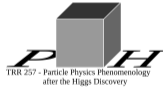


On the contribution of the electromagnetic dipole operator to the $B_s \rightarrow \mu^+ \mu^-$ decay amplitude

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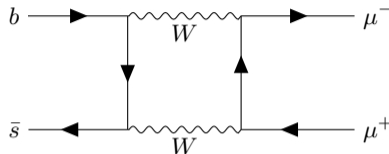


T. Feldmann, N. Gubernari, T. Huber, N. Seitz [PRD 107, 013007, arXiv: 2211.04209]

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The Decay $B_s \rightarrow \mu^+ \mu^-$

- Flavour changing neutral current (loop-induced)
- Rare decay \leftrightarrow sensitive to New Physics
- complementary to other decay modes
($\bar{B} \rightarrow X_s \gamma$, $B \rightarrow K^{(*)} \ell \ell$, ...)



Integrating out heavy W , Z , t , H leads to *effective weak Hamiltonian*.

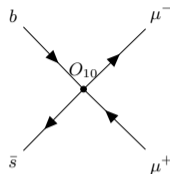
Most important operators for $b \rightarrow s \ell^+ \ell^-$ transitions:

semi-leptonic $(V - A) \times A$:	$\mathcal{O}_{10} = (\bar{s} \gamma_\mu P_L b) (\bar{\mu} \gamma^\mu \gamma_5 \mu)$
semi-leptonic $(V - A) \times V$:	$\mathcal{O}_9 = (\bar{s} \gamma_\mu P_L b) (\bar{\mu} \gamma^\mu \mu)$
electromagnetic dipole:	$\mathcal{O}_7^\gamma = \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma_{\mu\nu} b_R) F^{\mu\nu}$

The $B_s \rightarrow \mu^+ \mu^-$ decay amplitude

Without QED corrections:

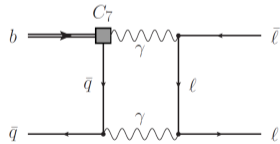
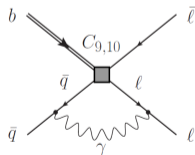
- only \mathcal{O}_{10} contributes
- only hadronic uncertainty from decay constant f_{B_s}
- helicity suppression with the muon mass



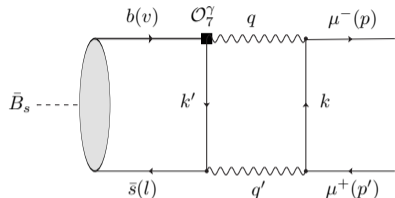
With QED corrections:

[Beneke, Bobeth, Szafron, 2017]

- also \mathcal{O}_9 and \mathcal{O}_7^γ contribute
- resolves momentum distribution of the strange quark
- helicity suppression lifted by strange-quark propagator
- single-log enhancement for \mathcal{O}_9 contribution
- double-log enhancement for \mathcal{O}_7^γ contribution from endpoint-divergent convolution integrals



\mathcal{O}_7^γ contribution to the exclusive decay amplitude



$$\propto -\frac{\alpha_{em}^2}{2\pi} C_7^{\text{eff}} m_\mu f_{B_s} [\bar{u}(p) (1 + \gamma_5) v(p')] \mathcal{F}^{\text{LO}}(E, m_\mu)$$

Hadronic information contained in form factor:

$$\mathcal{F}^{\text{LO}}(E, m_\mu) = \int_0^\infty \frac{d\omega}{\omega} \phi_+(\omega) \left[\frac{1}{2} \ln^2 \frac{m_\mu^2}{2E\omega} + \ln \frac{m_\mu^2}{2E\omega} + \frac{\pi^2}{3} \right]$$

- $\phi_+(\omega)$: light-cone distribution amplitude (LCDA) for B_s meson
- ω : light-cone projection of strange-quark momentum ($\langle \omega \rangle \sim \mathcal{O}(\Lambda_{\text{had}})$)
- logarithmic enhancement in small expansion parameter $\lambda^2 = \frac{m_\mu^2}{2E\omega} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_b}\right) \sim \mathcal{O}\left(\frac{m_\mu}{m_b}\right)$

