Recoil and Radiative-Recoil Corrections in Muonium

Greg Adkins
Franklin & Marshall College
Lancaster, Pennsylvania, USA
24 May 2023

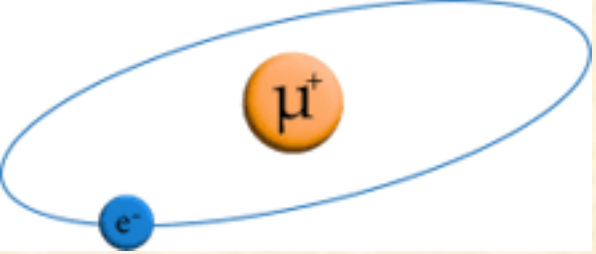
FFK2023

Recoil and Radiative-Recoil Corrections in Muonium

Motivation
Experimental Situation
Method of Calculation
Calculation of Recoil Corrections
Recoil Correction Results
Radiative Recoil Corrections
Conclusion

Collaborators
Jon Gomprecht
Yannis Li
Evan Shinn

Acknowledgments
NSF PHY-2011762
Franklin & Marshall College



Pre-Summary

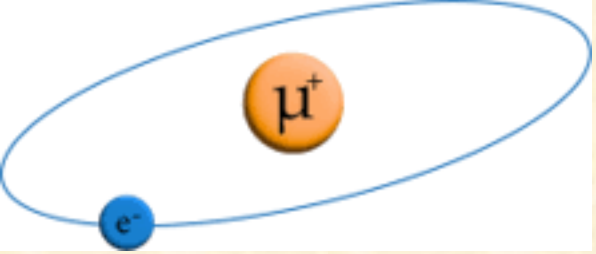
Calculations of corrections to positronium energy levels from the “hard” (relativistic) region depend on a single mass parameter, which can be factored out, leaving a pure number that is difficult to calculate.

Surprisingly, it is better to generalize and let the two particles have different masses. One can be factored out, leaving a single parameter $x=m_1/m_2$.

Test at order α^6 : The Feynman integrals F(x) satisfy a first-order ODE $dF/dx = M\,F$. A convenient solution for F(x) can be found. We use a BC at x=0 since it is easier to calculate F(0) than F(1). With the BC, the function F(x) is completely determined.

Now F(1) gives the result for positronium, but as a benefit, F(x) is available for all x. Specifically, results for muonium are found.

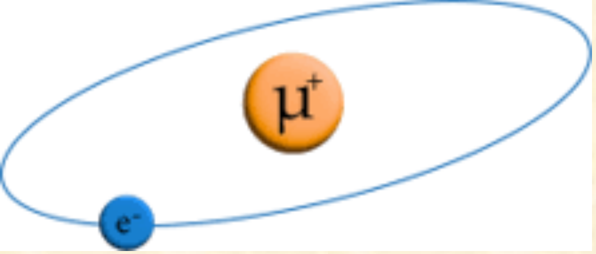
Hopefully, the same approach will work at order α^7 and give recoil corrections to muonium energy levels at that order.



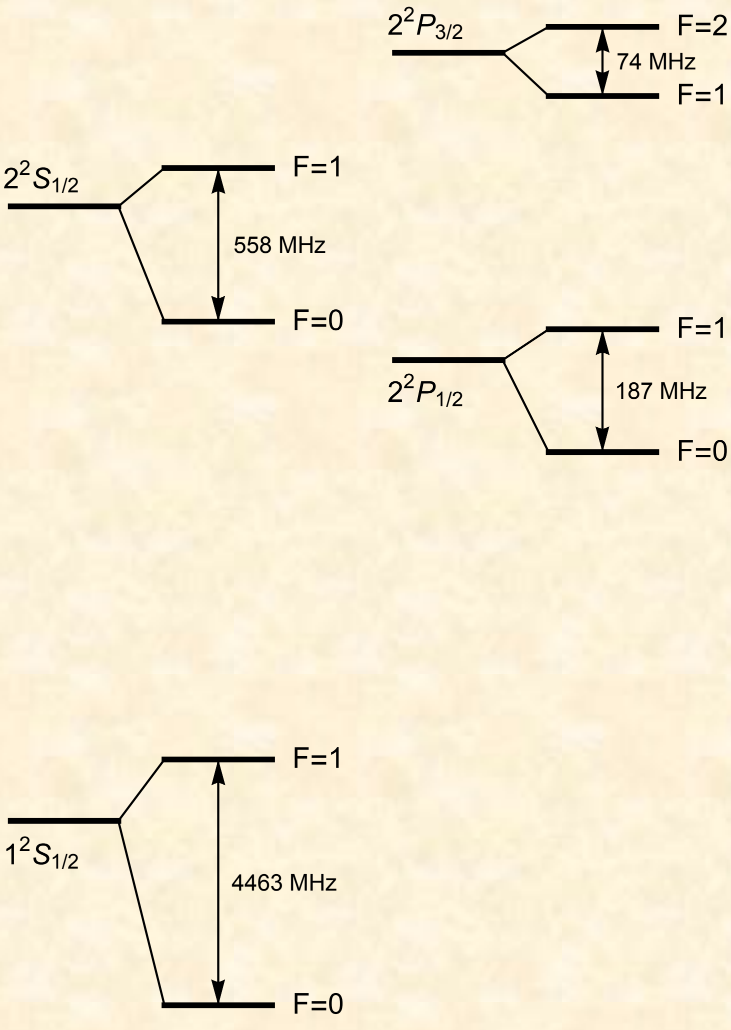
Muonium: a purely leptonic exotic atom

Muonium, the μ^+e^- bound system, is closely analogous to hydrogen but with several important differences. First, both of it's constituents are structureless point-like particles. Compared to hydrogen, where the proton size and internal structure matter, the theoretical analysis of muonium is relatively straightforward. Recoil effects are more important in muonium than in hydrogen, given that $m_e/m_\mu=1/207$ while $m_e/m_p=1/1837$. The finite muon lifetime of $\tau=2.2\ \mu s$ leads to a natural minimum linewidth through the uncertainty principle.

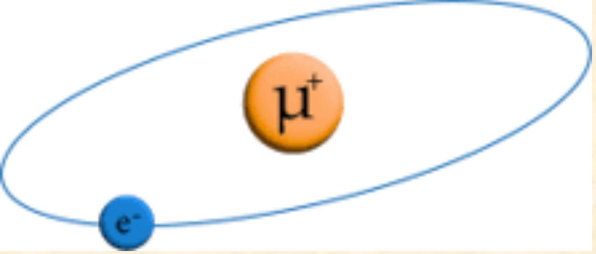
High precision measurements of many of the muonium n=1 and n=2 transitions combined with the possibility of high precision calculation of those transition frequencies based mainly on QED make muonium an attractive system for the determination of fundamental constants and testing the limits of current theory.



Muonium Spectrum



The n=1 and n=2 muonium energy levels are shown, along with the hyperfine intervals.



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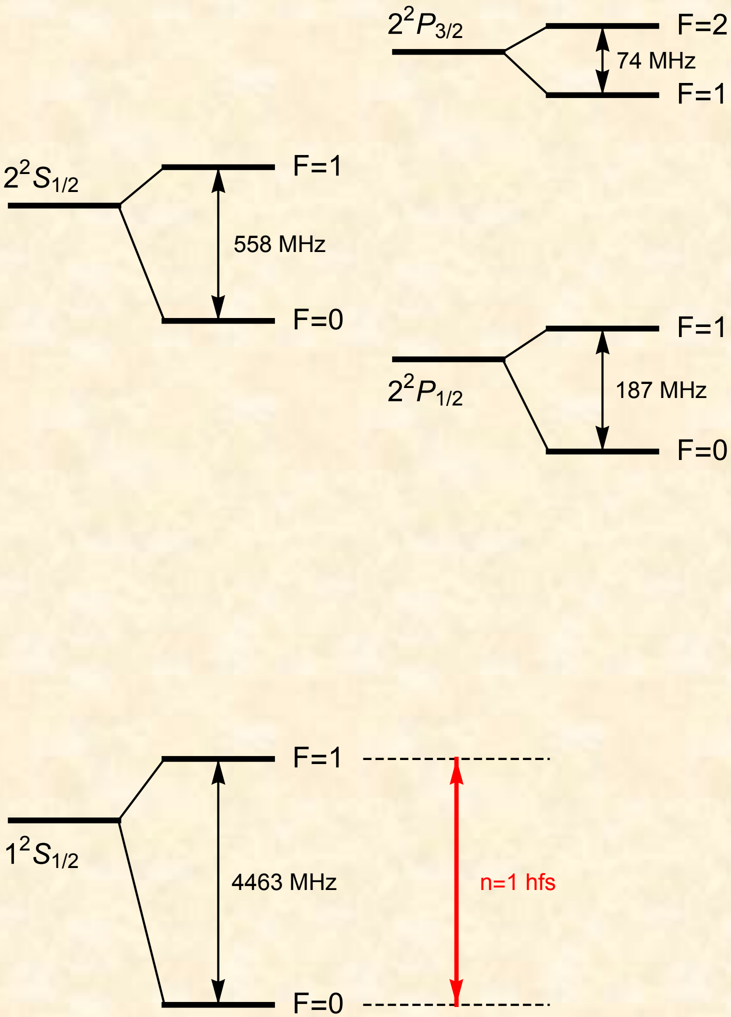


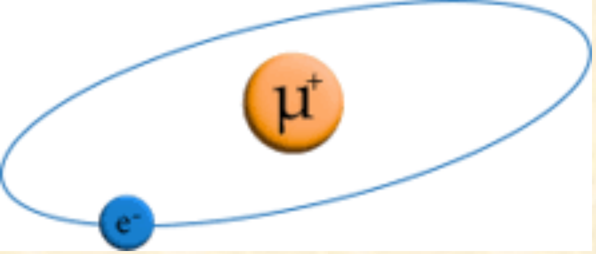
The n=1 hyperfine interval has been measured to high precision:

$$\Delta E = 4\,463\,302.88(16)\,\mathrm{kHz}$$

F.G.Marion et al., Phys. Rev. Lett. 49, 993 (1982)

$$\Delta E = 4\,463\,302.765(53)\,\mathrm{kHz}$$

W.Liu et al., Phys. Rev. Lett. 82, 711 (1999)



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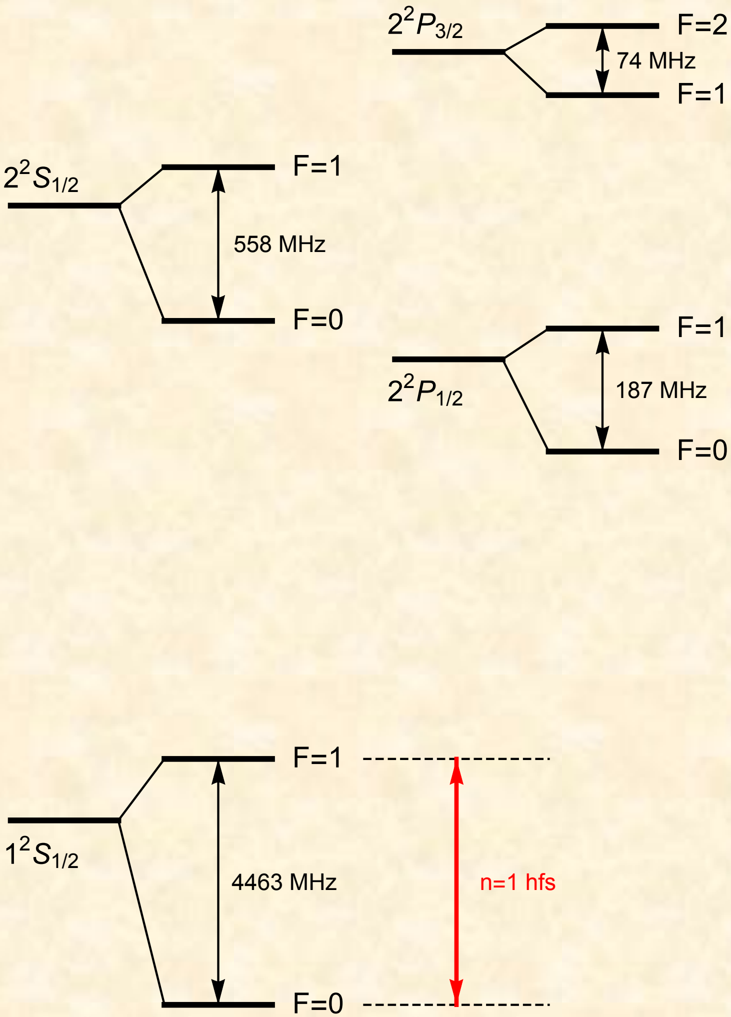

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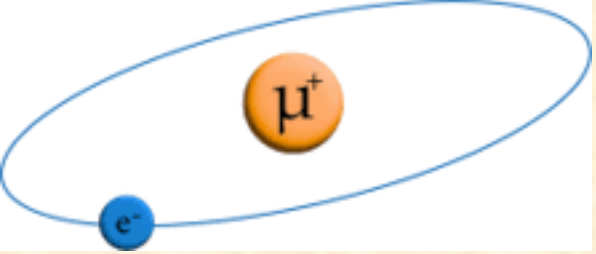
S.Kanda et al. (MuSEUM), Phys. Lett. B **815**, 136154 (2021)



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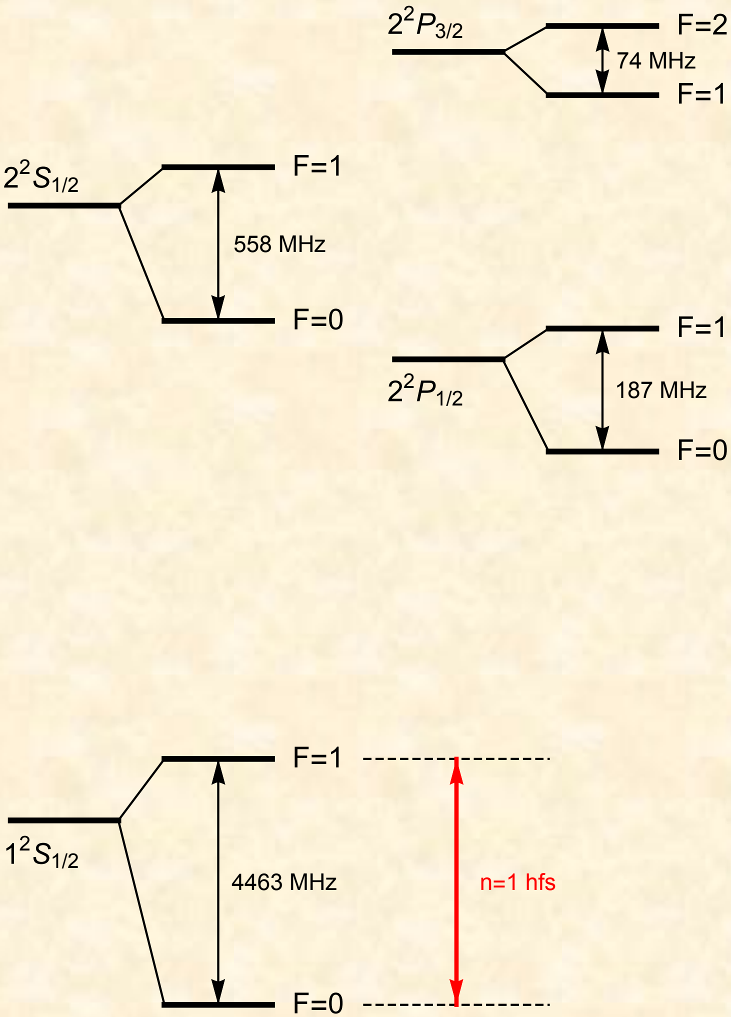

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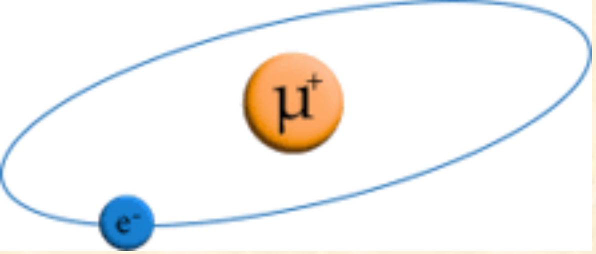
The new work has an uncertainty goal of ±5 Hz

P.Strasser et. al. (MuSEUM) EPJ Web of Conferences 198, 00003 (2019)



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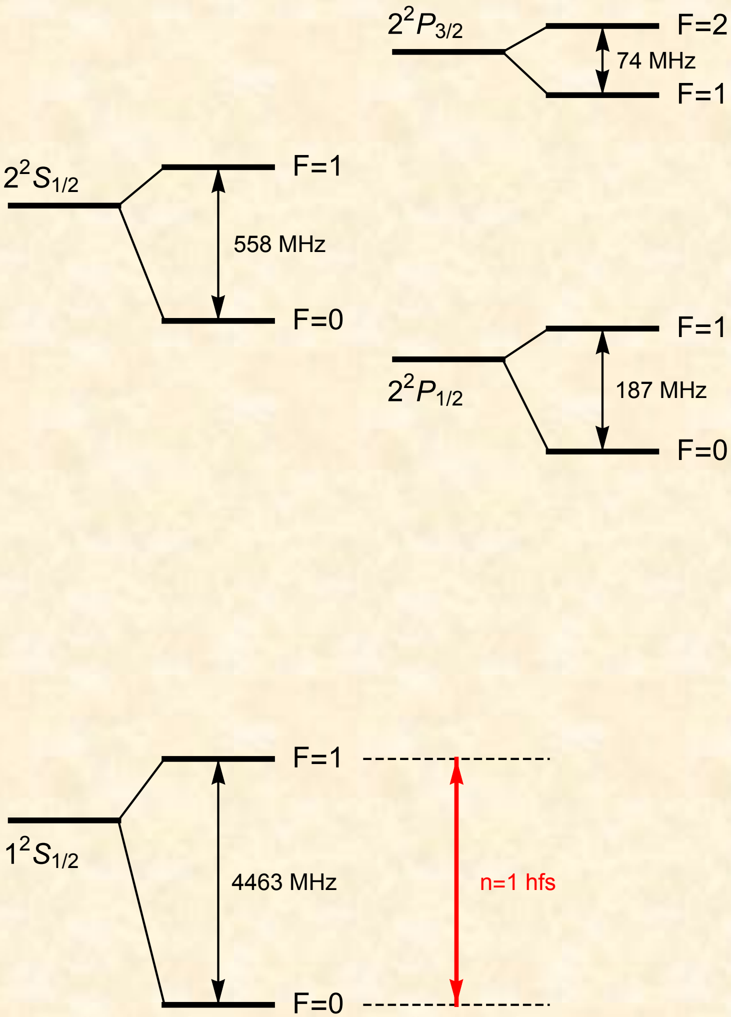

