The right way to introduce complex numbers in damped harmonic oscillators

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Abstract. In 1930, Born and Jordan wrote a quantum mechanics textbook. In that work, they used a strategy to convert the harmonic oscillator equations of motion into two uncoupled first-order equations. In classical mechanics, this mapping explicitly introduces complex numbers into the motion of a harmonic oscillator and directly shows how to solve for the position and momentum observables. I will explain how this mapping works and show how to demystify complex numbers use in damped-driven harmonic oscillators. As an added bonus, this approach also shows how energy varies with time and makes a strong connection to quantum mechanics.

Introduction

When I was originally taught the damped harmonic oscillator in my freshman physics class, the instructor told us to use complex exponentials and "*just take the real part*" at the end of the calculation to determine the position. I was bothered by that discussion then and it has stood with me now nearly 40 years later. When working on a completely unrelated problem in quantum mechanics, I was reading the 1930 textbook by Born and Jordan called *Elementare Quantenmechanik* [1]. In it, Born and Jordan show how one can go from the Hamilton equations of motion (two coupled linear first-order differential equations) to two decoupled first-order linear differential equations. In their work they then moved into using this construction to factorize the quantum harmonic oscillator Hamiltonian.

These ideas laid fallow for many, many years. More recently, Gauthier [2] used them to provide an alternative solution to the harmonic oscillator, which was further discussed by Tisdell [3]. Alves [4] rediscovered the connection to quantum mechanics. While this talk will focus solely on the simple harmonic oscillator, these ideas can also be employed to efficiently determine Kepler orbits and the orbits of a central field harmonic oscillator. The connection to quantum mechanics continues with these examples as well.

Experts may note that the approach given here is not new at all—it is just the standard matrix approach used to deal with coupled first-order differential equations. But, the ideas given here go far beyond that formal approach. Indeed, it is best done with no matrices at all, as we summarize our talk.

Complex numbers and the classical damped harmonic oscillator

In the United States, nearly every introductory textbook avoids the use of complex numbers, even for problems such as the damped harmonic oscillator, for which they are a natural. It does appear in a small number of textbooks, such as the *Feynman Lectures in Physics* [5], but it is not widespread. It is probably more common for instructors to introduce it anyway in their courses, because it is a natural way to proceed, even if the "just take the real part" at the end of the calculation. Instead, we find that it is more likely to appear in junior-level classical mechanics courses, such as those taught with Marion and Thorton's [6], or Taylor's [7] textbook. Nevertheless, the treatment always appears to be the same. When confronted with the differential equation of motion for the damped driven harmonic oscillator, where one cannot easily substitute in with sines or cosines multiplied by exponentials, the idea of using a complex exponential is motivated by the fact that its derivative does not change the form of the function, so it can quickly be shown to convert the differential equation to an algebraic equation. Then, when it comes time to determine the final motion, the student is often told that position is a real variable, so we must take the real part of the solution in order to have a physical solution.

The approach we propose is similar, but does not require any of the "leaps of faith" used in the standard pedagogy. Instead, we start from the equations of motion $\dot{x} = p/m$ and $\dot{p} = -m\omega^2 x$. These two coupled differential equations can be decoupled by expanding in a linear combination $A = p + \alpha x$, with α a constant. Then, computing the time derivative of A yields

$$\dot{A} = \dot{p} + \alpha\dot{x} = -m\omega^2 x + \frac{\alpha p}{m} = \frac{\alpha}{m}\left(p - \frac{m^2\omega^2}{\alpha}x\right)$$

which requires $-m^2\omega^2 = \alpha^2$, or $\alpha = \pm i m\omega$. We choose the positive sign. Then we find $\dot{A} = i\omega A$, which is solved by $A = A_0 e^{i\omega t}$, with $A_0 = p_0 + i m\omega x_0$. Using the solution for A^* as well, we then find $x(t) = (A - A^*)/2im\omega$. This gives the standard result $x(t) = x_0 \cos\omega t + \frac{p_0}{m}\sin\omega t$. Note how the complex numbers arise naturally in the course of the solution.

Furthermore, one can show that the energy satisfies

$$\frac{1}{2m}A^*(t)A(t) = \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t) = |A_0|^2 = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2 x_0^2 = const.$$

So, energy is conserved. Furthermore, if we generalize to quantum mechanics, using $[\hat{x}, \hat{p}] = i\hbar$, we find that $\hat{H} = \frac{1}{2m}\hat{A}^\dagger\hat{A} + \frac{1}{2}\hbar\omega$. So, one can employ this approach to make a bridge to quantum mechanics as well.

These results also hold for the damped and damped-driven harmonic oscillator and will be described in the talk. Even more surprising, this approach can be employed with central-force problems, providing an efficient derivation of the orbit equation for the Kepler problem and for an isotropic central-force harmonic oscillator. The extension to quantum mechanics holds there as well too.

In summary, these results provide a more natural way to discuss the driven-damped harmonic oscillator and provide a simple way to connect quantum mechanics to traditional classical mechanics problems. We feel it is worthwhile to make the effort to change classical mechanics instruction into this new form and use it to ease the transition into quantum mechanics.

References

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