Sum rule techniques for flavour physics

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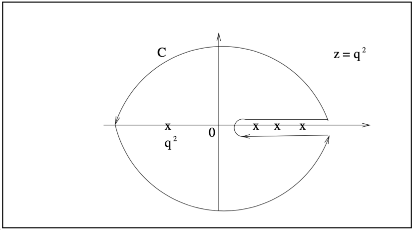
□ A glance at history

- In 1831, the Cauchy formula was derived,

A.-L. Cauchy, "Oeuvres completes, Ser. 1", 4, Paris (1890)

we will use it for a function $\Pi(q^2)$, in a complex plane $q^2 \to z$,

$$\Pi(q^2) = \frac{1}{2\pi i} \int_C dz \frac{\Pi(z)}{z - q^2}$$



here $q^2 = q_0^2 - \vec{q}^2$, q is a four-momentum

- In 1969, the operator product expansion (OPE) in quantum field theory was formulated, K. G. Wilson, Phys. Rev. **179** (1969), 1499-1512

we will use it in the momentum representation: at $q^2 \to -\infty$, $(x \to 0)$

$$i \int d^4x \, e^{iqx} \, T\{j_A(x) \, j_B(0)\} = \sum_n C_{AB}(q^2) \mathcal{O}_n(0),$$

- both Cauchy formula and Wilsonian OPE are the underlying elements of the QCD sum rule method

[M. Shifman, A. Vainshtein, V. Zakharov (1979)]

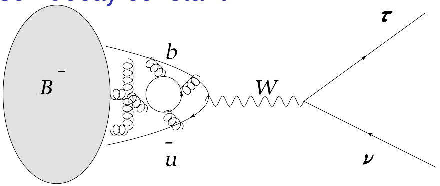
Outline of these lectures

- **Part 1: QCD (SVZ) sum rules based on local OPE**
  - calculation of the B-meson decay constant
- **Part 2: QCD Light-cone sum rules (LCSRs) for B meson semileptonic form factors**
  - $B \to \pi \ell \nu_\ell$, $B_s \to K \ell \nu_\ell$
  - $B \to 2\pi \ell \nu_\ell$, $B \to K \pi \ell \ell$
- **Part 3: Various applications**
  - nonlocal effects in $b \to s \ell \ell$ exclusive transitions
  - CP violation in charmed meson decays
  - B meson decays into dark matter

Part 1:
QCD (SVZ) sum rules based on the local OPE

□ $B \to \tau\nu_\tau$ decay and the B-meson decay constant

- the decay amplitude:



$$A(B^- \to \tau^- \bar{\nu}_\tau) = \frac{G_F}{\sqrt{2}} V_{ub} \langle 0|\bar{u}\gamma_\mu\gamma_5 b|B\rangle \, \bar{\tau}\gamma^\mu(1-\gamma_5)\nu_\tau$$

- hadronic matrix element ⇒ decay constant

$\langle 0|\bar{u}\gamma^\mu\gamma_5 b|B(p_B)\rangle = i\, p_B^\mu f_B, \quad p_B^2 = m_B^2$

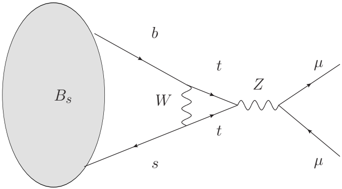
- partial width: (suppressed for $\ell = \mu, e$)

$$BR(B^- \to \tau^- \bar{\nu}_\tau)_{SM} = \frac{G_F^2 |V_{ub}|^2}{8\pi} m_\tau^2 m_B \left(1 - \frac{m_\tau^2}{m_B^2}\right)^2 f_B^2 \tau_{B^-} ,$$

$\{b \to u$ flavour-changing transition$\} \otimes \{$QCD $\}$

- V_{ub} determination, BSM search/limits from $B \to \tau\nu$ measurements are impossible without precise knowledge of f_B

□ Rare leptonic decays: $B_{s,d} \to \ell^+\ell^-$



- in SM t, W, Z-loops, sensitive to $V_{ts}V_{tb}^*$, potentially also to new physics
- after integrating out heavy loops: (the effective quark-lepton coupling C_{10}),