Momentum regions in QED box diagram

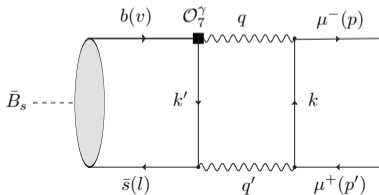
- introduce light-cone coordinates for muon momentum,

$$k^\mu = (\bar{n} \cdot k) \frac{n^\mu}{2} + (n \cdot k) \frac{\bar{n}^\mu}{2} + k_\perp^\mu$$

with $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$, and $(\bar{n} \cdot p) = (n \cdot p') \simeq 2E$, and $\omega = (\bar{n} \cdot \ell)$

- relevant momentum regions depend on regulator for endpoint divergences !
we considered two alternatives

$$\mathcal{R}_a(k) = \left(\frac{\nu^2}{-(n \cdot k)(\bar{n} \cdot l) + i0} \right)^\delta, \quad \mathcal{R}_b(k) = \left(\frac{\nu^2}{(\bar{n} \cdot k)(n \cdot p') - (n \cdot k)(\bar{n} \cdot l) + i0} \right)^\delta$$



region	$\{(\bar{n} \cdot k), k_\perp, (n \cdot k)\} / m_b$	regulator
\overline{hc}	$(\lambda^2, \lambda, 1)$	$\mathcal{R}_{a,b}$
\bar{c}	$(\lambda^4, \lambda^2, 1)$	$\mathcal{R}_{a,b}$
s	$(\lambda^2, \lambda^2, \lambda^2)$	$\mathcal{R}_{a,b}$
\overline{sc}	$(\lambda^3, \lambda^2, \lambda)$	\mathcal{R}_b

The bare QCD factorization theorem

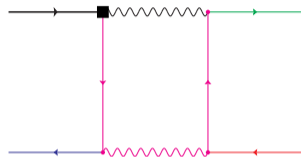
- Considering version (a) for the analytic regulator, the **bare** QCD factorization theorem for the \mathcal{O}_7^γ contribution takes the form:

$$\mathcal{F}(E, m_\mu) = \int_0^\infty \frac{d\omega}{\omega} \phi_+(\omega) \left\{ \int_0^1 \frac{du}{u} H_1(u) \bar{J}_1(u; \omega) + \bar{J}_2(1, \omega) \int_0^1 \frac{du}{u} H_1(u) \bar{C}(u; \omega) + H_1(0) \int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho} S(u, \rho; \omega) \bar{J}_2(1 + \rho, \omega) \right\}_{\text{bare}}$$

- Each term contains an endpoint-divergent convolution integral, that is regularised for finite δ and ϵ .
- Individual functions in $\{ \dots \}$ follow from momentum regions for given loop diagram \dots

- Feynman integral takes the form:

$$I_{hc}(\omega) = \int_0^1 \frac{du}{u} H_1^{(0)}(u) \bar{J}_1^{(1)}(u; \omega)$$



with

$$H_1^{(0)}(u) = 1, \quad \bar{J}_1^{(1)}(u; \omega) = -\Gamma(\epsilon) \left(\frac{\mu^2 e^{\gamma_E}}{2E\omega u(1-u)} \right)^\epsilon (1-u)$$

- endpoint divergence from $u \rightarrow 0$ regularized for $\epsilon < 0$ [δ has been set to zero]
- expanding in ϵ yields:

$$I_{hc}(\omega) = \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{2E\omega} + \frac{1}{2} \ln^2 \frac{\mu^2}{2E\omega} - \frac{\pi^2}{12} + \frac{1}{\epsilon} + \ln \frac{\mu^2}{2E\omega} + 2$$

Anti-collinear region: with $k \sim (\lambda^4, \lambda^2, 1)$

- Feynman integral takes the form:

