that were designed to measure covariational reasoning. However, neither population saturates the test. We suggest that these early results are indicative that there may be some things that we can learn from the Belgian school system, and there is still significant room for improvement in both the U.S. and Belgium.

Methods and Findings

One concern with our preliminary findings was that both populations took the PIQL in English. To see if taking the assessment in ones' non-native language had a significant impact on the difference in student scores, we have translated and validated the PIQL into Flemish. During the translation process, it became clear that there were ways both versions – English and Flemish –

have high levels of reading comprehension. Therefore, we adjusted the language use in both versions to be more easily understood. These results contribute to an on-going effort to ensure the assessments measure mathematical reasoning, and do not accidentally measure as students' proficiency in test taking more generally. In our presentation of the translation, we will also discuss patterns observed during faculty focus groups and individual student interviews used to validate the Flemish PIQL and compare these to the patterns observed when validating the English version in the United States.

The ultimate aim of this project is to identify ways in which physics instructors can help their students learn to reason mathematically in physics (and other science) contexts. We are using the results from the U.S. and Belgian administrations of the PIQL to inform the development of instructional activities that may help students learn to reason this way. These activities are being used as part of physics instruction and physics teacher preparation programs in the U.S., Ireland, and Belgium as an initial step towards evaluating whether they improve students' PQL and to expand our goal of comparing the development of PQL across cultural contexts. In this presentation, we will share reflections from the activities and compare assessment scores across the teacher preparation populations.

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