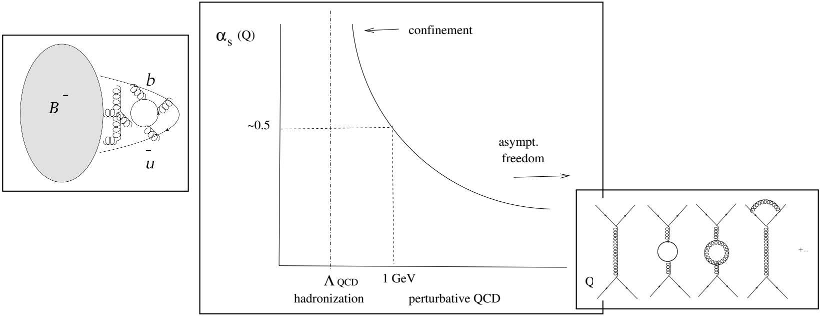
  the hadronic matrix element in decay amplitude is reduced to

  $\langle 0|\bar{s}\gamma^\mu\gamma_5 b|B_s(p_B)\rangle = ip_B^\mu f_{B_s}$, $(s \to d)$
- $f_{B_d} \simeq f_{B_u} \equiv f_B$ (isospin symmetry), but $f_{B_s} \neq f_B$, $(SU(3)_{fl}$ violation)

$\square$ *B*-meson annihilation from the point of view of in QCD

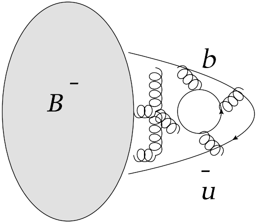
- $\bar{\Lambda} \sim m_B - m_b \sim$ 500-700 MeV,

the energy scale of quark-gluon interactions binding b and $\bar{u}$ inside the B



- $\alpha_s(\bar{\Lambda})$ too large for a perturbative expansion
- domain of nonperturbative QCD

B-meson annihilation in nonperturbative QCD



- $|B^-\rangle = |b\bar{u} \oplus \text{gluons} \oplus \text{soft quark-antiquark pairs}\rangle$
- $\langle 0|$, the QCD vacuum,

□ QCD Vacuum

- the lowest energy state, no hadrons
  contains fluctuating quark-antiquark and gluon fields:
  vacuum condensates
- e.g., $\langle 0|\overline{q}q|0\rangle \neq 0, \quad q = u, d, s$
  -spontaneous breaking of chiral symmetry
- $\langle 0|G_{\mu\nu}G^{\mu\nu}|0\rangle \neq 0$, $\langle 0|\overline{q}\sigma_{\mu\nu}G^{\mu\nu}q|0\rangle \neq 0$,...
- universal set of vacuum condensate densities with dimension $d = 3, 4, 5, ..$

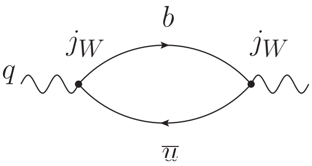
□ Correlation function of $\overline{u}b$ currents

- formal definition of the vacuum correlation function:

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\langle 0|T\{j_\mu^W(x)j_\nu^{W\dagger}(0)\}|0\rangle\,,$$

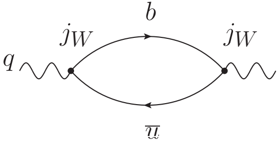
  a quantum amplitude of emission and absorbtion of $\overline{u}b$ pair in vacuum by the external current:

$$j_\mu^W = \bar{u}\gamma_\mu\gamma_5 b,$$
$$j_\mu^{W\dagger} = \bar{b}\gamma_\mu\gamma_5 u$$

$\Pi_{\mu\nu}(q^2) =$ 

- the flavour and J^P of the current can vary
  currents with other meson quantum numbers (B_s, D, D_s,π ρ, ...),
  (any Lorentz-covariant and colour-invariant local operator)

□ Correlation function far below the B-meson threshold



- 4-momentum of the $b\bar{u}$ pair: $q = (q_0, \vec{q})$, $q^2 = q_0^2 - \vec{q}^2$, rest frame: $\vec{q} = 0$, $q^2 = q_0^2$, fix the energy $q_0 \ll m_b$
- the $b\bar{u}$-pair is virtual: $\Delta E \Delta t \sim 1$, the energy deficit $\Delta E \sim m_b$, $\Delta t \sim 1/m_b$