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Theoretical prediction:

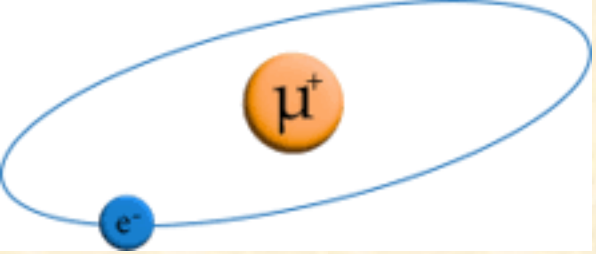
$$\Delta E = 4\,463\,302.868(271)\,\mathrm{kHz}$$

Mohr, Newell, Taylor, Rev. Mod. Phys. 88, 035009 (2016)

$$\Delta E = 4\,463\,302.872(515)\,\mathrm{kHz}$$

Eides, Phys. Lett. B 795, 113 (2019)

See also Karshenboim and Korzinin, Phys. Rev. A 103, 022805 (2021)



Muonium Spectrum

$$\tilde{E}_F = \frac{8 m_e \alpha^4}{3} \left(\frac{m_r}{m_e}\right)^3 \left(\frac{m_e}{m_\mu}\right) = 4454\,\mathrm{MHz}$$

$$\alpha^2 \tilde{E}_F \left(\frac{m_e}{m_\mu}\right) = 1147\,\mathrm{Hz}\,,$$

$$\alpha^2 \tilde{E}_F \left(\frac{m_e}{m_\mu}\right)^2 = 5.5\,\mathrm{Hz}\,,$$

$$\alpha^3 \tilde{E}_F \left(\frac{m_e}{m_\mu}\right) = 8.4\,\mathrm{Hz}$$

$$\ln\left(\frac{1}{\alpha}\right) = 4.92 \;, \quad \ln\left(\frac{m_\mu}{m_e}\right) = 5.33$$

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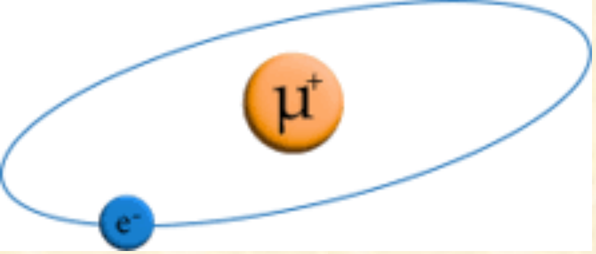
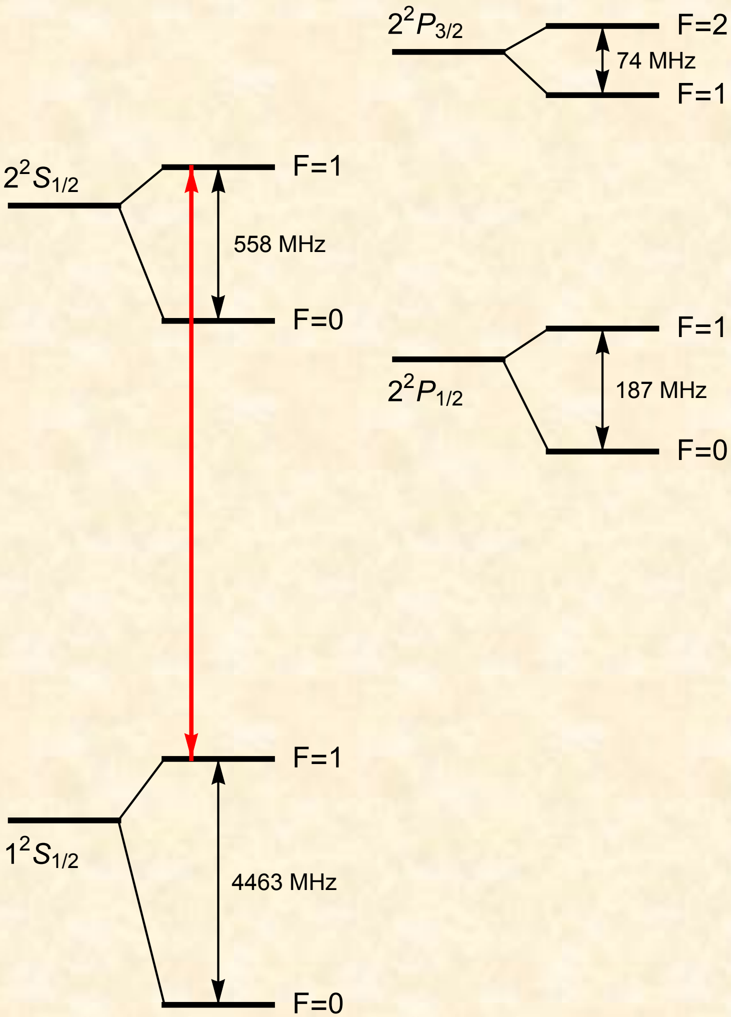
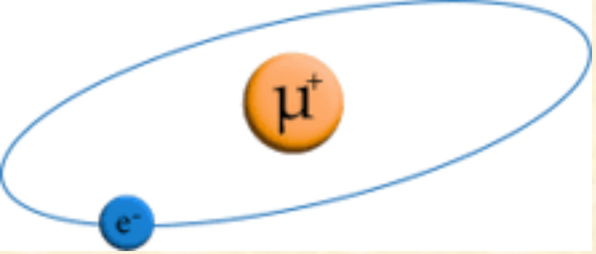


Muonium Spectrum: 1S-2S Interval



The 1S-2S interval has a natural linewidth of 145 kHz and can be measured with great precision.

V. Meyer et al., Phys. Rev. Lett. 84, 1136 (2000)



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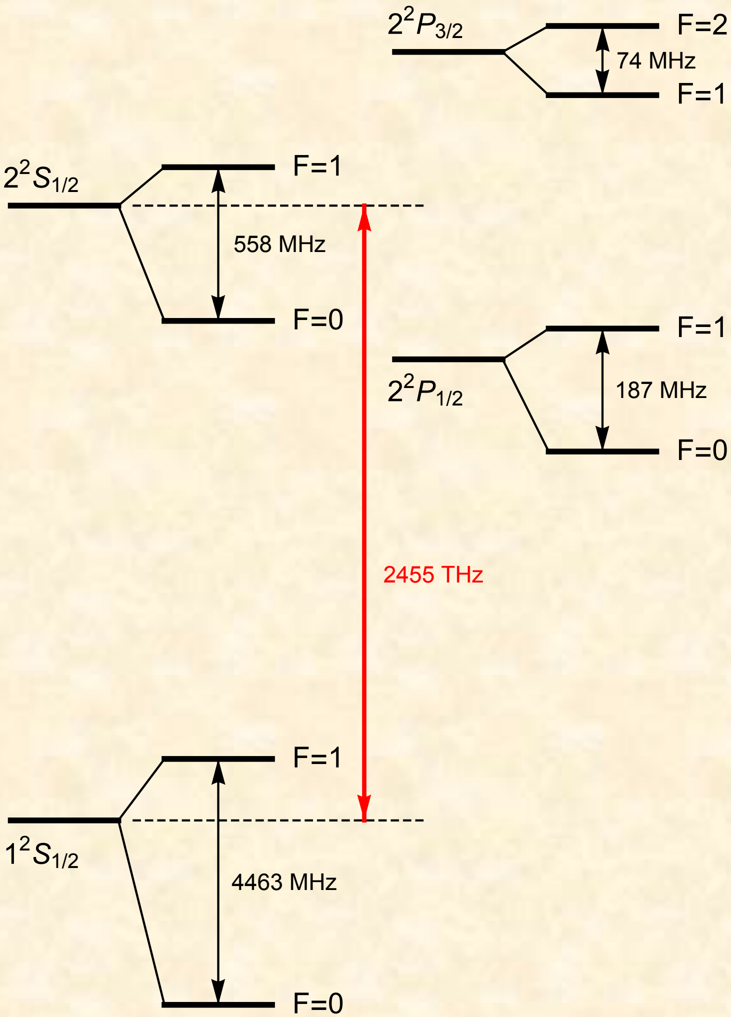

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$\Delta\nu$=2,455,528,941.0(9.8) MHz (4 ppb)

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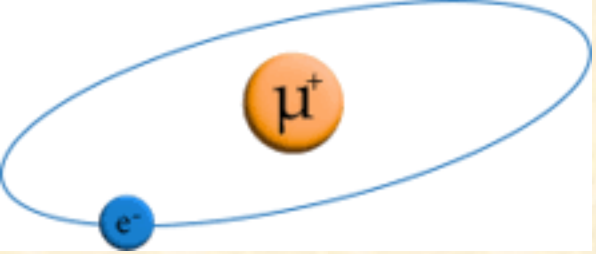

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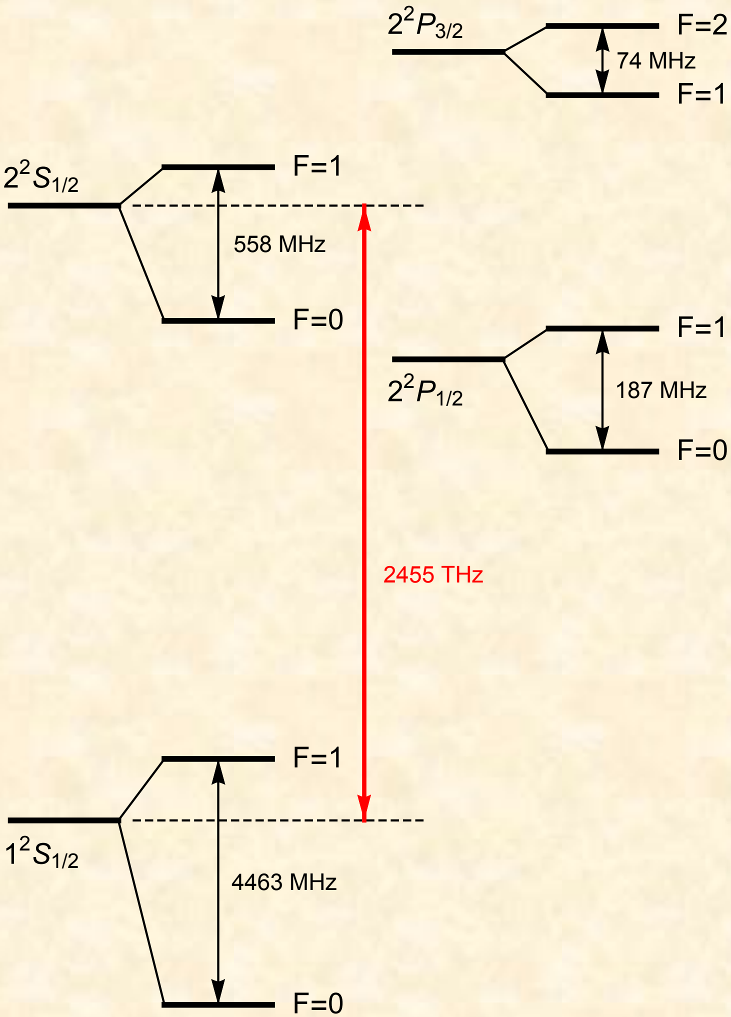

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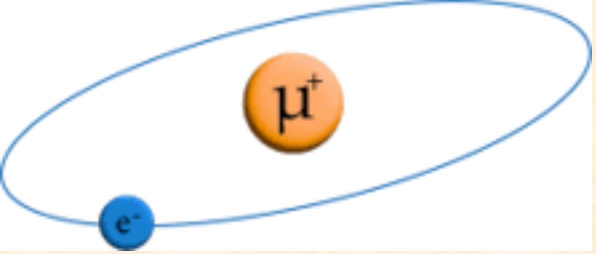

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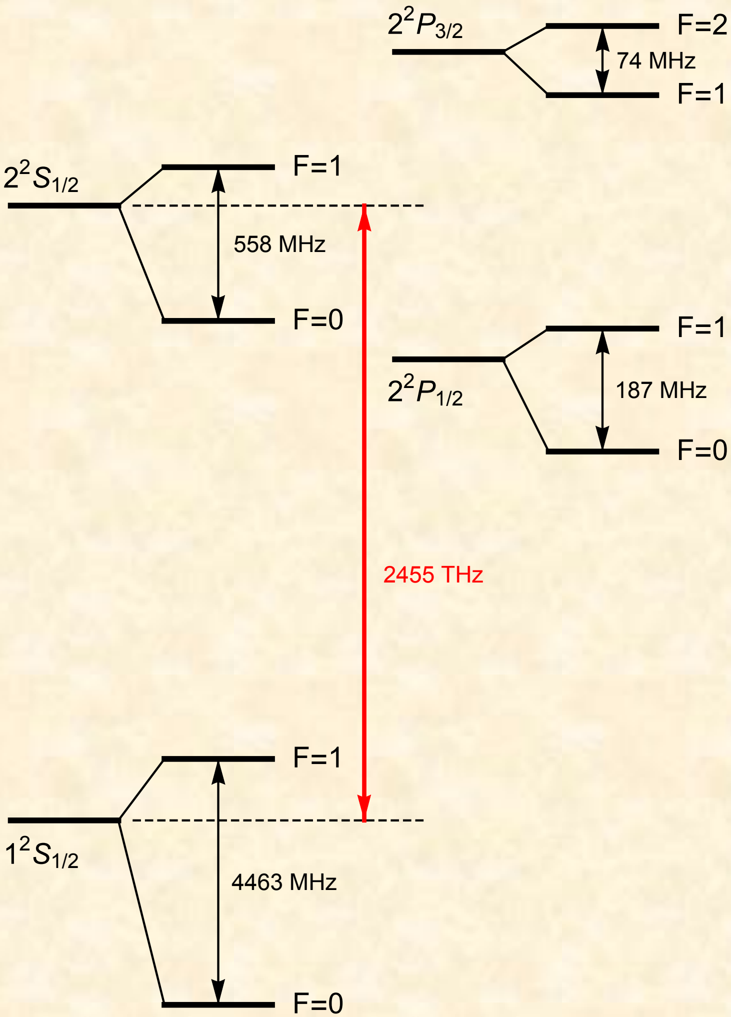

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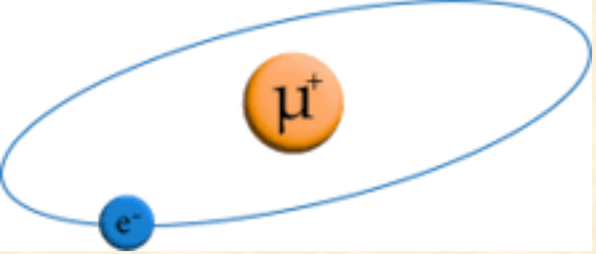
The present Mu-MASS goal is for an uncertainty of ±10 kHz (4 ppt)
I. Cortinovis et al., Eur. Phys. J. D 77, 66 (2023).



