A relationship between collisions and the Betz limit

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Abstract. The so-called Betz limit is commonly used to calculate how much mechanical power can be collected from the wind. According to this limit, no mechanism can capture more than (16/27) of the kinetic energy in the wind flow. In this work, a first principles derivation of the Betz limit is presented. It includes elements of linear momentum transfer in the fluid dynamics context in a collision between a fluid and a solid collector.

Introduction

Wind power has become increasingly important in the last few decades because of the climate change emergency. It is usually associated with the so called Betz limit that was published in 1919 by A. Betz in a book on flow machines [1]. The purpose of this presentation is to adapt the content of Betz´s work turning it into to a classroom example in a mechanics course for instance. It is worth mention that wind is a renewable and sustainable source to produce electricity with virtually no carbon emission process in all production chain (fabrication, transport, assembly and operation). Moreover, wind power can be explored without the need of costly prospection procedures and of fancy technology of fabrication.

Discussion

The derivation consider a fluid with density ρ flowing with velocity v (a uniform velocity field is assumed). The kinetic energy of an infinitesimal portion of fluid of mass dm is then $dK = ½\, dm\, v^2$. Considering the fluid contained in a volume with infinitesimal width dx and cross sectional area A, one can rewrite the kinetic energy in terms of the density as $dK = ½\, \rho A dx\, v^2$ since $dm = \rho A dx$. The mechanical power carried by the fluid is then $P = dK/dt$ which is readily shown to be $P = ½\, \rho A v^3$ where dx/dt is identified with the fluid velocity v. This is the mechanical power available in a fluid flow. What the Betz limit shows is that only the fraction (16/27) of the power P can be collected by a flow machine.

The fluid has its energy collected when interacts with an object transferring energy and linear momentum to it. It is assumed that this object (the collector) has the shape of a plate of cross sectional area A and is free to move. Because of the momentum transfer from the fluid, the collector acquires a velocity V. The force that act on the collector can be calculated by the linear momentum exchange in the process. It is considered that the fluid decelerate from its initial velocity, v, to the collector velocity, V. In other words, the fluid suffers a completely inelastic collision. As a result, there is a linear momentum variation of magnitude $dp = dm\,(v - V)$ where dm is the fluid´s mass that collide with the collector. If the time interval, dt, in which the deceleration process occur is small enough, the kinematics of uniformly accelerated motion can be used so the distance travelled by the colliding fluid is $dx = (v + V)\, dt\, /\, 2$. From this, one can calculate the mass that interacted with the collector as $dm = \rho A dx = \rho A(v + V)\, dt\, /\, 2$ during the time interval dt and the force is then $F = dp/dt = \rho A(v + V)\,(v - V)\, /\, 2$. The transferred power can be evaluated from the work the fluid does on the collector ($dW = F dx$). The final expression corresponds to the power transferred from the flow to the collector:

$$P_{collected} = \rho A(v - V)\,(v + V)^2\, /\, 4 \qquad (1)$$

where $P_{collected} = dW/dt$ was used.

Equation (1) can be written in terms of the power carried by the fluid flow if the parameter $\xi = V/v$ is introduced: $P_{collected} = f(\xi)\, P$, where $f(\xi) = (1-\xi)(1+\xi)^2/2$. The variable ξ is known in the literature as the Betz parameter and the function $f(\xi)$ has a maximum when $\xi = 1/3$. The value of $f(1/3)$ is exactly the Betz limit of (16/27).

Application

The development of green technologies has become increasingly important in our days especially due to the anthropogenic greenhouse effect. The seek of carbon-free energy production methods is an important issue to take into consideration. From the educational point of view, part of the efforts to accomplish the necessary changes (the global energy transition for instance) should include amendments on the Physics contents presented to students. However, most courses or text books in basic Physics lacks subjects concerning such applications since they are tied to the technology existing at the time they were written. The aim of this contribution is then to provide resources for discussions in classroom on how much power can be extracted from natural fluid flow in a quantitave and simple way so that students can appreciate the problem critically on well based grounds and with firm conviction. It can be used either in mechanics or fluid dynamics courses at university level as a complement to the existing text-book materials.

Conclusion

The above derivation of the Betz limit links the maximum energy transfer to the occurrence of a completely inelastic collision during the energy collection. This derivation can be used to calculate the energy transfer process with other fluids like water for instance. Since water is much heavier than air (the water density is nearly 1000 times larger than the air density) the collected power can go from kW to MW just by changing the nature of the working fluid. This contribution shows an example on how basic Physics can be linked to problems of environmental interest. Wind power is a quite simple virtually carbon-free method to obtain reasonable amounts of electricity. Among its advantages one can mention that wind is easy to find (oil and uranium depends on costly prospection methods for instance) and the wind collection does not need high technology (such as in the semiconductor industry for solar power exploration).

References

[1] A. Betz, *Introduction to the theory of flow machines* (D. G. Randall, Trans.). Oxford: Pergamon Press, 1983.