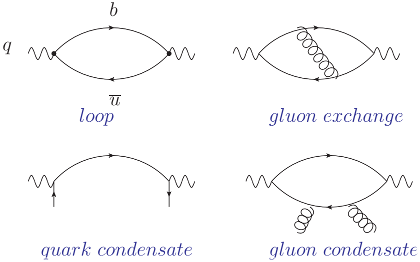
  $m_b \gg \Lambda_{QCD}$: $\Delta t \ll 1/\Lambda_{QCD}$
- virtual quarks propagate during short times, are asymptotically free,
- at $q^2 \ll m_b^2, m_B^2$,

  $\Pi_{\mu\nu}(q^2) \simeq$ simple loop diagram

  $\oplus$\{ calculable QCD corrections\} $\Leftarrow$ to be added

□ Calculating the correlation function at $q^2 \ll m_B^2$

- adding perturbative gluon exchanges to the simple loop , $\alpha_s(m_B) \ll 1$
- including nonperturbative effects due to condensates
- typical diagams



- technically, using Feynman rules of QCD and considering the vacuum quark-antiquarks and gluons as external static fields.
- The result: analytical expression for $\Pi_5(q^2)$ in terms of m_b, m_u and universal QCD parameters α_s, $\langle \bar{q}q \rangle$,...

□ Expansion of the operator-product in local operators

- interpreting the calculation as an operator-product expansion:

$$T\{j_\mu^W(x) j_\nu^{W\dagger}(0)\} = \sum_{d=0,3,4,..} C_{\mu\nu(d)}(x^2, m_b, m_u, \alpha_s) O_d(0)$$

  in local operators with the quantum numbers of vacuum

  (Lorentz-scalar, C-,P-,T-invariant, colorless) and growing dimensions:

  $O_0 = 1,\ O_3 = \bar{q}q,\ O_4 = G^{\mu\nu}G_{\mu\nu}, \ldots.$ (no operator of dimension 2 in QCD !)

- vacuum average

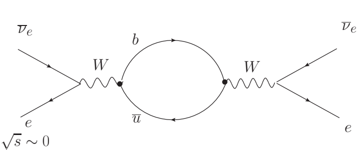
  integrating over

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx} \langle 0|T\{j_\mu^W(x) j_\nu^{W\dagger}(0)\}|0\rangle$$
$$= \sum_{d=0,3,4,..} \overline{C}_{\mu\nu(d)}(q^2, m_b, m_u, \alpha_s)\langle 0|O_d|0\rangle$$

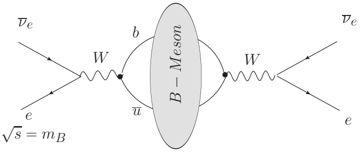
- perturbative loops $\rightarrow$ Wilson coefficeints $\overline{C}_d$ as series in α_s,
  $d \neq 0$, $\langle 0|O_d|0\rangle \sim (\Lambda_{QCD})^d$ - vacuum condensate densities,
- at $q^2 \ll m_b^2$, high-d terms suppressed by $O[(\Lambda_{QCD}/m_b)^d]$
  the OPE can safely be truncated

□ Correlation function above B threshold

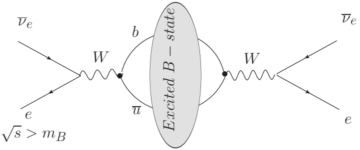
- Hypothetical neutrino-electron scattering, varying c.m. energy $\sqrt{s} = \sqrt{q^2}$,
- $\Pi_{\mu\nu}(q^2)$ is the part of the scattering amplitude



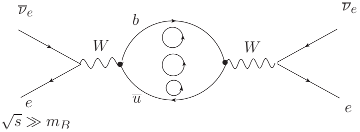
highly virtual quark pair,



B-meson, resonance



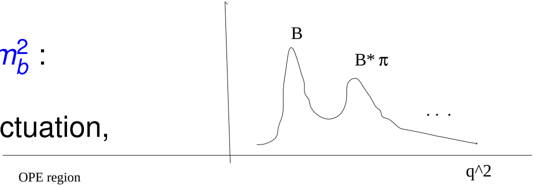
excited B mesons



multiple hadrons (continuum)