Muonium Spectrum: 1S-2S Interval

$$m_e\alpha^6\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right) = 88.9\,\mathrm{kHz}\,,$$

$$m_e\alpha^6\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right)^2 = 0.43\,\mathrm{kHz}\,,$$

$$m_e\alpha^7\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right) = 0.65\,\mathrm{kHz}$$

$$\ln\left(\frac{1}{\alpha}\right) = 4.92 \quad , \quad \ln\left(\frac{m_\mu}{m_e}\right) = 5.33$$

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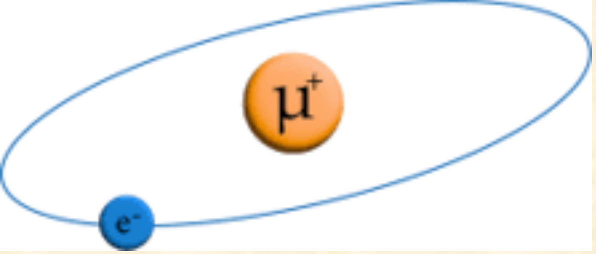
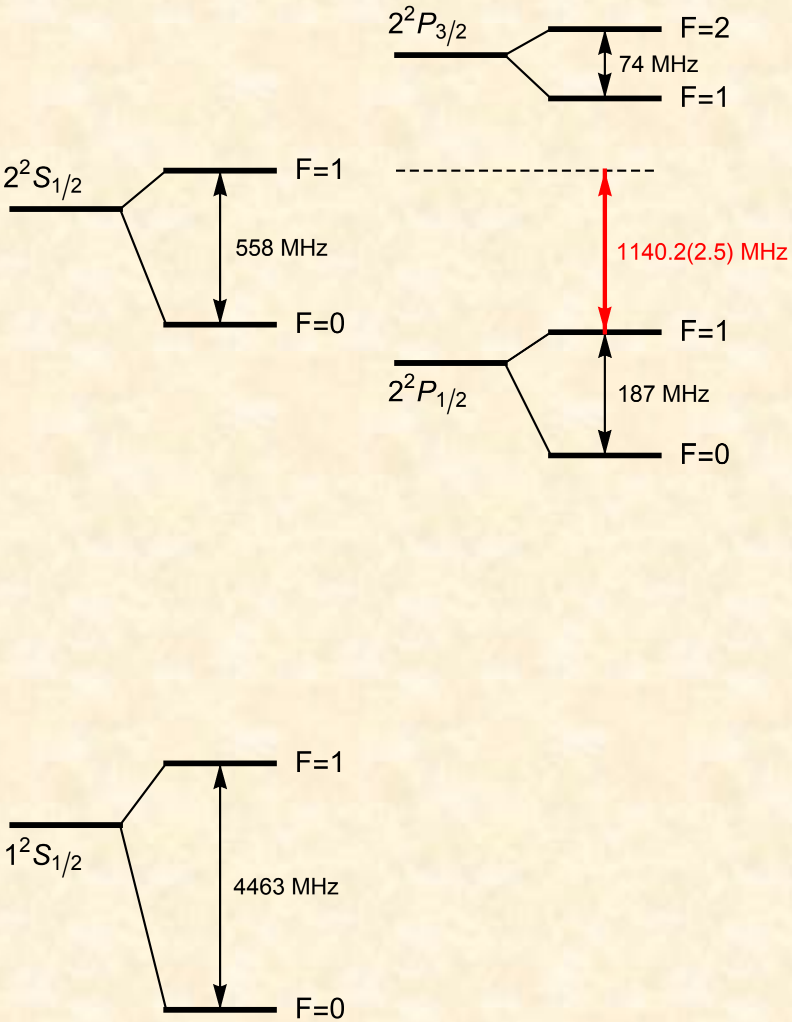
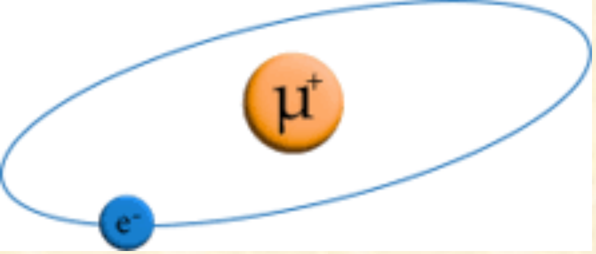


Muonium Spectrum: Lamb Shift

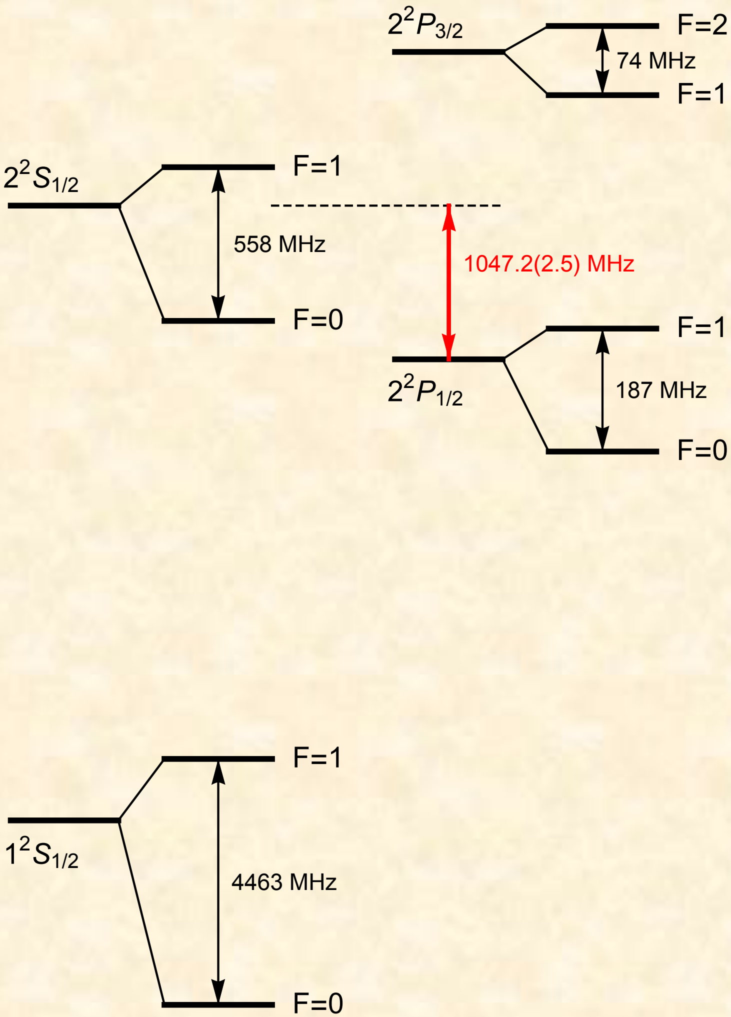


A recent measurement of one of the n=2 transitions, combined with hfs values, allows the determination of the Lamb Shift.

B. Ohayon et al., Phys. Rev. Lett. 128, 011802 (2022)



Muonium Spectrum:
Lamb Shift



A recent measurement of one of the n=2 transitions, combined with hfs values, allows the determination of the Lamb Shift.

$$\Delta E = 1047.2(2.5)\,\mathrm{MHz}$$

B. Ohayon et al., Phys. Rev. Lett. 128, 011802 (2022)

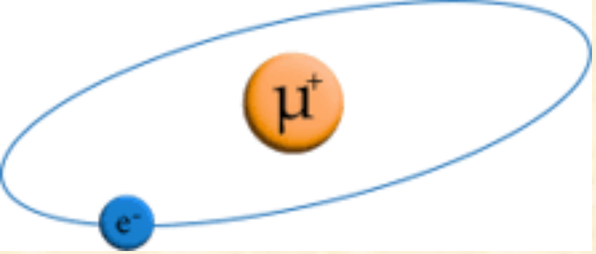
Theoretical prediction:

$$\Delta E = 1047.498(1)\,\mathrm{MHz}$$

Janka, Ohayon, Crivelli, EPJ Conf. 262, 01001 (2022)

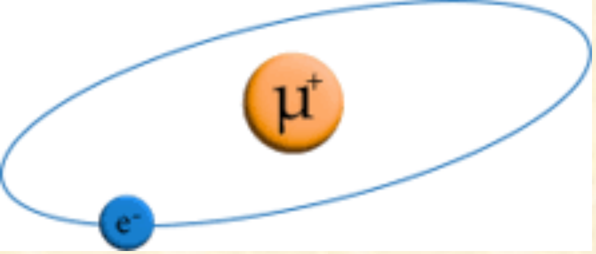
$$\Delta E = 1047.284(2)\,\mathrm{MHz}$$

Frugiuele, Pérez-Ríos, Peset, Phys. Rev. D 100, 015010 (2019)



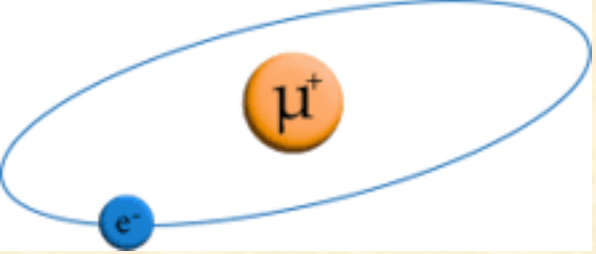
Method of Calculation

1. Use Non-Relativistic QED (NRQED) and dimensional regularization
2. Obtain all required matching coefficients. (Finding the contact term matching coefficients is an essential part of the recoil calculation, and is a significant challenge.)
3. Describe two-body bound states using the NRQED Bethe-Salpeter equation. Energies appear as poles in the Green function.
4. Build a perturbation scheme based on an exact lowest-order solution to the NRQED Bethe-Salpeter equation
5. Use the "method of regions" to identify contributions at various powers of the expansion parameter α
6. Express all contributions in terms of expectation values of various operators in states of the D-dimensional non-relativistic Schrödinger-Coulomb equation



Method of Calculation

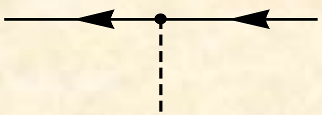
For our calculation of recoil and radiative-recoil corrections at order α^6, there were two main classes of terms: hard (relativistic) and soft (non-relativisitic). The recoil correction has contributions of both types, but the radiative-recoil correction was from the hard region only. Contributions coming from both hard and soft regions have the possiblilty of $\ln(1/\alpha)$ factors. Contributions involving the hard region have the possibility of $\ln(m_\mu/m_e)$ factors.



NRQED Feynman Rules

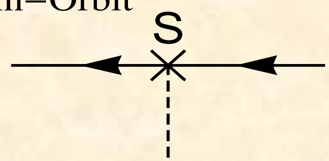
Interaction Vertices:

Coulomb



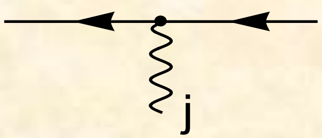
$-\mathrm{iq}$

Spin–Orbit



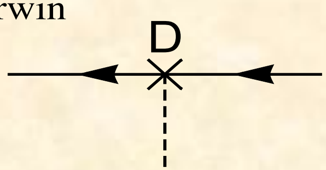
$c_S \frac{q}{4\,m^2}(\vec{p}_2 \times \vec{p}_1)\cdot\vec{\sigma}$

Convection



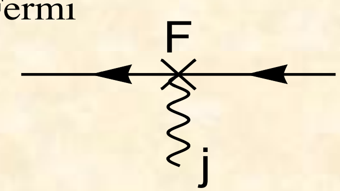
$\frac{\mathrm{iq}}{2\,m}(p_2+p_1)_j$

Darwin



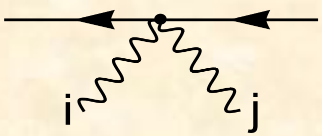
$c_D \frac{\mathrm{iq}}{8\,m^2}(\vec{p}_2-\vec{p}_1)^2$

Fermi



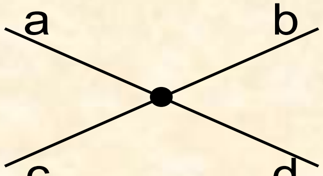
$c_F \frac{q}{2\,m}[(\vec{p}_2-\vec{p}_1)\times\vec{\sigma}]_j$

Seagull



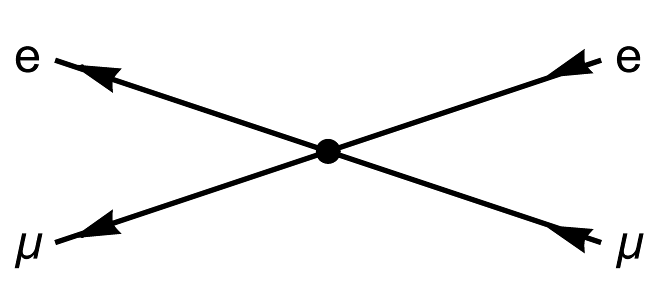
$\frac{-\mathrm{iq}^2}{m}\delta_{\mathrm{ij}}$

Contact



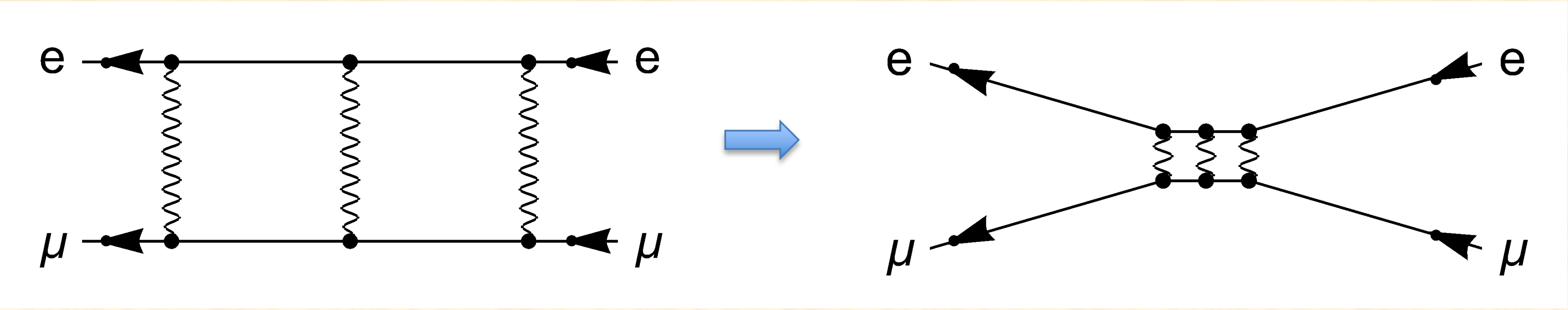
$\frac{d_s}{m^2}\delta_{\mathrm{ab}}\delta_{\mathrm{cd}} + \frac{d_v}{m^2}\vec{\sigma}_{\mathrm{ab}}\cdot\vec{\sigma}_{\mathrm{cd}}$

$+\cdots$



The NRQED contact term contains the contributions of all photon-exchange diagrams (possibly including radiative corrections) containing purely relativistic momenta.

The idea is that the space-time size of a relativistic process is small on an atomic scale.



The NRQED Contact Term

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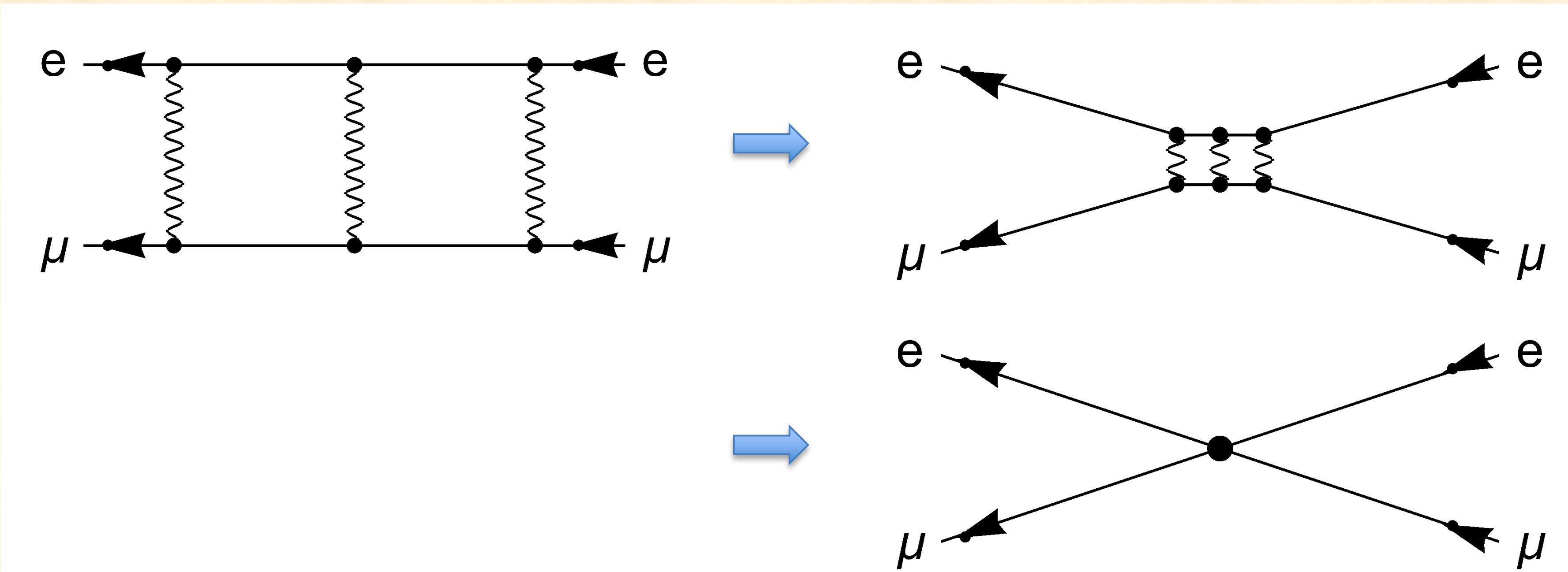
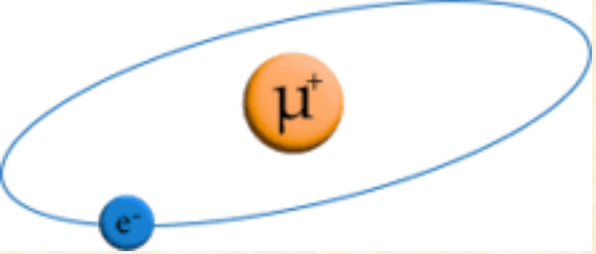
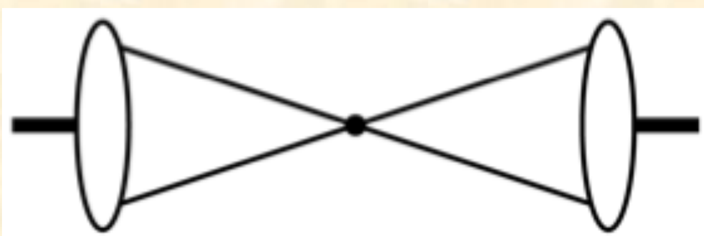
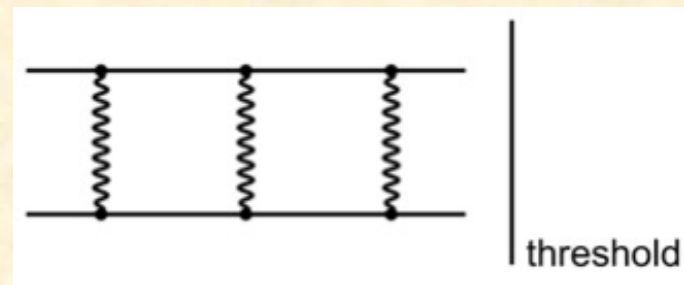
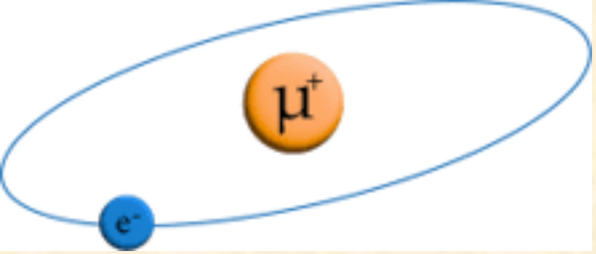




Energy Correction due to the Contact Term

The contact term contributes to an energy shift in NRQED by first order perturbation theory. The contact term matching coefficients are calculated from QED by taking the threshold limit of graphs where all loop momenta are hard (*i.e.* relativistic). Because the contact term has all particles meeting at a point, the energy shift is proportional to the square of the wave function at contact (*i.e.* at zero relative displacement).