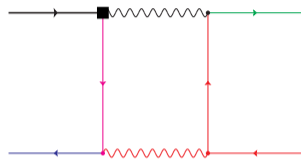
$$I_{\bar{c}}(\omega) = \bar{J}_2^{(0)}(1, \omega) \int_0^1 \frac{du}{u} H_1^{(0)}(u) \bar{C}^{(1)}(u; \omega)$$

with

$$\bar{J}_2^{(0)}(z, \omega) = \frac{1}{z}, \quad \bar{C}^{(1)}(u; \omega) = \left(\frac{\mu^2 e^{\gamma_E}}{m_\mu^2} \right)^\epsilon \left(\frac{\nu^2}{2E\omega} \right)^\delta (1-u)^{1-2\epsilon} u^{-\delta}$$

- endpoint divergence from $u \rightarrow 0$ requires analytic regulator $\delta \neq 0$
- expanding in δ and subsequently in ϵ yields

$$I_{\bar{c}}(\omega) = \left(-\frac{1}{\delta} - \ln \frac{\nu^2}{2E\omega} \right) \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{m_\mu^2} \right) + \frac{\pi^2}{3} - \frac{1}{\epsilon} - \ln \frac{\mu^2}{m_\mu^2} - 2$$



Soft region: with $k \sim (\lambda^2, \lambda^2, \lambda^2)$

- Feynman integral takes the form:

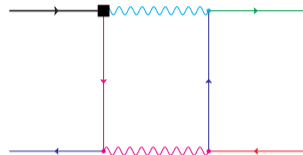
$$\tilde{I}_s(\omega) = H_1^{(0)}(0) \int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho} S^{(1)}(u, \rho; \omega) \bar{J}_2^{(0)}(1 + \rho, \omega)$$

with

$$S^{(1)}(u, \rho; \omega) = \theta(u\rho - \lambda^2) \left(\frac{\mu^2 e^{\gamma_E}}{2E\omega} \right)^\epsilon \left(\frac{\nu^2}{2uE\omega} \right)^\delta \frac{(u\rho - \lambda^2)^{-\epsilon}}{\Gamma(1 - \epsilon)}$$

- endpoint divergence requires analytic regulator $\delta \neq 0$
- expanding in δ and subsequently in ϵ yields

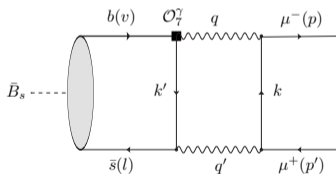
$$I_s(\omega) = \left(\frac{1}{\delta} + \ln \frac{\nu^2}{m_\mu^2} \right) \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{m_\mu^2} \right) - \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{m_\mu^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{m_\mu^2} + \frac{\pi^2}{12}$$



Summing up

$$I(\omega) = I_{hc}(\omega) + I_c(\omega) + I_s(\omega) = \frac{1}{2} \ln^2 \frac{m_\mu^2}{2E\omega} + \ln \frac{m_\mu^2}{2E\omega} + \frac{\pi^2}{3}$$

- all poles cancel ✓
- double-logarithmic enhancement ✓



$$\mathcal{F}^{\text{LO}} = \int_0^\infty \frac{d\omega}{\omega} I(\omega) \quad \checkmark$$

- but how to renormalize functions *before* the limit $\delta \rightarrow 0$???

Construction of the renormalized QCD factorization theorem

Crucial ingredients:

[Liu/Neubert, Bell/Böer/Feldmann ...]