□ Hadronic representation of $\Pi_{\mu\nu}(q^2)$

- $\Pi_{\mu\nu}(q^2)$ at $q^2 \ll m_b^2$:
  a short-distance
  short-lived $b\bar{u}$ -fluctuation,
  $\simeq$ loop diagram



- $\Pi_{\mu\nu}(q^2)$ at $q^2 \geq m_B^2$, :
  propagation of B meson and excited B states
  (infinite sum over resonant and multiparticle states)

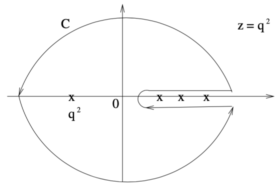
- the hadronic representation (dispersion relation):

$$\Pi_{\mu\nu}(q^2) = \frac{\langle 0|j_\mu^W|B\rangle\langle B|j_\nu^{W\dagger}|0\rangle}{m_B^2 - q^2} + \sum_{B_{exc}} \frac{\langle 0|j_\mu^W|B_{exc}\rangle\langle B_{exc}|j_\nu^{W\dagger}|0\rangle}{m_{B_{exc}}^2 - q^2}$$

- rigorous theory derivation is based on
  analyticity of $\Pi_{\mu\nu}(q^2)$ at $q^2 \to z$
  (valid in any local quantum-field theory)

Derivation of dispersion relation

- transforming the Cauchy formula



$$\Pi(q^2) = \frac{1}{2\pi i}\int\limits_{|z|=R} dz\frac{\Pi(z)}{z-q^2} + \frac{1}{2\pi i}\int\limits_{s_{min}}^{R} dz\frac{\Pi(z+i\delta)-\Pi(z-i\delta)}{z-q^2} + \frac{1}{2\pi}\int\limits_{\tilde{C}} dz\frac{\Pi(z)}{z-q^2}\,. \qquad (88)$$

Suppose the function decreases at $|q^2| \to \infty$, $\Pi(q^2) \sim 1/|q^2|^\lambda$, where $\lambda > 0$. Then the first integral vanishes at $R \to \infty$. Taking an infinitely small semicircle, one makes the third integral also vanishing. Furthermore, since there are no singularities of $\Pi(z)$ at Re $z < s_{min}$, the integrand in the second integral reduces to the imaginary part: $\Pi(q^2+i\delta) - \Pi(q^2-i\delta) = 2i\,\text{Im}\,\Pi(q^2)$ (due to Schwartz reflection principle). Finally, we obtain the desired *dispersion relation*

$$\Pi(q^2) = \frac{1}{\pi}\int\limits_{s_{min}}^{\infty} ds\frac{\text{Im}\,\Pi(s)}{s-q^2-i\delta} \qquad (89)$$

- unitarity relation for imaginary part:

$$\text{Im}\Pi_{\mu\nu}(s) = \sum_{h_B=B,B^8\pi,\dots} \langle 0|j^W_\mu|h_B\rangle\langle h_B|j^{W\dagger}_\nu|0\rangle d\tau_{h_B}$$

□ Quark-hadron duality

(omitting Lorentz indices everywhere)

$$\Pi(q^2) = \frac{\langle 0|j^W|B\rangle\langle B|j^{W\dagger}|0\rangle}{m_B^2 - q^2} + \sum_{B_{exc}} \frac{\langle 0|j^W|B_{exc}\rangle\langle B_{exc}|j^{W\dagger}|0\rangle}{m_{B_{exc}}^2 - q^2}$$

$$= \frac{1}{\pi}\int\limits_{(m_b+m_u)^2}^{s_0} ds \frac{\mathrm{Im}\Pi(s)}{s-q^2} + \frac{1}{\pi}\int\limits_{s_0}^{\infty} ds \frac{\mathrm{Im}\Pi(s)}{s-q^2}$$

- the sum of B_{exc}-states is approximated by the calculable integral over $\mathrm{Im}\Pi_5(s)$ $\Rightarrow$ s_0, the effective threshold

□ Deriving the sum rule for f_B^2

▶ isolating the ground-state B-state and introducing the spectral density of excited hadronic states

$$\Pi(q^2) = \frac{f_B^2 m_B^4}{m_B^2 - q^2} + \int_{s_h}^{\infty} ds \frac{\rho^h(s)}{s - q^2}$$