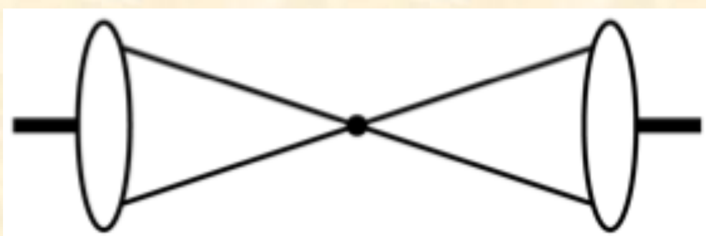
$\Delta E \propto$  $\propto \; |\psi(0)|^2$ 



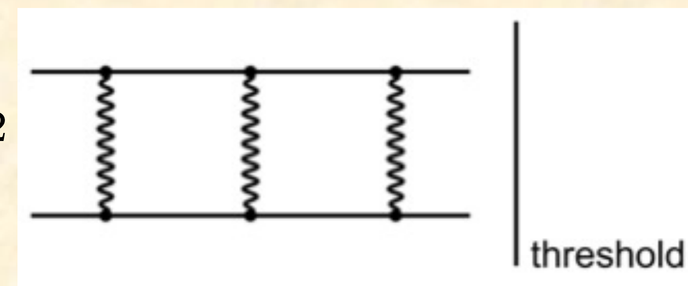
Energy Correction due to the Contact Term

The contact term contributes to an energy shift in NRQED by first order perturbation theory. The contact term matching coefficients are calculated from QED by taking the threshold limit of graphs where all loop momenta are hard (*i.e.* relativistic). Because the contact term has all particles meeting at a point, the energy shift is proportional to the square of the wave function at contact (*i.e.* at zero relative displacement).

$\Delta E \propto$

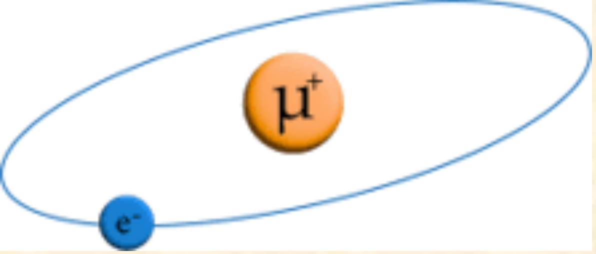


$\propto \; |\psi(0)|^2$

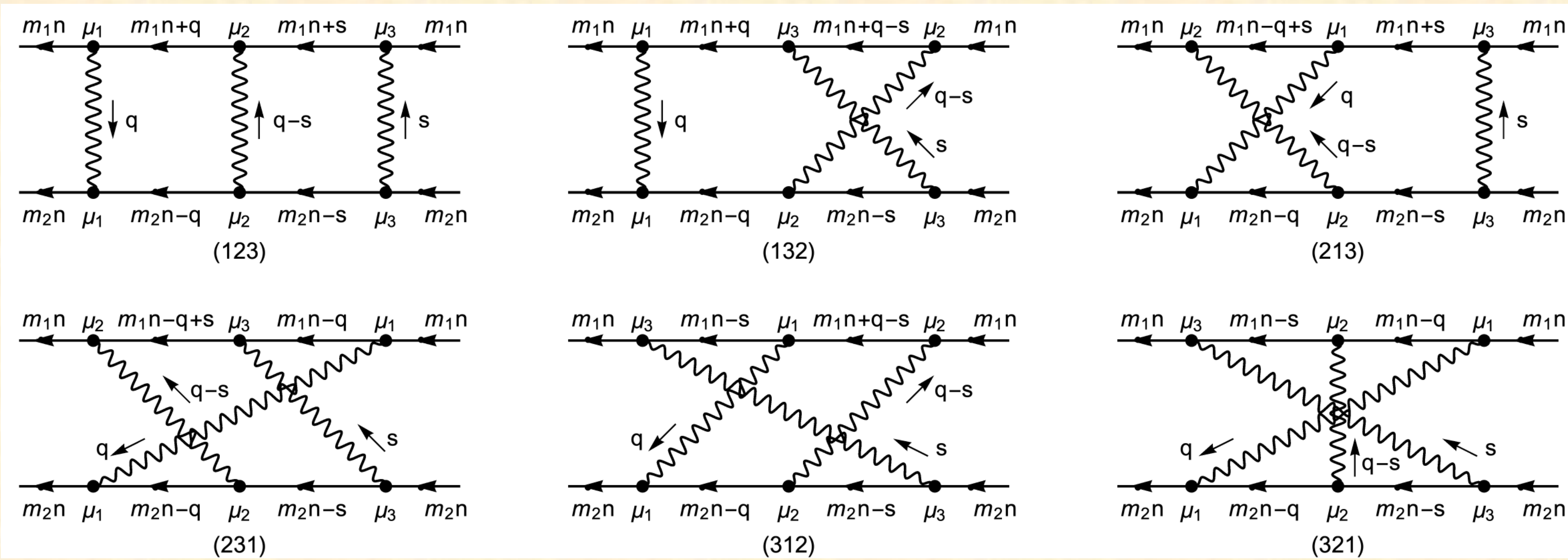


$$\Delta E = -|\psi(0)|^2 \mathcal{M} = -\frac{m_r^3 (Z\alpha)^3}{\pi n^3} \mathcal{M}$$

where $\mathcal{M}$ is the amplitude for hard corrections to QED threshold scattering



Recoil Diagrams at Order $(Z\alpha)^6$



Integration by Parts Reduction

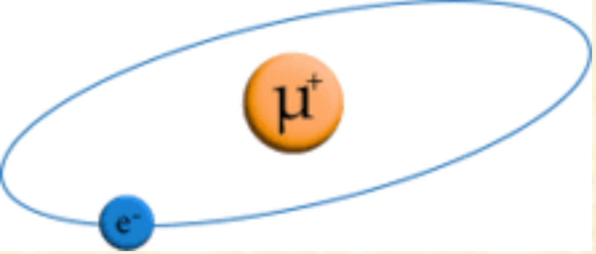
Integration by parts (IBP) identities are found using the fact that the integral of a divergence in d-dimensional space is zero.

$$0 = \int d^d q \, d^d s \, \frac{\partial}{\partial q^\mu} \left\{ \frac{v^\mu}{\text{dens}} \right\} = \int d^d q \, d^d s \, \frac{\partial}{\partial s^\mu} \left\{ \frac{v^\mu}{\text{dens}} \right\} \tag{.20}$$

where $v^\mu = q^\mu$, s^μ, or any external momentum vector. An example of the type of identity that is produced is

$$\begin{aligned} 0 &= \int d^d q \, d^d s \, \frac{\partial}{\partial q^\mu} \left\{ \frac{q^\mu}{(-q^2 + m_A^2)^\alpha (-(q-s)^2 + m_B^2)^\beta \cdots} \right\} \\ &= \int d^d q \, d^d s \left\{ \frac{d}{(\,)^\alpha (\,)^\beta \ldots} + \frac{2\alpha q^2}{(\,)^{\alpha+1} (\,)^\beta \ldots} + \frac{2\beta q \cdot (q-s)}{(\,)^\alpha (\,)^{\beta+1} \ldots} + \cdots \right\} \end{aligned} \tag{.21}$$

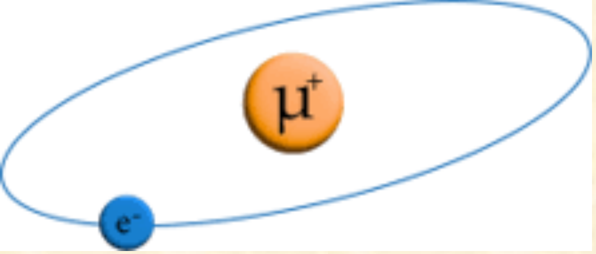
Many such identities are generated (using computer assistance) and are used to reduce a complicated integral containing very many terms to a linear combination of a few "master integrals". The relatively small set of master integrals are the only ones that must actually be evaluated.



Master Integrals Needed for the Recoil Correction

$$
\begin{aligned}
M_0(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^3 s \, \frac{1}{(-q^2)(-q^2 \pm 2m_1 qn)(-s^2 \pm 2m_2 sn)} \,, \\
M_1(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{1}{(-q^2 \pm 2m_1 qn)(-s^2 \pm 2m_2 sn)} \,, \\
M_2(m) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{1}{(-q^2)(-(q-s)^2)(-s^2 \pm 2msn)} \,, \\
M_3(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{1}{(-(q-s)^2)(-q^2 - 2m_1 qn)(-s^2 + 2m_2 sn)} \,, \\
M_4(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{(-q^2)}{(-(q-s)^2)(-q^2 - 2m_1 qn)(-s^2 + 2m_2 sn)} \,, \\
M_5(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{1}{(-(q-s)^2)(-q^2 + 2m_1 qn)(-s^2 + 2m_2 sn)} \,, \\
M_6(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{(-q^2)}{(-(q-s)^2)(-q^2 + 2m_1 qn)(-s^2 + 2m_2 sn)} \,, \\
M_7(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{1}{(-(q-s)^2 + 2m_2(q-s)n)(-q^2 - 2m_1 qn)(-s^2 - 2m_1 sn)} \,, \\
M_8(m_1,m_2) &= \Phi^2 \int \text{đ}^d q\, \text{đ}^d s \, \frac{(-q^2)}{(-(q-s)^2 + 2m_2(q-s)n)(-q^2 - 2m_1 qn)(-s^2 - 2m_1 sn)} \,.
\end{aligned}
$$

We use $d = 4 - 2\epsilon$. The factors $\Phi = -i(2\pi)^{2-\epsilon} e^{\epsilon\gamma_E}$ are included for convenience.



Master Integrals Needed for the Recoil Correction

Some of the master integrals can be done exactly:

$$
\begin{aligned}
M_0(m_1, m_2) &= m_1^{-2\epsilon} m_2^{2-2\epsilon} e^{2\epsilon\gamma_E} B(1, 1-2\epsilon)\Gamma(\epsilon)\Gamma(-1+\epsilon)\,, \\
M_1(m_1, m_2) &= (m_1 m_2)^{2(1-\epsilon)} e^{2\epsilon\gamma_E} \Gamma^2(-1+\epsilon)\,, \\
M_2(m) &= m^{2(1-2\epsilon)} e^{2\epsilon\gamma_E} B(1-\epsilon, 1-\epsilon) B(\epsilon, 3-4\epsilon)\Gamma(-1+2\epsilon)\,.
\end{aligned}
$$

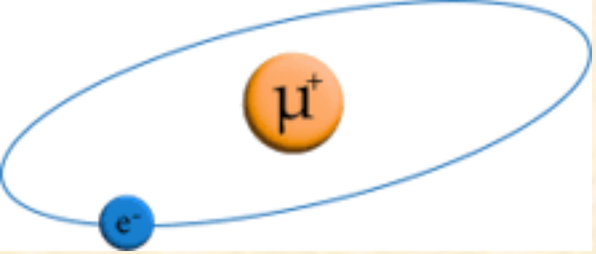
Master Integrals Needed for the Recoil Correction

But others are not so easily done. We note that each depends on two masses: m_1 and m_2. We can scale one mass (we choose m_2) out, leaving integrals that depend only on the mass ratio $x= m_1 / m_2$. We first consider two of the master integrals:

$$
\begin{aligned}
M_3(m_1,m_2) &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1}{(-(q-s)^2)(-q^2-2m_1 qn)(-s^2+2m_2 sn)} \,, \\
M_4(m_1,m_2) &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{(-q^2)}{(-(q-s)^2)(-q^2-2m_1 qn)(-s^2+2m_2 sn)} \,,
\end{aligned}
$$

We scale out m_2, leaving

$$
\begin{aligned}
M_3(m_1,m_2) &= m_2^{2-4\epsilon} J_2(x) \\
M_4(m_1,m_2) &= m_2^{4-4\epsilon} J_3(x)
\end{aligned}
$$

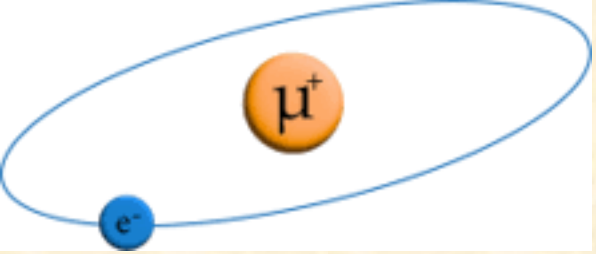


Method of Differential Equations

We focus on the integrals $J_1(x)$, $J_2(x)$, and $J_3(x)$, where $J_1(x)$ is an auxiliary integral that has been included to make a set that is closed under differentiation. We can get the value of $J_1(x)$ exactly. The $J_i(x)$ integrals are:

$$
\begin{aligned}
J_1(x) &\equiv \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1}{(-q^2 - 2xqn)(-s^2 + 2sn)} \, , \\
J_2(x) &\equiv \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1}{(-(q-s)^2)(-q^2 - 2xqn)(-s^2 + 2sn)} \, , \\
J_3(x) &\equiv \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{(-q^2)}{(-(q-s)^2)(-q^2 - 2xqn)(-s^2 + 2sn)} \, .
\end{aligned}
$$

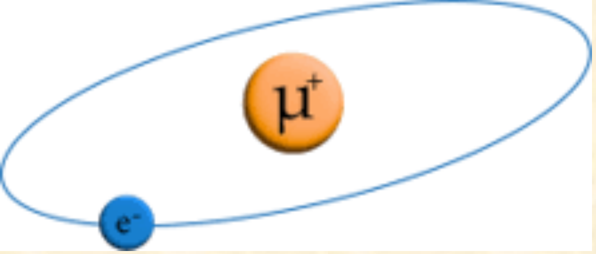
$$
J_1(x) = M_1(x,1) = x^{2(1-\epsilon)} e^{2\epsilon\gamma_E} \Gamma^2(-1+\epsilon)
$$



Method of Differential Equations

The x derivatives of the $J_i(x)$ integrals are easily found, and then the IBP identities were used to write the derivative in terms of $J_i(x)$:

$$
\begin{aligned}
\frac{dJ_1(x)}{dx} &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{2qn}{(-q^2 - 2xqn)^2(-s^2 + 2sn)} \\
&= \frac{2(1-\epsilon)}{x} J_1(x) \, , \\
\frac{dJ_2(x)}{dx} &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{2qn}{(-(q-s)^2)(-q^2 - 2xqn)^2(-s^2 + 2sn)} \\
&= \frac{(1-\epsilon)}{x(1+x)(1+2x)} J_1(x) + \frac{3 - 4\epsilon - 2\epsilon x - 2x^2}{x(1+x)(1+2x)} J_2(x) + \frac{3(-1+\epsilon)}{x^2(1+2x)} J_3(x) \, , \\
\frac{dJ_3(x)}{dx} &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{(-q^2)(2qn)}{(-(q-s)^2)(-q^2 - 2xqn)^2(-s^2 + 2sn)} \\
&= \frac{4(-1+\epsilon)x}{(1+x)(1+2x)} J_1(x) + \frac{2(-5+6\epsilon)x}{(1+x)(1+2x)} J_2(x) + \frac{1 + 2(6-5\epsilon)x + 8(1-\epsilon)x^2}{x(1+x)(1+2x)} J_3(x) \, .
\end{aligned}
$$



The set of coupled differential equations can be put in matrix form. This is a set of first order, ordinary, linear, homogeneous coupled differential equations. It is challenging to solve only because the coefficient matrix **A** depends on the independent variable x.

$$\frac{d}{dx}\vec{J} = \mathbf{A}\vec{J}$$

$$\vec{J} = \begin{pmatrix} J_1(x) \\ J_2(x) \\ J_3(x) \end{pmatrix} \quad , \quad \mathbf{A} = \begin{pmatrix} \frac{2(1-\epsilon)}{x} & 0 & 0 \\ \frac{(1-\epsilon)}{x(1+x)(1+2x)} & \frac{3-4\epsilon-2\epsilon x-2x^2}{x(1+x)(1+2x)} & \frac{3(-1+\epsilon)}{x^2(1+2x)} \\ \frac{4(-1+\epsilon)x}{(1+x)(1+2x)} & \frac{2(-5+6\epsilon)x}{(1+x)(1+2x)} & \frac{1+2(6-5\epsilon)x+8(1-\epsilon)x^2}{x(1+x)(1+2x)} \end{pmatrix}$$

Method of Differential Equations

As they stand, the differential equations for $J_1(x)$, $J_2(x)$ and $J_3(x)$ are challenging to solve. The trick is to make a change in dependent variable. The original differential equations are

$$\frac{d}{dx}\vec{J} = \mathbf{A}\vec{J}$$

then with the change of variable

$$\vec{J} = T\vec{I}$$

the differential equation becomes $\frac{d}{dx}\vec{I} = M\vec{I}$ with the new coefficient matrix M in "epsilon form":

$$M = T^{-1}AT - T^{-1}\frac{dT}{dx} = \epsilon \times (\text{function of } x \text{ only})$$

For that case at hand, we find

$$\frac{d}{dx}\vec{I} = \epsilon\left(\frac{a}{x} + \frac{b}{1+x}\right)\vec{I},$$

$$a = \begin{pmatrix} -2 & 0 & 0 \\ \frac{1413693}{146300} & -\frac{31857}{110} & \frac{32841}{220} \\ \frac{1393481}{73150} & -\frac{91427}{165} & \frac{31417}{110} \end{pmatrix} \quad , \quad b = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{32643}{4180} & \frac{1877}{22} & -\frac{1989}{44} \\ -\frac{227509}{14630} & \frac{5311}{33} & -\frac{1877}{22} \end{pmatrix}$$