- use refactorization conditions,

$$[[\bar{C}(u; \omega)]] \equiv \bar{C}(u; \omega)|_{u \rightarrow 0} = \int_0^\infty \frac{d\rho}{\rho} S(u, \rho; \omega) + \mathcal{O}(\alpha)$$

to redistribute endpoint-divergent contributions within factorization theorem

- take into account that scaleless integrals vanish for analytic regulators,

$$\int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho} S(u, \rho; \omega) \Big|_{\lambda^2 \rightarrow 0} = 0,$$

Construction of the renormalized QCD factorization theorem

$$\begin{aligned}
 \mathcal{F}(E, m_\mu) = & \int_0^\infty \frac{d\omega}{\omega} \phi_+(\omega) \left\{ \int_0^\infty \frac{du}{u} \left[H_1(u) \bar{J}_1(u; \omega) \theta(1-u) - H_1(0) \bar{J}_2(1, \omega) \theta(u-1) \int_1^\infty \frac{d\rho}{\rho} S(u\rho; \omega) \right] \Big|_{\lambda^2 \rightarrow 0} \right. \\
 & + \bar{J}_2(1, \omega) \int_0^1 \frac{du}{u} \left[H_1(u) \bar{C}(u) - H_1(0) [[\bar{C}(u)]] \right] \\
 & + H_1(0) \int_0^\infty \frac{du}{u} \int_0^\infty \frac{d\rho}{\rho} \left[\bar{J}_2(1+\rho, \omega) - \theta(1-\rho) \bar{J}_2(1, \omega) \right] S(u\rho; \omega) \\
 & \left. + \bar{J}_2(1, \omega) H_1(0) \int_0^1 \frac{du}{u} \int_0^1 \frac{d\rho}{\rho} S(u\rho; \omega) \Big|_{\lambda^2 \rightarrow 0} \right\}
 \end{aligned}$$

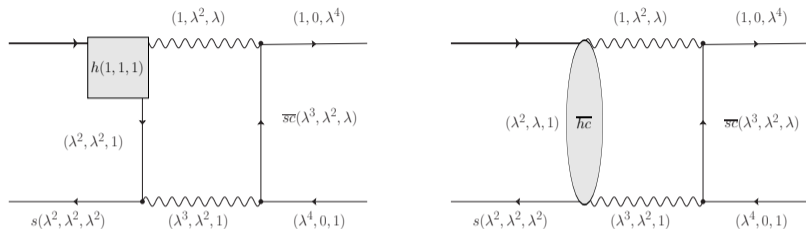
- Each individual line in the factorization theorem is now endpoint finite!
- The last line contains the large double-logarithmic enhancement, and corresponds to *anti-soft-collinear* region with cut-off longitudinal integrals

[see also: Liu/Penin, [1809.04950].]

$$\propto \int_0^1 \frac{du}{u} \int_0^1 \frac{d\rho}{\rho} \theta(u\rho - \lambda^2) = \frac{1}{2} \ln^2 \lambda^2$$

- In the following we concentrate on this term of the factorization theorem ...

Leading-Logarithmic QCD-Corrections



- 1-loop RGE is multiplicative in so-called dual space, where

$$\phi_+(\omega) = \int \frac{d\omega'}{\omega'} \sqrt{\frac{\omega}{\omega'}} J_1 \left(2\sqrt{\frac{\omega}{\omega'}} \right) \rho_+(\omega')$$

- leading-logarithmic QCD corrections to double-logarithmic term in form factor take the form

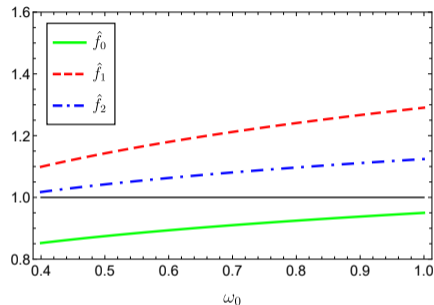
$$\mathcal{F}(E, m_\mu) \Big|_{\text{LL}} = \frac{1}{2} e^{V(\mu_{hc}, \mu_h)} e^{V(\mu_{hc}, \mu_0)} \int \frac{d\omega'}{\omega'} \ln^2 \frac{m_\mu^2}{2E\hat{\omega}'} \left(\frac{\hat{\mu}_0 e^{2\gamma_E}}{\omega'} \right)^{-g(\mu_{hc}, \mu_0)} \rho_+(\omega', \mu_0),$$