Method of Differential Equations

We assume the existence of a perturbative solution

$$\vec{I} = \vec{I}^{(0)} + \epsilon \vec{I}^{(1)} + \epsilon^2 \vec{I}^{(2)} + \cdots$$

where the $I^{(n)}$ are ϵ-independent function of x, and plug into the differential equation:

$$\frac{d}{dx}\left\{\vec{I}^{(0)} + \epsilon \vec{I}^{(1)} + \epsilon^2 \vec{I}^{(2)} + \cdots\right\} = \epsilon \left(\frac{a}{x} + \frac{b}{1+x}\right)\left\{\vec{I}^{(0)} + \epsilon \vec{I}^{(1)} + \epsilon^2 \vec{I}^{(2)} + \cdots\right\}$$

By considering the various orders individually, we find

$$\begin{aligned}
\frac{d}{dx}\vec{I}^{(0)} &= 0\,, \\
\frac{d}{dx}\vec{I}^{(1)} &= \left(\frac{a}{x} + \frac{b}{1+x}\right) I^{(0)}\,, \\
\frac{d}{dx}\vec{I}^{(2)} &= \left(\frac{a}{x} + \frac{b}{1+x}\right) I^{(1)}\,, \\
\frac{d}{dx}\vec{I}^{(3)} &= \left(\frac{a}{x} + \frac{b}{1+x}\right) I^{(2)}\,, \\
&\cdots
\end{aligned}$$

Method of Differential Equations

At $O(\epsilon^0)$ one has

$$\frac{d}{dx}\vec{I}^{(0)} = 0$$

so that

$$\vec{I}^{(0)} = \vec{h}^{(0)} = \left(h_1^{(0)}, h_2^{(0)}, h_3^{(0)}\right)^{\mathrm{T}}, \text{ a constant vector}$$

Method of Differential Equations

At $O(\epsilon^0)$ one has

$$\frac{d}{dx}\vec{I}^{(0)} = 0$$

so that

$$\vec{I}^{(0)} = \vec{h}^{(0)} = \left(h_1^{(0)}, h_2^{(0)}, h_3^{(0)}\right)^{\mathrm{T}}, \text{ a constant vector}$$

At $O(\epsilon^1)$ one has

$$\frac{d}{dx}\vec{I}^{(1)} = \left\{\frac{1}{x}\,a.\vec{I}^{(0)} + \frac{1}{1+x}\,b.\vec{I}^{(0)}\right\}$$

so that

$$\begin{aligned}\vec{I}^{(1)} &= \int dx \left\{\frac{1}{x}a.\vec{I}^{(0)} + \frac{1}{1+x}b.\vec{I}^{(0)}\right\} \\ &= \mathrm{HPL}(0;x)\,a.\vec{h}^{(0)} + \mathrm{HPL}(-1;x)\,b.\vec{h}^{(0)} + \vec{h}^{(1)}\end{aligned}$$

Method of Differential Equations

The "harmonic polylogarithm" functions HPL$(a;x)$ are defined as a set of iterated integrals, starting with

$$\mathrm{HPL}(1;x) \equiv \int_0^x \frac{dt}{1-t} = -\ln(1-x) \quad , \quad \mathrm{HPL}(0;x) \equiv \ln x \quad , \quad \mathrm{HPL}(-1;x) \equiv \int_0^x \frac{dt}{1+t} = \ln(1+x)$$

and in general

$$\mathrm{HPL}(0,\cdots 0;x) \equiv \frac{1}{n!}\ln^n x \quad \text{for the HPL with first argument consisting of n zeros}$$

and

$$\mathrm{HPL}(a,a_1,\cdots,a_k;x) = \int_0^x dt\, f_a(t)\,\mathrm{HPL}(a_1,\cdots,a_k;t)$$

with

$$f_1(x) = \frac{1}{1-x} \quad , \quad f_0(x) = \frac{1}{x} \quad , \quad f_{-1}(x) = \frac{1}{1+x}$$

E. Remiddi and J. A. M. Vermaseran, Int. J. Mod. Phys. A **15**, 725 (2000)
T. Gehrmann and E. Remiddi, Comp. Phys. Commun. **141**, 296 (2001)
D. Maître, Comp. Phys. Commun. **174**, 222 (2006)

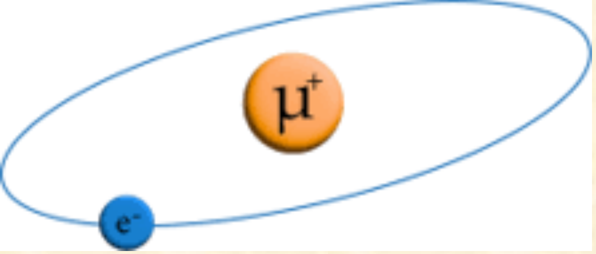
Method of Differential Equations

At $O(\epsilon^2)$ one has

$$\frac{d}{dx}\vec{I}^{(2)} = \left\{ \frac{1}{x}\, a.\vec{I}^{(1)} + \frac{1}{1+x}\, b.\vec{I}^{(1)} \right\}$$

so that

$$\begin{aligned}
\vec{I}^{(2)} &= \int dx \left\{ \frac{1}{x} a.\vec{I}^{(1)} + \frac{1}{1+x} b.\vec{I}^{(1)} \right\} \\
&= \mathrm{HPL}(0,0;x)\, a.a.\vec{h}^{(0)} + \mathrm{HPL}(0,-1;x)\, a.b.\vec{h}^{(0)} \\
&+ \mathrm{HPL}(-1,0;x)\, b.a.\vec{h}^{(0)} + \mathrm{HPL}(-1,-1;x)\, b.b.\vec{h}^{(0)} \\
&+ \mathrm{HPL}(0;x)\, a.\vec{h}^{(1)} + \mathrm{HPL}(-1,x)\, b.\vec{h}^{(1)} \\
&+ \vec{h}^{(2)}
\end{aligned}$$



Method of Differential Equations

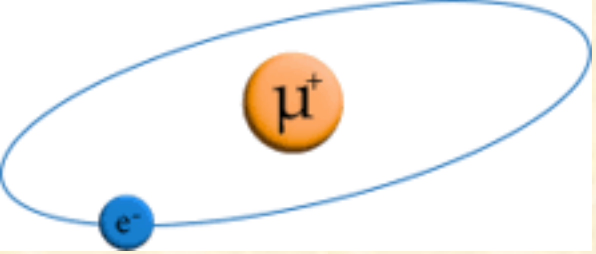
At $O(\epsilon^3)$ one has

$$\frac{d}{dx}\vec{I}^{(3)} = \left\{ \frac{1}{x}\, a.\vec{I}^{(2)} + \frac{1}{1+x}\, b.\vec{I}^{(2)} \right\}$$

so that

$$\begin{aligned}
\vec{I}^{(3)} &= \int dx \left\{ \frac{1}{x} a.\vec{I}^{(2)} + \frac{1}{1+x} b.\vec{I}^{(2)} \right\} \\
&= \mathrm{HPL}(0,0,0;x)\, a.a.a.\vec{h}^{(0)} + 2^3 \text{ terms total involving } \vec{h}^{(0)} \\
&+ \mathrm{HPL}(0,0;x)\, a.a.\vec{h}^{(1)} + 2^2 \text{ terms total involving } \vec{h}^{(1)} \\
&+ \mathrm{HPL}(0;x)\, a.\vec{h}^{(2)} + \mathrm{HPL}(-1,x)\, b.\vec{h}^{(2)} \\
&+ \vec{h}^{(3)}
\end{aligned}$$

So finally $\vec{I} = \vec{I}^{(0)} + \epsilon \vec{I}^{(1)} + \epsilon^2 \vec{I}^{(2)} + \epsilon^3 \vec{I}^{(3)} + \cdots$



Method of Differential Equations

The integration constants are found by computing the expansion of the integrals about the point x=0 using the method of regions. The integrals are much simpler when x=0 that for non-zero values of x.

$$J_2(x) = \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1}{(-(q-s)^2)(-s^2+2sn)(-q^2-2xqn)}.$$

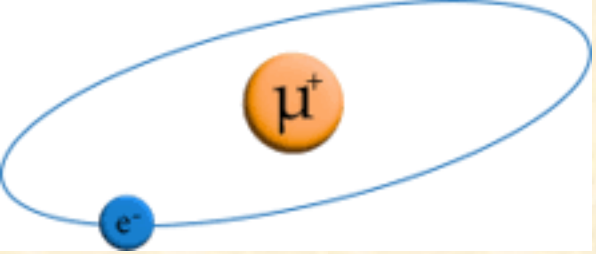
In Region 1, $q \sim 1$, $s \sim 1$, so the expansion looks like

$$\begin{aligned} J_{21}(x) &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1}{(-(q-s)^2)(-s^2+2sn)(-q^2)\left(1 - x\left(\frac{2qn}{-q^2}\right)\right)} \\ &= \Phi^2 \int \bar{d}^d q \bar{d}^d s \, \frac{1 + x\left(\frac{2qn}{-q^2}\right) + x^2\left(\frac{2qn}{-q^2}\right)^2 + x^3\left(\frac{2qn}{-q^2}\right)^3 + O(x^4)}{(-(q-s)^2)(-s^2+2sn)(-q^2)} \\ &= J_{210} + xJ_{211} + x^2 J_{212} + x^3 J_{213} + O(x^4). \end{aligned}$$

There are also contribution from regions where $q \sim x$, $s \sim 1$ and $q \sim x$, $s \sim x$.

We need the expansion out to $O(\epsilon)$. The complete result is a bit lengthy, but the first two terms are

$$J_2(x) = \frac{-1 - x^2 + O(x^4)}{2\epsilon^2} + \frac{-5 + 2x - 5x^2 + 8x^2 \ln x + O(x^4)}{4\epsilon} + O(\epsilon^0)\,.$$

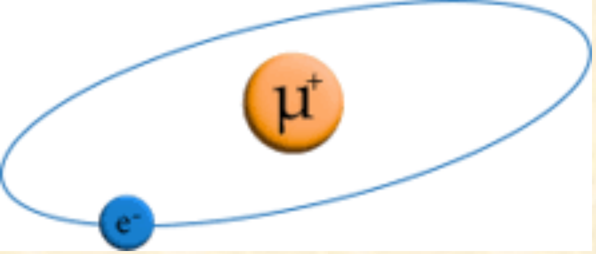


Recoil Correction to Muonium Energies

$$\Delta E_{\rm hfs} = \frac{8m_r^3(Z\alpha)^6}{3m_1m_2n^3}\left\{\left(\frac{11}{6}+\frac{3}{2n}-\frac{11}{6n^2}+\frac{3}{8}h_{\rm hfs}(x)\right)\right.$$
$$\left.+\frac{m_r^2}{m_1m_2}\left(2\ln\left(\frac{n}{2Z\alpha}\right)-2{\rm H}_n+\frac{31}{36}+\frac{3}{n}+\frac{4}{3n^2}\right)\right\}$$

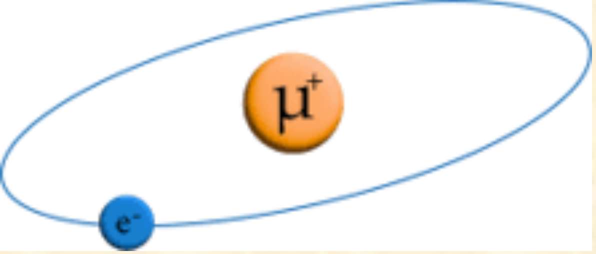
$$\Delta E_{\rm avg} = \frac{m_r(Z\alpha)^6}{n^3}\left\{\left(-\frac{1}{8}-\frac{3}{8n}+\frac{3}{4n^2}-\frac{5}{16n^3}\right)\right.$$
$$+\frac{m_r^2}{m_1m_2}\left(-\frac{1}{4n^2}+\frac{3}{16n^3}+h_{\rm avg}(x)\right)$$
$$\left.+\frac{m_r^4}{m_1^2m_2^2}\left(-\frac{4}{9}-\frac{2}{n}-\frac{1}{3n^2}-\frac{1}{16n^3}\right)\right\}$$

where H_n is the $n^{\rm th}$ harmonic number $H_n = \sum_{i=1}^{n}\frac{1}{i}$



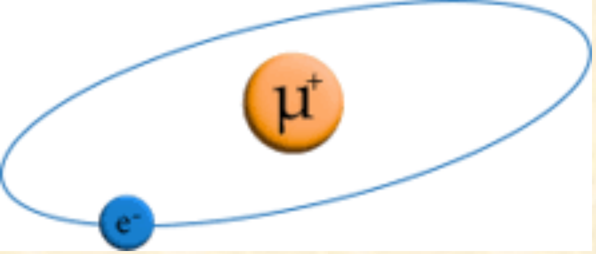
Recoil Correction to Muonium Energies: Hyperfine Splitting

$$
\begin{aligned}
h_{\rm hfs}(x) \;=\; & \frac{1}{9\pi^2(1-x^2)^2}\Big\{2\pi^2 x\left(x^2-29\right)(x-1) \\
&+144\pi^2 x\left(x^2-1\right)\log(2) - 72x^2\left(3x^2-11\right)\zeta(3) \\
&+8\pi^2x^2\left(x^2+3\right)\log(x) + 12x^2\left(x^2-3\right)\log^2(x) \\
&+12\pi^2(x+1)^3(x-1)\mathrm{HPL}(\{1\},x) \\
&+12\pi^2\left(x-1\right)^2\left(x^2-3\right)\mathrm{HPL}(\{-1\},x) \\
&+12\left(x^2-1\right)\left(x^2-10x+1\right)\mathrm{HPL}(\{1,0\},x) \\
&-12\left(x^2-1\right)\left(x^2+10x+1\right)\mathrm{HPL}(\{-1,0\},x) \\
&-48(x+1)^2\left(x^2+2x-2\right)\mathrm{HPL}(\{1,0,0\},x) \\
&+48(x-1)^2\left(x^2-2x-2\right)\mathrm{HPL}(\{-1,0,0\},x) \\
&-48(x+1)\left(2x^3-6x^2+3x-1\right)\mathrm{HPL}(\{2,0\},x) \\
&+48(x-1)\left(2x^3+6x^2+3x+1\right)\mathrm{HPL}(\{-2,0\},x) \\
&+144(x+1)(x-1)^3\mathrm{HPL}(\{-1,1,0\},x) \\
&+144(x-1)(x+1)^3\mathrm{HPL}(\{1,-1,0\},x)\Big\}
\end{aligned}
$$



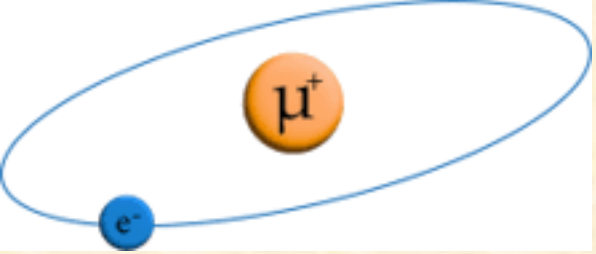
Recoil Correction to Muonium Energies: Average Energy

$$
\begin{aligned}
h_{\rm avg}(x) \;=\; & \frac{1}{18\pi^2 x(1-x^2)^2}\Big\{ -72\pi^2 x\left(x^2-1\right)\log(2) \\
& +3\pi^2 x(x-1)\left(3x^3+19x^2-5x+9\right) \\
& -108x^2\left(x^2-2\right)\left(x^2-1\right)\zeta(3) \\
& +12x^2\left(x^2-1\right)\left[\pi^2\left(x^2-4\right)-3\right]\log(x)+36x^4\log^2(x) \\
& +6\pi^2(x-1)(x+1)^3\left(x^2-3x+1\right)\mathrm{HPL}(\{1\},x) \\
& -6\pi^2(x-1)^2(x+1)\left(x^3-x^2+7x+5\right)\mathrm{HPL}(\{-1\},x) \\
& +36x\left(x^2-1\right)\left(2x^2+x+2\right)\mathrm{HPL}(\{1,0\},x) \\
& +36x\left(x^2-1\right)\left(2x^2-x+2\right)\mathrm{HPL}(\{-1,0\},x) \\
& +36(x-1)(x+1)^2\left(x^2+3x-2\right)\mathrm{HPL}(\{1,0,0\},x) \\
& +36(x+1)(x-1)^2\left(x^2-3x-2\right)\mathrm{HPL}(\{-1,0,0\},x) \\
& -36x(x+1)(x-1)^2\left(2x^2+3x-1\right)\mathrm{HPL}(\{2,0\},x) \\
& +36x(x-1)(x+1)^2\left(2x^2-3x-1\right)\mathrm{HPL}(\{-2,0\},x) \\
& +72(x+1)(x-1)^3\left(x^2+3x+1\right)\mathrm{HPL}(\{-1,1,0\},x) \\
& +72(x-1)(x+1)^3\left(x^2-3x+1\right)\mathrm{HPL}(\{1,-1,0\},x)\Big\}
\end{aligned}
$$



Recoil Correction to Muonium Energies

There are a few practical consequences of our calculation for muonium. For the hyperfine spitting, we have found an additional 11 Hz beyond what was previously known from the order x^2 recoil correction. This will be important when the improved experimental results are available. For the average shift, we have completed the calculation of soft contributions to the higher order recoil correction, reducing the theoretical uncertainty.



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Recoil Correction to Positronium Energies

As a bonus, our results also allow the positronium limit $x \to 1$ to be easily obtained :

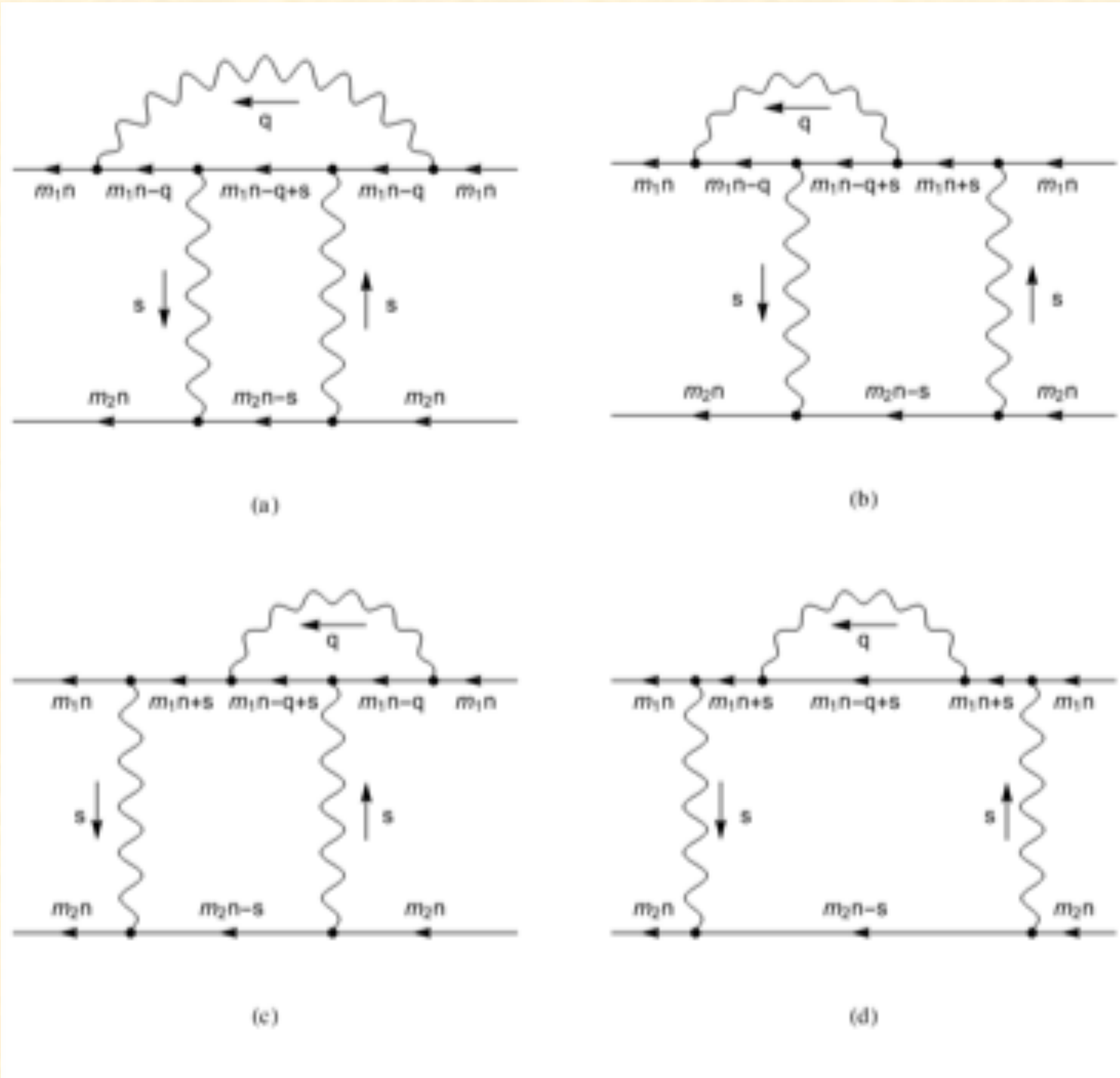
$$h_{\text{hfs}}(1) = \frac{1}{\pi^2}\left\{-17\zeta(3) - \frac{2}{3}\pi^2 \ln 2 + \frac{10}{3}\right\}$$

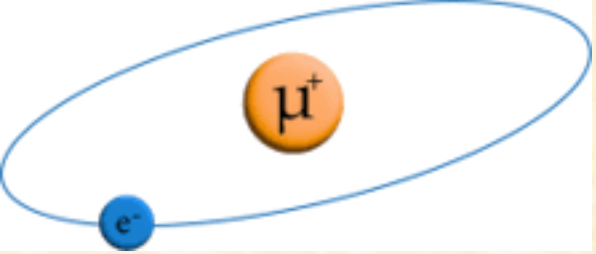
$$h_{\text{avg}}(1) = \frac{1}{\pi^2}\left\{-3\zeta(3) - \frac{13\pi^2}{24} - \frac{11}{2}\right\}$$

$$\begin{aligned}\Delta E_{\text{hfs}} &= \frac{m\alpha^6}{n^3}\left\{\frac{1}{6}\ln\left(\frac{1}{\alpha}\right) - \frac{17\zeta(3)}{8\pi^2} + \frac{5}{12\pi^2} - \frac{\ln 2}{4}\right. \\ &\quad \left. + \frac{295}{432} + \frac{1}{6}\ln n - \frac{1}{6}\mathrm{H}_n + \frac{3}{4n} - \frac{1}{2n^2}\right\}\end{aligned}$$

$$\begin{aligned}\Delta E_{\text{avg}} &= \frac{m\alpha^6}{n^3}\left\{-\frac{3\zeta(3)}{8\pi^2} - \frac{11}{16\pi^2} - \frac{83}{576}\right. \\ &\quad \left. - \frac{1}{4n} + \frac{1}{3n^2} - \frac{69}{512n^3}\right\}\end{aligned}$$

Radiative – Recoil Diagrams at Order $\alpha(\mathbf{Z}\alpha)^5$

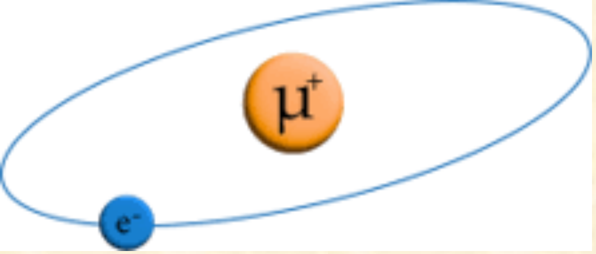




Radiative-Recoil Correction to Muonium hfs

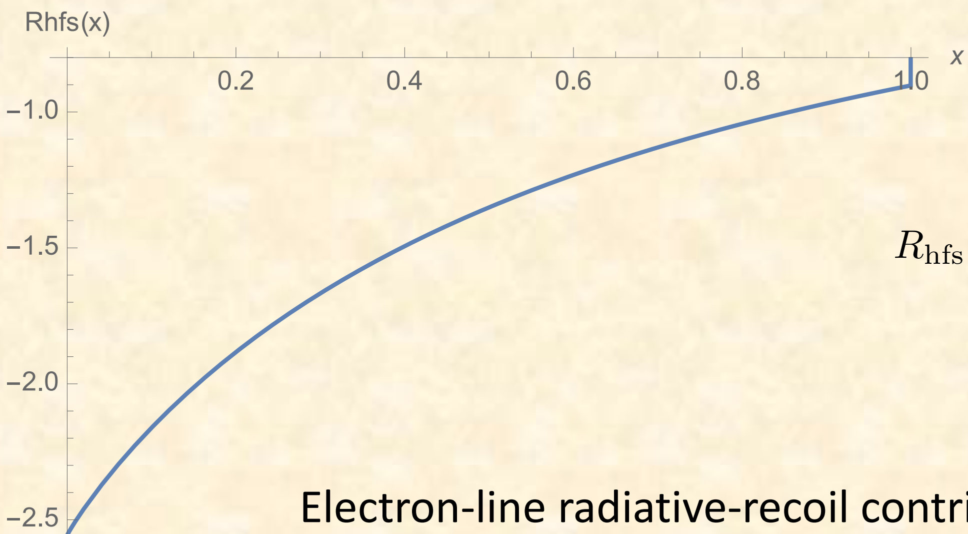
$$\Delta E_{\rm hfs} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha (Z\alpha)}{n^3} R_{\rm hfs} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$

$$\begin{aligned}
R_{\rm hfs}(x) \;=\; & \frac{1}{24\pi^2 x(1-x^2)^2} \Big\{ \pi^2 x \left(-78 + 14x + 159x^2 - 2x^3 - 99x^4 + 6x^5\right) \\
& +36x^2\left(4 - 5x^2 + x^4\right)\zeta(3) + 24\pi^2 x\left(1 + 3x - 8x^2 - 3x^3 + 7x^4\right)\log(2) \\
& +24x^2\left(-1 + x^2\right)\log(x) + 12\left(x^2 - x^4 + 3x^6\right)\log^2(x) \\
& +6(1-x)^2\left(3 - 5x - 33x^2 - 19x^3 + 6x^4\right)\mathrm{HPL}(\{1,0\},x) \\
& -6(1+x)^2\left(3 + 5x - 33x^2 + 19x^3 + 6x^4\right)\mathrm{HPL}(\{-1,0\},x) \\
& +x(1+x)^2\left(1 - 9x + 9x^2 - x^3\right)\Big[2\pi^2\mathrm{HPL}(\{1\},x) + 12\,\mathrm{HPL}(\{-2,0\},x) \\
& \qquad -12\,\mathrm{HPL}(\{1,0,0\},x) + 24\,\mathrm{HPL}(\{1,-1,0\},x)\Big] \\
& +x(1-x)^2\left(1 + 9x + 9x^2 + x^3\right)\Big[-4\pi^2\mathrm{HPL}(\{-1\},x) + 12\,\mathrm{HPL}(\{2,0\},x) \\
& \qquad -12\,\mathrm{HPL}(\{-1,0,0\},x) - 24\,\mathrm{HPL}(\{-1,1,0\},x)\Big]\Big\}
\end{aligned}$$



Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\rm hfs} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha(Z\alpha)}{n^3} R_{\rm hfs} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$

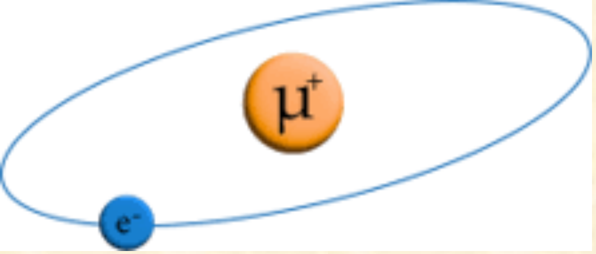


Positronium limiting case:

$$R_{\rm hfs}(1) = \frac{1}{\pi^2} \left\{ \frac{3}{4}\zeta(3) + 2\pi^2 \ln 2 - \frac{79\pi^2}{32} + \frac{7}{8} \right\}$$

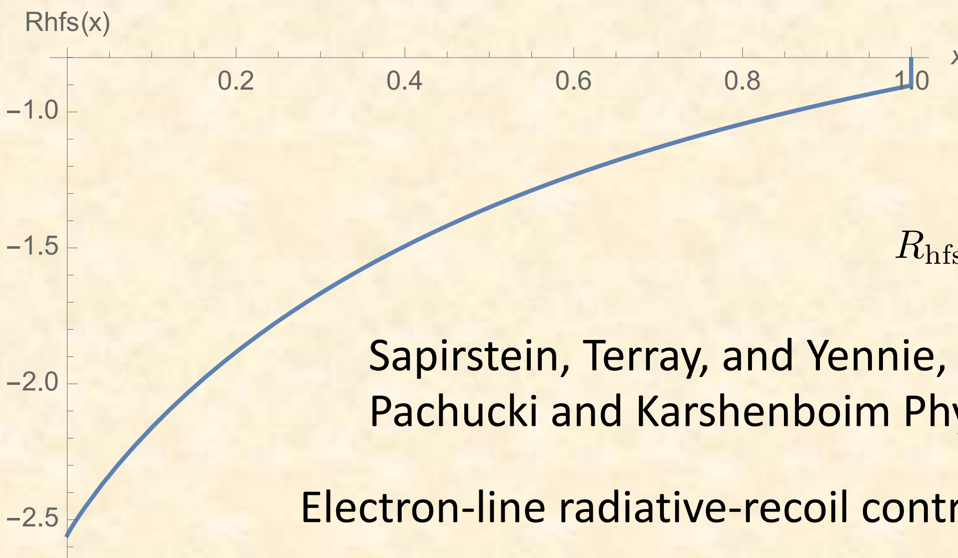
Electron-line radiative-recoil contribution for muonium:

$$R_{\rm hfs}(x) = \left(-\frac{13}{4} + \ln 2 \right) + \frac{x}{\pi^2} \left(-\frac{15}{4} \ln x + 6\zeta(3) + 3\pi^2 \ln 2 + \frac{\pi^2}{2} + \frac{17}{8} \right) + x^2 \left(-\frac{3}{2} - 6 \ln 2 \right) + O(x^3)$$



Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\rm hfs} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha(Z\alpha)}{n^3} R_{\rm hfs} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$



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Sapirstein, Terray, and Yennie, Phys. Rev. D 29, 2290 (1984) – numerical
Pachucki and Karshenboim Phys. Rev. Lett. 80, 2101 (1998) – analytical

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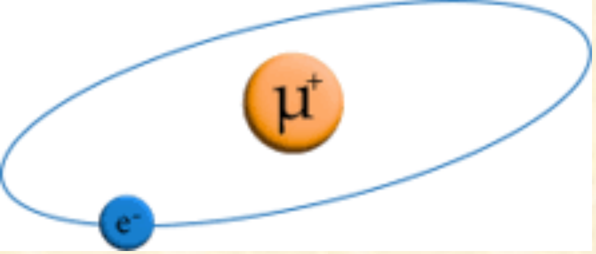
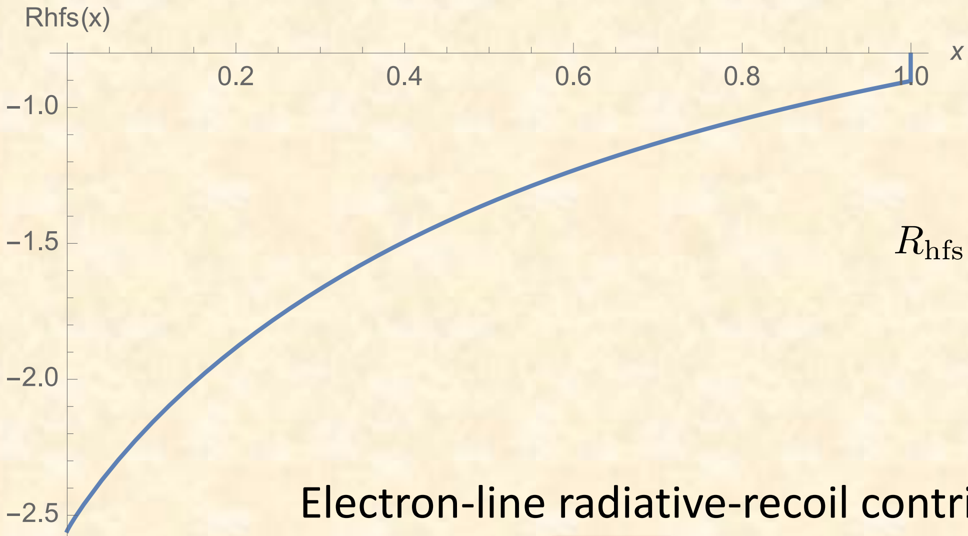


Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\text{hfs}} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha(Z\alpha)}{n^3} R_{\text{hfs}} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$



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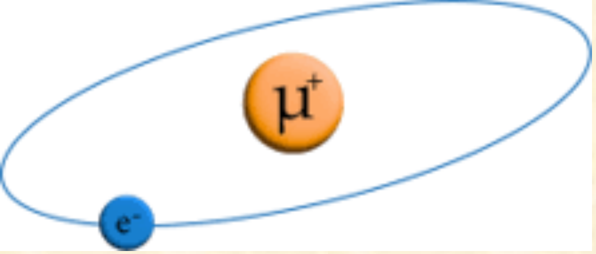
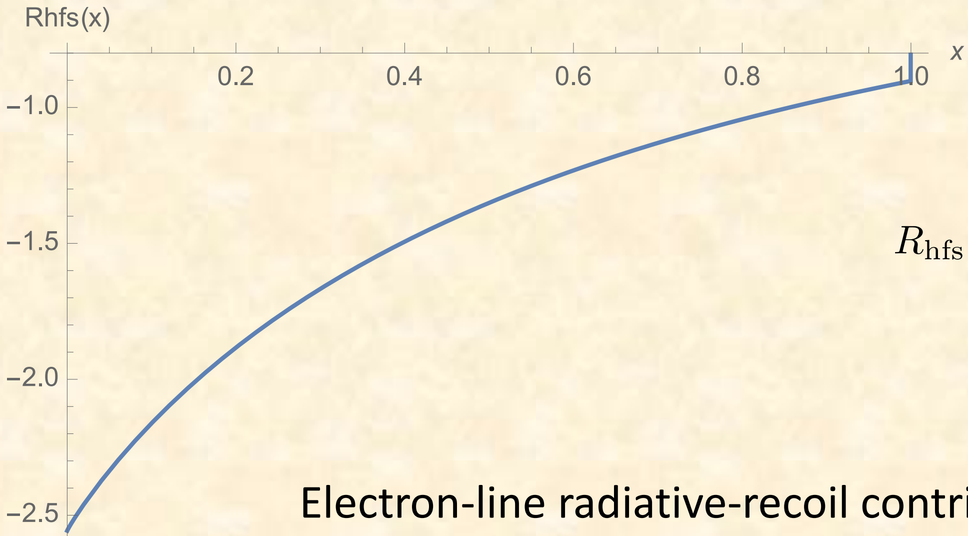
Terray and Yennie, Phys. Rev. Lett. 48, 1803 (1982)



Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\text{hfs}} = \left\{\frac{8(Z\alpha)^4 m_r^3}{3m_1 m_2}\right\}\frac{\alpha(Z\alpha)}{n^3}R_{\text{hfs}} \qquad m_r = \frac{m_1 m_2}{m_1+m_2} \quad , \quad x = \frac{m_1}{m_2}$$