Leading-Logarithmic QCD-Corrections

- numerical estimate requires systematic expansion of LCDA
- we employ the parametrization as in [Feldmann/Lüghausen/van Dyk 2022]

$$\rho_+(\omega', \mu_0) = \frac{e^{-\omega_0/\omega'}}{\omega'} \sum_{k=0}^K \frac{(-1)^k a_k(\mu_0)}{1+k} L_k^{(1)}(2\omega_0/\omega')$$

- depending on the shape of the LCDA, the RG effects can be of the order 10 – 30% (see figure)
- Note, however, that the overall effect of the electromagnetic dipole operator is already small!



Next steps:

- confirm factorization formula by calculating anti-hard-collinear functions at 2 loops !
- conceptually understand factorization of QED corrections for this case ?

Backup Slides

Soft Region. Details

Taking the analytic regulator

$$\mathcal{R}_a(k) \rightarrow \left(\frac{\nu^2}{-\omega(n \cdot k) + i0} \right)^\delta,$$

we can first perform the k_\perp integration, to end up with

$$I_s(\omega) = \frac{\Gamma(\epsilon)}{2\pi i} \int \frac{d(n \cdot k)}{-n \cdot k + i0} \int \frac{d(\bar{n} \cdot k)}{\bar{n} \cdot k + i0} \frac{\omega}{\bar{n} \cdot k - \omega + i0} \\ \times \left(\frac{\mu^2 e^{\gamma_E}}{-(n \cdot k)(\bar{n} \cdot k) + m^2 - i0} \right)^\epsilon \mathcal{R}_a(k).$$

Now consider the analytic structure in $(\bar{n} \cdot k)$ -plane:

- There are two poles at

$$\text{Re}(\bar{n} \cdot k) = 0, \quad \omega \quad \textit{below} \text{ the real axis.}$$

- There is a branch cut from the dimensional regulator,

$$\begin{aligned} \text{for } (n \cdot k) > 0 & : & m^2/(n \cdot k) \leq \text{Re}(\bar{n} \cdot k), & \textit{below} \text{ the real axis,} \\ \text{for } (n \cdot k) < 0 & : & \text{Re}(\bar{n} \cdot k) \leq m^2/(n \cdot k), & \textit{above} \text{ the real axis.} \end{aligned}$$

- For $(n \cdot k) > 0$, the integrand is analytic in the upper half plane, and therefore the integral vanishes.
- For $(n \cdot k) < 0$, the branch cut from the dimensional regulator is in the upper half plane.

$$\text{Disc} \left(\frac{\mu^2}{-(n \cdot k)(\bar{n} \cdot k) + m^2 - i0} \right)^\epsilon = \frac{2\pi i \theta((n \cdot k)(\bar{n} \cdot k) - m^2)}{\Gamma(\epsilon)\Gamma(1-\epsilon)} \left(\frac{\mu^2}{(n \cdot k)(\bar{n} \cdot k) - m^2} \right)^\epsilon$$

- we arrive at the representation

$$\begin{aligned} I_s(\omega) &= \frac{1}{\Gamma(1-\epsilon)} \int_{-\infty}^0 \frac{d(n \cdot k)}{n \cdot k} \int_{-\infty}^{m^2/n \cdot k} \frac{d(\bar{n} \cdot k)}{\bar{n} \cdot k} \frac{\omega}{\omega - \bar{n} \cdot k} \left(\frac{\mu^2 e^{\gamma_E}}{(n \cdot k)(\bar{n} \cdot k) - m^2} \right)^\epsilon \mathcal{R}_a(k) \\ &= \frac{1}{\Gamma(1-\epsilon)} \left(\frac{\mu^2 e^{\gamma_E}}{2E\omega} \right)^\epsilon \left(\frac{\nu^2}{2E\omega} \right)^\delta \int_0^\infty \frac{du}{u} \int_{\lambda^2/u}^\infty \frac{d\rho}{\rho} \frac{1}{1+\rho} (u\rho - \lambda^2)^{-\epsilon} u^{-\delta} \end{aligned}$$