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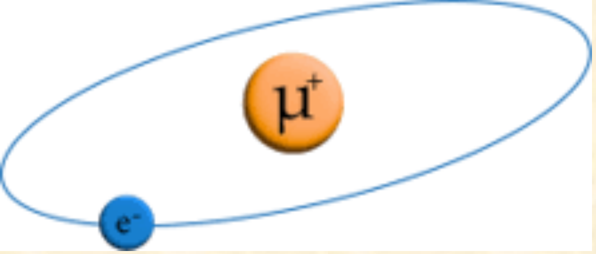
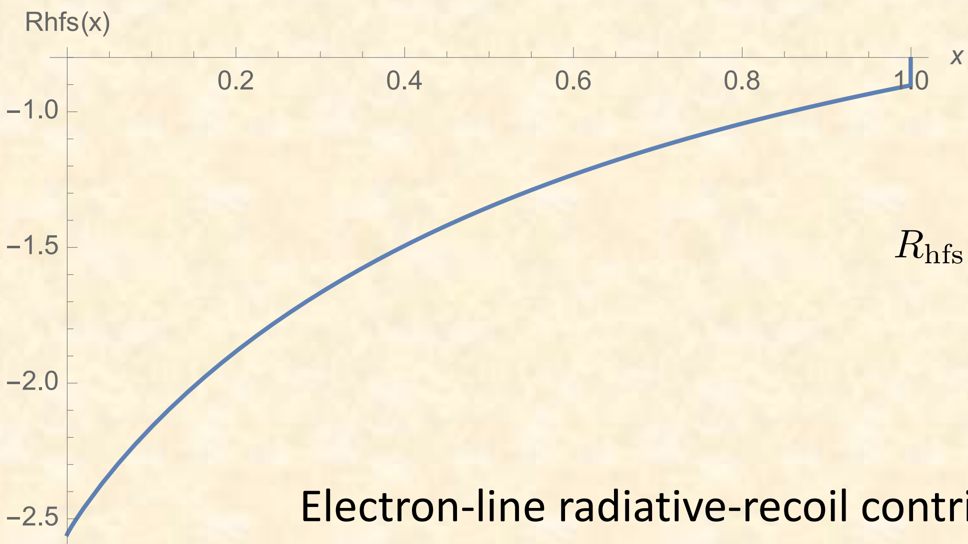
Sapirstein, Terray, and Yennie, Phys. Rev. Lett. 48, 1803 (1982) – numerical
Eides, Karshenboim, and Shelyuto, Phys. Lett. B 177, 425 (1986) – analytical



Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\rm hfs} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha(Z\alpha)}{n^3} R_{\rm hfs} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$



Positronium limiting case:

$$R_{\rm hfs}(1) = \frac{1}{\pi^2} \left\{ \frac{3}{4}\zeta(3) + 2\pi^2 \ln 2 - \frac{79\pi^2}{32} + \frac{7}{8} \right\}$$

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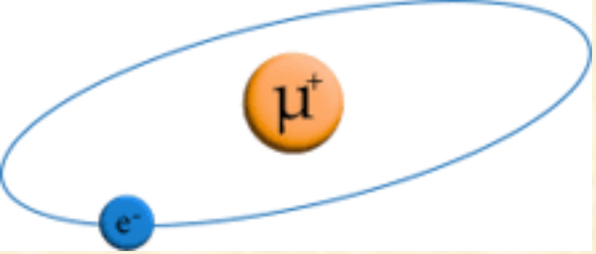
Eides, Grotch, and Shelyuto, Phys. Rev. D 58,013008 (1998)



Radiative-Recoil Correction to Muonium hfs

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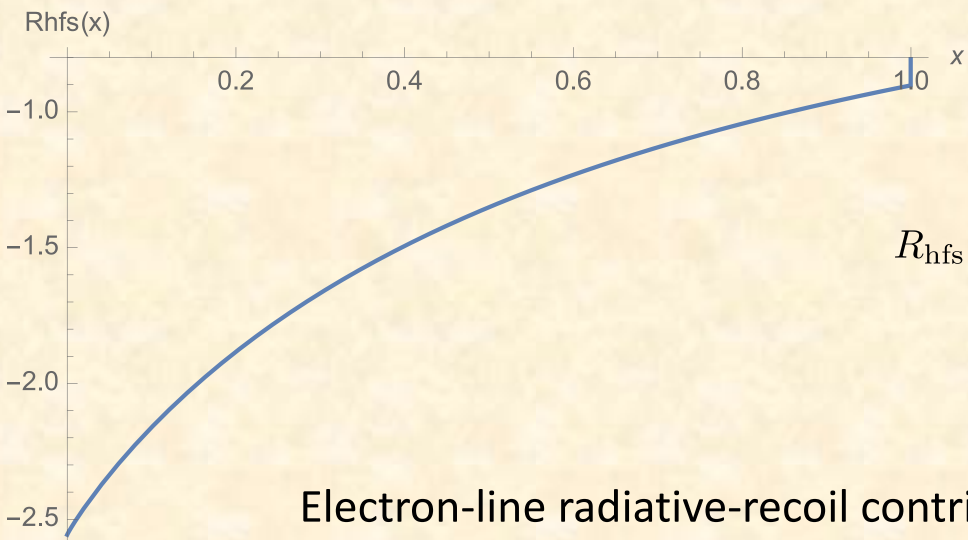

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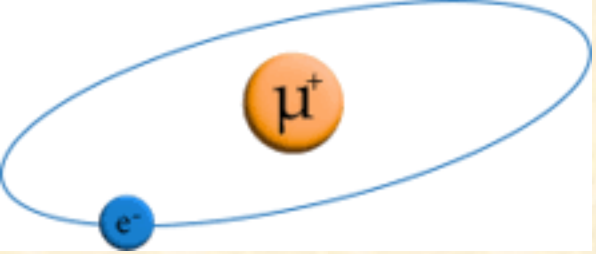
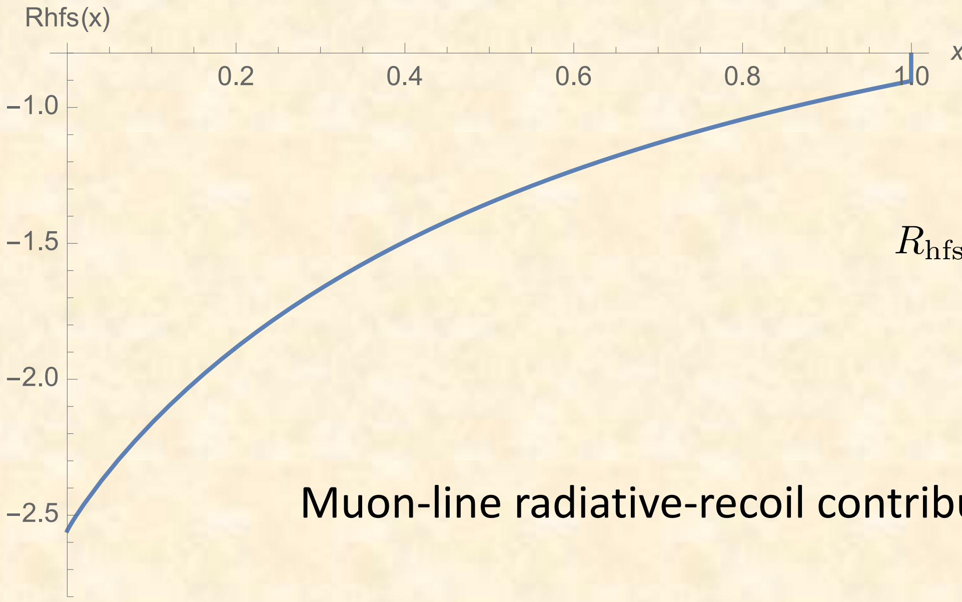
Expansion out to O(x^6): Blokland, Czarnecki, and Melnikov, Phys. Rev. D 65,073015 (2002)



Radiative-Recoil Correction to Muonium hfs

$$\Delta E_{\rm hfs} = \left\{ \frac{8(Z\alpha)^4 m_r^3}{3 m_1 m_2} \right\} \frac{\alpha(Z\alpha)}{n^3} R_{\rm hfs} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$

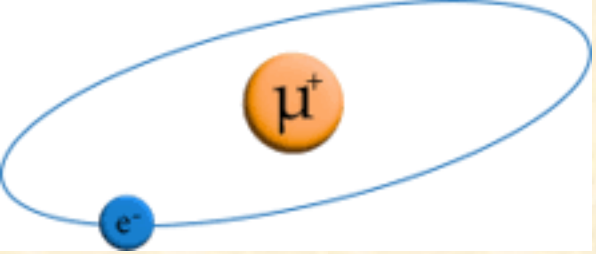


Positronium limiting case:

$$R_{\rm hfs}(1) = \frac{1}{\pi^2} \left\{ \frac{3}{4}\zeta(3) + 2\pi^2 \ln 2 - \frac{79\pi^2}{32} + \frac{7}{8} \right\}$$

Muon-line radiative-recoil contribution: let $x \to 1/x$

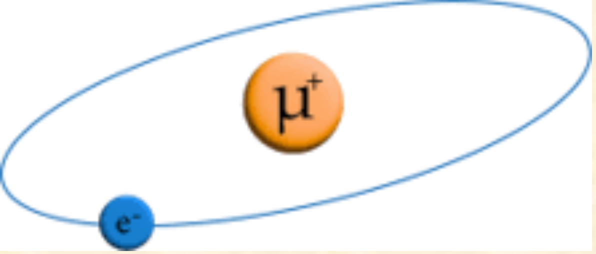
Eides, Karshenboim, and Shelyuto, Ann. Phys. 205, 291 (1991)



Radiative-Recoil Correction to Muonium Average Energy Shift

$$\Delta E_{\rm avg} = \frac{3\Delta E_{s=1} + \Delta E_{s=0}}{4} = \frac{\alpha (Z\alpha)^5 m_r^3}{n^3 m_1^2} R_{\rm avg} \qquad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$

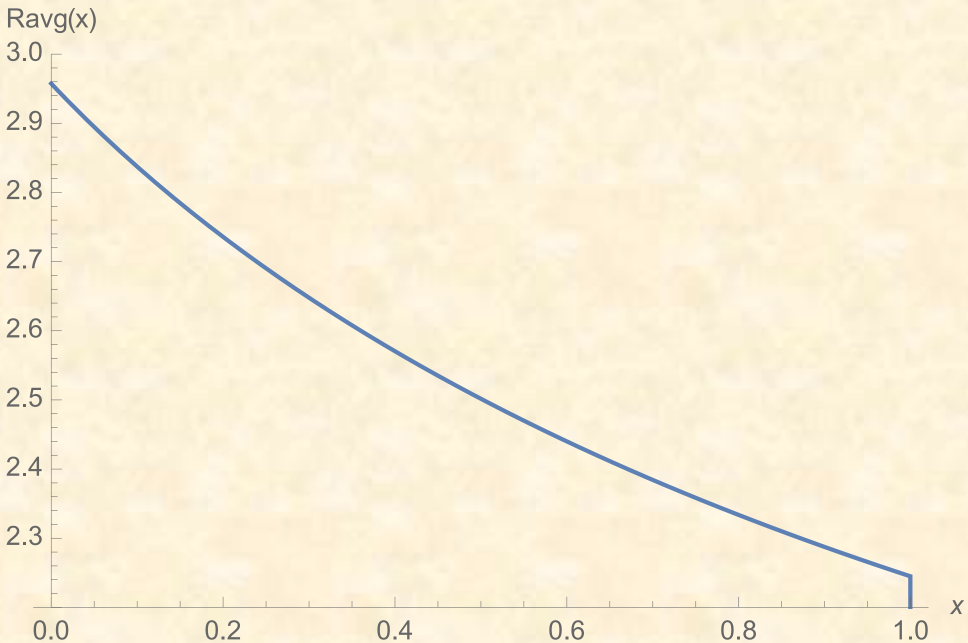
$$\begin{aligned}
R_{\rm avg}(x) \;=\; & \frac{1}{24\pi^2 x(1-x^2)^2}\Big\{ -828x(-1+x)^2 \\
& +\pi^2 x\left(417+56x-1143x^2+64x^3+651x^4+40x^5-69x^6-16x^7\right) \\
& +192\pi^2\left(-1-x+6x^2+x^3-7x^4+2x^6\right)\log(2)+288x\left(2-5x^2+3x^4\right)\zeta(3) \\
& -12x\left(59-92x^2+33x^4\right]\log(x)-48x\left(2-5x^2-5x^4+2x^6\right)\log^2(x) \\
& -32\pi^2(-1+x)^2\left(-1-x+5x^2+7x^3+2x^4\right)\mathrm{HPL}(\{-1\},x) \\
& +16\pi^2(1+x)^2\left(-1+9x+5x^2-7x^3+2x^4\right)\mathrm{HPL}(\{1\},x) \\
& +6\left(91-40x-285x^2+96x^3+217x^4-72x^5-23x^6+16x^7\right)\mathrm{HPL}(\{-1,0\},x) \\
& -6\left(-91-40x+285x^2+96x^3-217x^4-72x^5+23x^6+16x^7\right)\mathrm{HPL}(\{1,0\},x) \\
& -96(-1+x)^2\left(-1-x+5x^2+7x^3+2x^4\right)\mathrm{HPL}(\{-1,0,0\},x) \\
& -96(1+x)^2\left(-1+x+5x^2-7x^3+2x^4\right)\mathrm{HPL}(\{1,0,0\},x) \\
& -192(-1+x)^2\left(-1-x+5x^2+7x^3+2x^4\right)\mathrm{HPL}(\{-1,1,0\},x) \\
& +192(1+x)^2\left(-1+x+5x^2-7x^3+2x^4\right)\mathrm{HPL}(\{1,-1,0\},x) \\
& +96(1+x)^2\left(-1+x+5x^2-7x^3+2x^4\right)\mathrm{HPL}(\{-2,0\},x) \\
& +96(-1+x)^2\left(-1-x+5x^2+7x^3+2x^4\right)\mathrm{HPL}(\{2,0\},x)\Big\}
\end{aligned}$$



Radiative-Recoil Correction to Muonium Average Energy Shift

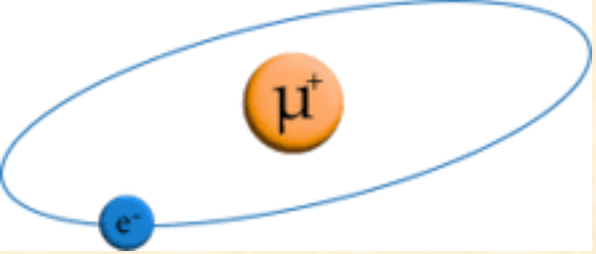
$$\Delta E_{\rm avg} = \frac{3\Delta E_{s=1} + \Delta E_{s=0}}{4} = \frac{\alpha(Z\alpha)^5 m_r^3}{n^3 m_1^2} R_{\rm avg}$$

$$m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x = \frac{m_1}{m_2}$$



$$R_{\rm avg}(1) = \frac{1}{\pi^2}\left\{\frac{9}{2}\zeta(3) + \frac{31\pi^2}{12} - \frac{35}{4}\right\}$$

$$R_{\rm avg}(x) = \left(-2\ln 2 + \frac{139}{32}\right) + \frac{x}{\pi^2}\left(6\zeta(3) - 2\pi^2\ln 2 + \frac{3\pi^2}{4} - 14\right) + x^2\left(8\ln 2 - \frac{127}{32}\right) + O(x^3)$$



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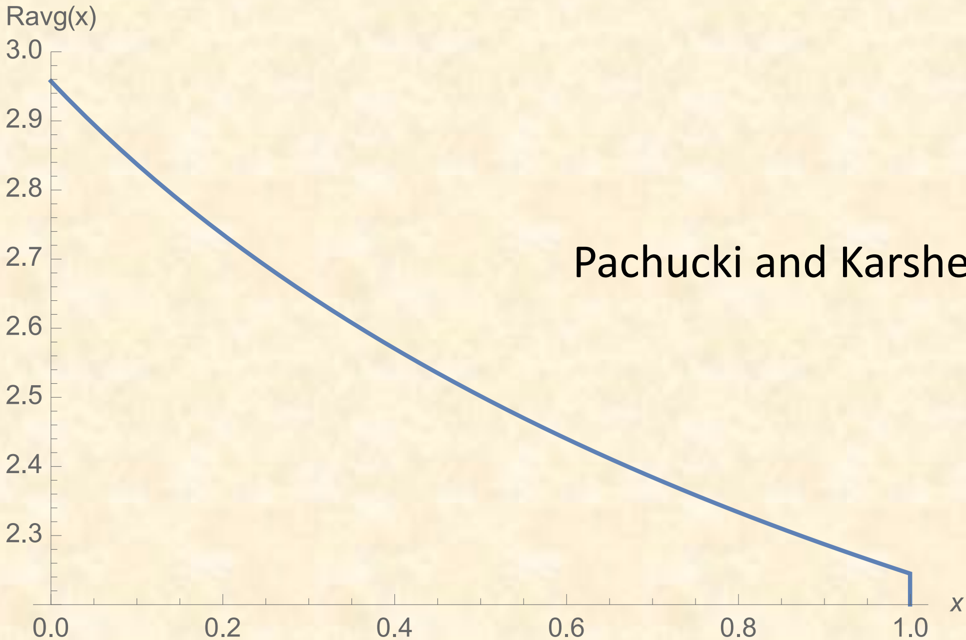


$$R_{\rm avg}(1) = \frac{1}{\pi^2}\left\{ \frac{9}{2}\zeta(3) + \frac{31\pi^2}{12} - \frac{35}{4} \right\}$$

(must double for positronium)