- Concentrating on the anti-soft-collinear region which generates the leading double-logarithmic term
- Include the leading logarithmic radiative QCD corrections the QCD effects in the factorization formula

$$\mathcal{F}(E, m) \Big|_{\text{LL}} = \frac{1}{2} H_1(0; \mu) \int \frac{d\omega'}{\omega'} \rho_+(\omega'; \mu) \ln^2 \frac{m^2 e^{2\gamma_E}}{2E\omega'} \times \bar{\mathcal{J}}_2(1, \omega'; \mu)$$

- Hard QCD corrections at the $b \rightarrow s\gamma^*$ vertex ($H_1(0, \mu)$)
- Hard-collinear corrections to the strange-quark jet function $\bar{\mathcal{J}}_2(1, \omega', \mu)$
- Soft corrections to the B -meson LCDA

- Inserting the net leading-logarithmic RG factor into the form factor for the \mathcal{O}_7 contribution, we end up with

$$\mathcal{F}(E, m) \Big|_{\text{LL}} = \frac{1}{2} e^{V(\mu_{hc}, \mu_h)} e^{V(\mu_{hc}, \mu_0)} \int \frac{d\omega'}{\omega'} \ln^2 \frac{m^2}{2E\hat{\omega}'} \left(\frac{\hat{\mu}_0}{\omega'} \right)^{-g(\mu_{hc}, \mu_0)} \rho_+(\omega', \mu_0),$$

with $\hat{\omega}' = \omega' e^{-2\gamma_E}$.

- If we define the generating function for logarithmic moments of $\rho_+(\omega', \mu_0)$ as

$$F_{[\rho_+]}(t; \mu_0, \mu_m) = \int_0^\infty \frac{d\omega'}{\omega'} \left(\frac{\hat{\mu}_m}{\omega'} \right)^{-t} \rho_+(\omega', \mu_0),$$

- This can also be written as ($\hat{\mu}_m = \mu_m e^{2\gamma_E}$)

$$\mathcal{F}(E, m) \Big|_{\text{LL}} = \frac{1}{2} e^{V(\mu_{hc}, \mu_h)} e^{V(\mu_{hc}, \mu_0)} \left(\frac{2E\mu_0}{m^2} \right)^{-g(\mu_{hc}, \mu_0)} \frac{d^2}{dt^2} F_{[\rho_+]}(t + g(\mu_{hc}, \mu_0); \mu_0, \frac{m^2}{2E}) \Big|_{t=0}.$$

- Using the explicit parametrization as in [Feldmann/Lüghausen/van Dyk 2022]

$$\rho_+(\omega', \mu_0) = \frac{e^{-\omega_0/\omega'}}{\omega'} \sum_{k=0}^K \frac{(-1)^k a_k(\mu_0)}{1+k} L_k^{(1)}(2\omega_0/\omega')$$

- Truncating the parametrization for $\rho_+(\omega', \mu_0)$ at $K = 2$, we obtain

$$\begin{aligned} \mathcal{F}(E, m) \Big|_{\text{LL}} &\simeq \frac{\Gamma(1-g)}{2\omega_0} e^{V(\mu_{hc}, \mu_h) + V(\mu_{hc}, \mu_0)} \left(\frac{\hat{\mu}_0}{\omega_0} \right)^{-g} \\ &\times \left\{ \left(a_0 - g a_1 + \frac{1+2g^2}{3} a_2 \right) \left[(\ln \hat{\lambda}_0^2 + \psi(1-g))^2 + \psi'(1-g) \right] \right. \\ &\quad \left. + \left(2a_1 - \frac{8}{3} g a_2 \right) (\ln \hat{\lambda}_0^2 + \psi(1-g)) + \frac{4a_2}{3} \right\} \\ &\equiv \sum_{k=0}^2 a_k f_k(\omega_0), \end{aligned}$$