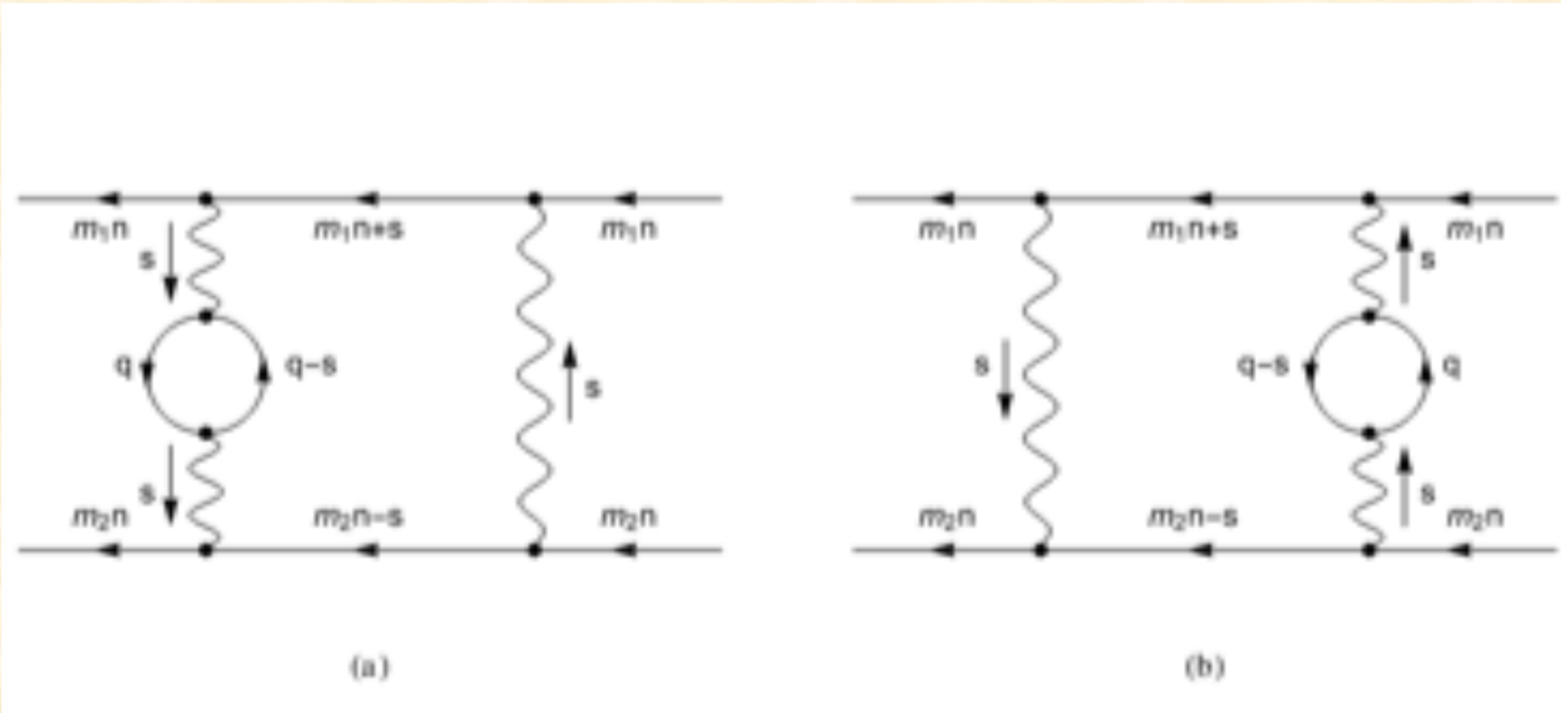
Pachucki and Karshenboim, Phys. Rev. Lett. 80, 2101 (1998)

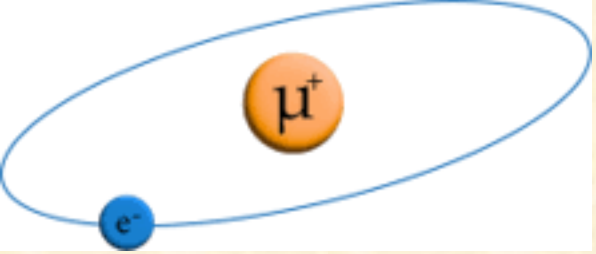
$$R_{\rm avg}(x) = \left(-2\ln 2 + \frac{139}{32} \right) + \frac{x}{\pi^2}\left(6\zeta(3) - 2\pi^2 \ln 2 + \frac{3\pi^2}{4} - 14 \right) + x^2 \left(8\ln 2 - \frac{127}{32} \right) + O(x^3)$$

Agrees with the high order expansion of Blokland, Czarnecki, and Melnikov, Phys. Rev. D 65, 073015 (2002)

Radiative – Recoil Diagrams at Order $\alpha(Z\alpha)^5$

Vacuum Polarization



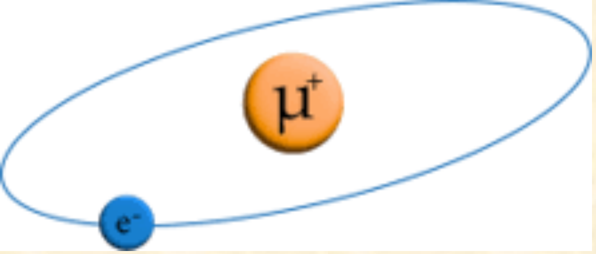


Radiative-Recoil Correction to Muonium Energy Levels from Vacuum Polarization

$$\Delta E = \frac{\alpha (Z\alpha)^5 m_r^3}{m_{\rm VP}^2} R \quad , \quad m_r = \frac{m_1 m_2}{m_1 + m_2} \quad , \quad x_i = \frac{m_i}{m_{\rm VP}}$$

$$\begin{aligned} R_{\rm hfs} \;=\; & \frac{16}{9\pi^2 x_1^4 x_2^4 (x_2^2 - x_1^2)} \Big\{ 3x_1(1 - 4x_1^3 + 3x_1^4) x_2^4 {\rm HPL}[(1,0), x_1] + 3x_1^4(-1 + 4x_2^3 - 3x_2^4) x_2 {\rm HPL}[(1,0), x_2] \\ & + 3x_1(1 + 4x_1^3 + 3x_1^4) x_2^4 {\rm HPL}[(-1,0), x_1] + 3x_1^4(-1 - 4x_2^3 - 3x_2^4) x_2 {\rm HPL}[(-1,0), x_2] \\ & + 2x_1^2(3 + x_1^2) x_2^4 \ln x_1 + 2x_1^4 (3 + x_2^2) x_2^2 \ln x_2 + 6x_1^2 x_2^2 (x_2^2 - x_1^2) \Big\} , \end{aligned}$$

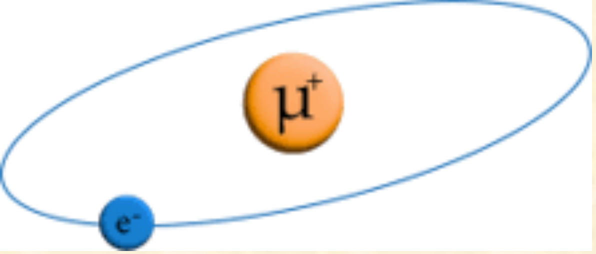
$$\begin{aligned} R_{\rm avg} \;=\; & \frac{1}{216\pi^2 x_1^4 x_2^4 (x_2^2 - x_1^2)} \\ & \times \Big\{ 9(17 - 45x_1^2 + 32x_1^3 - 9x_1^4 + 5x_1^6) x_2^5 {\rm HPL}[(1,0), x_1] - 9x_1^5 (17 - 45x_2^2 + 32x_2^3 - 9x_2^4 + 5x_2^6) {\rm HPL}[(1,0), x_2] \\ & + 9(17 - 45x_2^2 - 32x_2^3 - 9x_2^4 + 5x_2^6) x_2^5 {\rm HPL}[(-1,0), x_1] - 9x_1^5 (17 - 45x_2^2 - 32x_2^3 - 9x_2^4 + 5x_2^6) {\rm HPL}[(-1,0), x_2] \\ & - 6x_1 (51 - 118x_1^2 + 15x_1^4) x_2^5 \ln x_1 + 6x_1^5 (51 - 118x_2^2 + 15x_2^4) x_2 \ln x_2 + 306 x_1 x_2 (x_2^4 - x_1^4) - 776 x_1^3 x_2^3 (x_2^2 - x_1^2) \Big\} \end{aligned}$$



Radiative-Recoil Correction to Muonium Energy Levels from Vacuum Polarization

Muonium hyperfine splitting due to vacuum polarization:

$$\Delta E_{\rm hfs}^{\rm M;VP} = \frac{\alpha(Z\alpha)}{n^3} E_{\rm F} \left\{ \frac{3}{4} + \frac{x}{\pi^2} \left(-2\ln^2 x + \frac{8}{3}\ln x - \frac{\pi^2}{3} - \frac{28}{9} \right) + \frac{3x^2}{4} + O(x^3) \right\}$$



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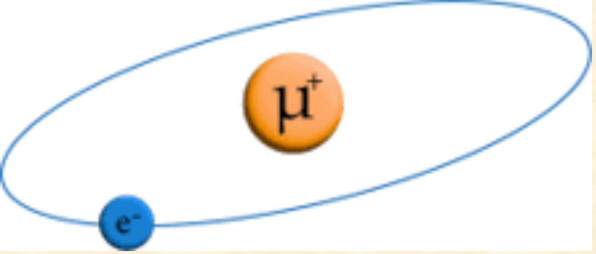
Terray and Yennie: Phys. Rev. Lett. 48, 1803 (1982). (-9004 Hz)



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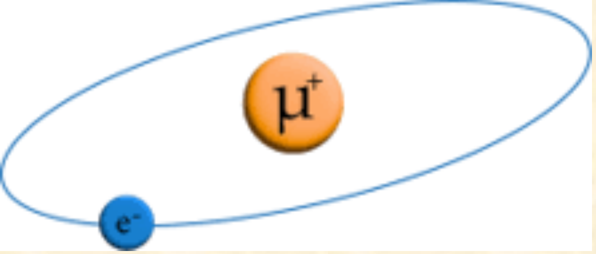
Eides, Grotch, and Shelyuto, Phys. Rev. D 58, 013008 (1998). (4.16 Hz)

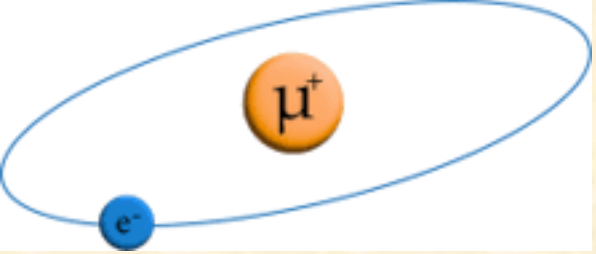


Radiative-Recoil Correction to Muonium Energy Levels from Vacuum Polarization

$$\Delta E^{\mathrm{M;VP}}_{\mathrm{hfs}} = \frac{\alpha(Z\alpha)}{n^3} E_{\mathrm{F}} \left\{ \frac{3}{4} + \frac{x}{\pi^2} \left(-2\ln^2 x + \frac{8}{3}\ln x - \frac{\pi^2}{3} - \frac{28}{9} \right) + \frac{3x^2}{4} + O(x^3) \right\}$$

Average energy level shift due to vacuum polarization:

$$\Delta E^{\mathrm{M;VP}}_{\mathrm{avg}} = \frac{\alpha(Z\alpha)^5 m_r^3}{m_e^2 n^3} \left\{ \frac{5}{48} + \frac{x}{\pi^2} \left(\frac{2\pi^2}{9} - \frac{70}{27} \right) - \frac{x^2}{12} + \frac{x^3}{\pi^2} \left(\frac{4}{3}\ln^2 x + \frac{8}{3}\ln x + \frac{4\pi^2}{9} + \frac{46}{27} \right) + O(x^4) \right\}$$

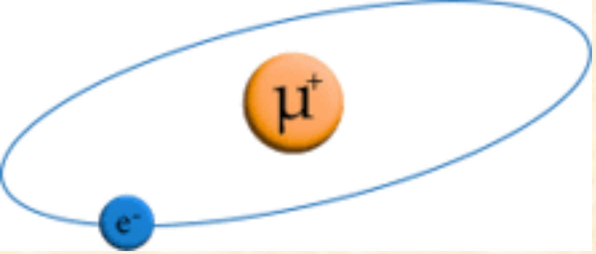


Radiative-Recoil Correction to Muonium Energy Levels from Vacuum Polarization

$$\Delta E^{\mathrm{M;VP}}_{\mathrm{hfs}} = \frac{\alpha(Z\alpha)}{n^3} E_{\mathrm{F}} \left\{ \frac{3}{4} + \frac{x}{\pi^2}\left(-2\ln^2 x + \frac{8}{3}\ln x - \frac{\pi^2}{3} - \frac{28}{9} \right) + \frac{3x^2}{4} + O(x^3) \right\}$$

$$\Delta E^{\mathrm{M;VP}}_{\mathrm{avg}} = \frac{\alpha(Z\alpha)^5 m_r^3}{m_e^2 n^3} \left\{ \frac{5}{48} + \frac{x}{\pi^2}\left(\frac{2\pi^2}{9} - \frac{70}{27} \right) - \frac{x^2}{12} + \frac{x^3}{\pi^2}\left(\frac{4}{3}\ln^2 x + \frac{8}{3}\ln x + \frac{4\pi^2}{9} + \frac{46}{27} \right) + O(x^4) \right\}$$

Eides and Grotch: Phys. Rev. A 52, 1757 (1995). (-7043 Hz)



Radiative-Recoil Correction to Muonium Energy Levels from Vacuum Polarization

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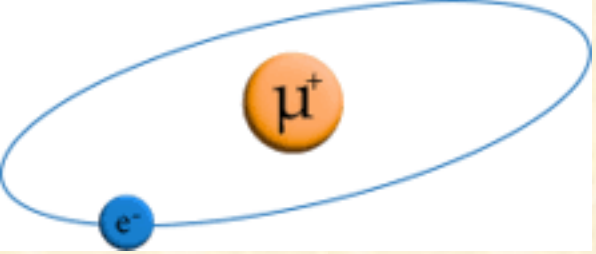
$$\Delta E^{\mathrm{M;VP}}_{\mathrm{avg}} = \frac{\alpha(Z\alpha)^5 m_r^3}{m_e^2 n^3} \left\{ \frac{5}{48} + \frac{x}{\pi^2} \left(\frac{2\pi^2}{9} - \frac{70}{27} \right) - \frac{x^2}{12} + \frac{x^3}{\pi^2} \left(\frac{4}{3}\ln^2 x + \frac{8}{3}\ln x + \frac{4\pi^2}{9} + \frac{46}{27} \right) + O(x^4) \right\}$$

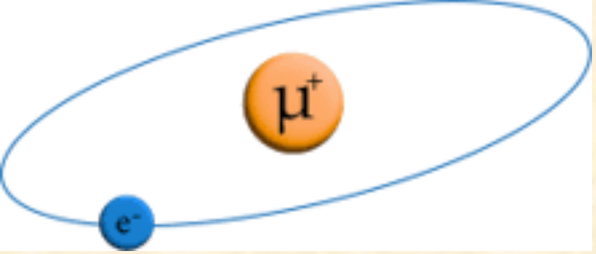
The order x^2 term gives a contribution -70.15 Hz



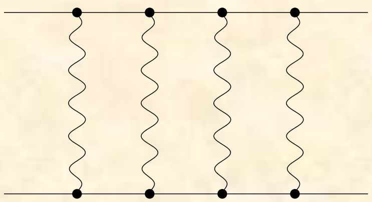
Summary

We have calculated all recoil and radiative-recoil corrections to muonium energy levels at order α^6 without approximation in the particle masses. Our results are expressed in terms of the harmonic polylogarithm functions HPL[a_1,a_2,…;x], which are iterated integrals that generalize the standard dilogarithm and polylogarithm functions. The corresponding results for positronium are obtained as a limiting case.

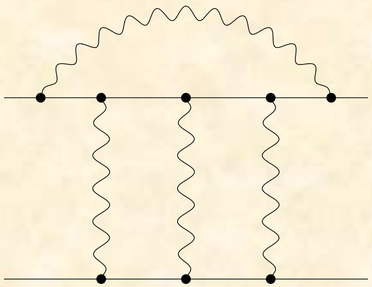
We used the methods of dimensionally regularized NRQED to separate contributions involving relativistic momenta from those involving purely non-relativistic momenta, NRQED bound state theory to calculate soft contributions, integration by parts identities to express hard contributions in terms of a few master integrals, and the method of differential equations to evaluate those master integrals. These methods are robust and will hopefully allow for the evaluation of higher order corrections.



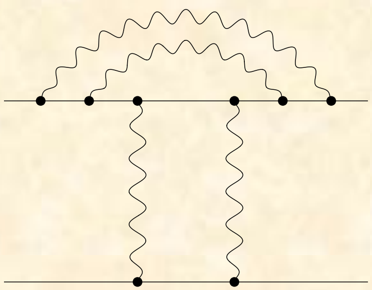
Next Steps: Recoil and Radiative-Recoil Corrections at order α^7



Pure recoil at order $\frac{m_e(Z\alpha)^7}{n^3}\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right)$



Radiative-recoil at order $\frac{m_e\alpha(Z\alpha)^6}{n^3}\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right)$



Radiative-recoil at order $\frac{m_e\alpha^2(Z\alpha)^5}{n^3}\left(\frac{m_r}{m_e}\right)^3\left(\frac{m_e}{m_\mu}\right)$

Quantum
Electrodynamics
Atoms, Lasers and Gravity

This book introduces readers to a variety of topics surrounding quantum field theory, notably its role in bound states, laser physics, and the gravitational coupling of Dirac particles. It discusses some rather sophisticated concepts based on detailed derivations which cannot be found elsewhere in the literature.

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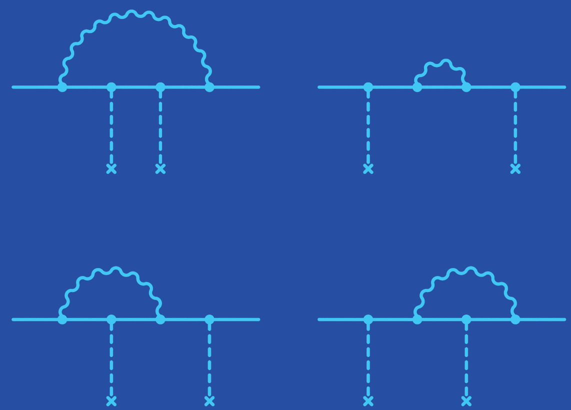

Ulrich D Jentschura
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Thank you!

Greg Adkins
Franklin & Marshall College