▶ expressing the OPE result as a dispersion relation

$$\Pi(q^2)^{(OPE)} = \frac{1}{\pi} \int_{m_b^2}^{s_0} ds \frac{\mathrm{Im}\Pi^{(OPE)}(s)}{s - q^2} + \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\mathrm{Im}\Pi^{(OPE)}(s)}{s - q^2}$$

equating the two representations at $q^2 \ll m_b^2$
- global quark-hadron duality

▶ at sufficiently large s the local duality is also valid:

$$\rho^h(s) \simeq \frac{1}{\pi} \mathrm{Im}\Pi^{(OPE)}(s),$$

□ Deriving the sum rule for f_B^2

- semilocal quark-hadron duality is used, the effective threshold s_0

$$\int_{s_h}^{\infty} ds \frac{\rho^h(s)}{s-q^2} \simeq \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}\Pi^{(OPE)}(s)}{s-q^2}$$

- this yields approximate analytical relation for decay constant:

$$\frac{f_B^2 m_B^4}{m_B^2-q^2} = \frac{1}{\pi}\int_{m_b^2}^{s_0} ds \frac{\mathrm{Im}\Pi^{(OPE)}(s)}{s-q^2}$$

- Borel transformation

$$\Pi(M^2) \equiv \mathcal{B}_{M^2}\Pi(q^2) = \lim_{\substack{-q^2,n\to\infty \\ -q^2/n=M^2}} \frac{(-q^2)^{(n+1)}}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\,.$$

suppresses the higher-state contributions to the hadronic sum, the sum rule less sensitive to the duality approximation

$\mathcal{B}_{M^2}\left(\frac{1}{m^2-q^2}\right) = \exp(-m^2/M^2)$

The resulting QCD sum rule

$$f_B^2 m_B^4 e^{-m_B^2/M^2} = \int\limits_{m_b^2}^{s_0} ds e^{-s/M^2} \; \mathrm{Im}\Pi^{(OPE)}(s, m_b, m_u, \alpha_s, \langle 0|\bar{q}q|0\rangle, ...)$$

- current accuracy of $\Pi^{(OPE)}(q^2)$ at $q^2 \ll m_b^2$:
  vacuum condensates with $d \leq 6$
  loop $\oplus$ $O(\alpha_s) \oplus O(\alpha_s^2)$
  [K.Chetyrkin, M.Steinhauser (2001)]
- standard way to fix s_0:
  calculate the mass of B-meson from the same sum rule:

$$m_B^2 = -\frac{\frac{d}{d(1/M^2)}[SR]}{SR}$$

☐ Input parameters

parameter	input value	[Ref.]	rescaled values
quark-gluon coupling and quark masses			
$\alpha_s(m_Z)$	0.1179 ± 0.0010	[3]	$\alpha_s(1.5\,\text{GeV}) = 0.3479^{+0.0100}_{-0.0096}$ $\alpha_s(3.0\,\text{GeV}) = 0.2531^{+0.0050}_{-0.0048}$
$\overline{m}_c(\overline{m}_c)$	1.280 ± 0.025 GeV	[3]	$\overline{m}_c(1.5\,\text{GeV}) = 1.202 \pm 0.023$ GeV
$\overline{m}_b(\overline{m}_b)$	4.18 ± 0.03 GeV	[3]	$\overline{m}_b(3.0\,\text{GeV}) = 4.46 \pm 0.04$ GeV
$(\overline{m}_u + \overline{m}_d)(2\ \text{GeV})$	6.78 ± 0.08 MeV	[3, 41]	$(\overline{m}_u + \overline{m}_d)(1.5\ \text{GeV}) = 7.40 \pm 0.09$ MeV $(\overline{m}_u + \overline{m}_d)(3.0\ \text{GeV}) = 6.14 \pm 0.07$ MeV
condensates			
$\langle \bar{q}q \rangle(2\,\text{GeV})$	$-(286 \pm 23\ \text{MeV})^3$	[41]	$\langle \bar{q}q \rangle(1.5\,\text{GeV}) = -(279 \pm 22\ \text{MeV})^3$ $\langle \bar{q}q \rangle(3.0\,\text{GeV}) = -(295 \pm 24\ \text{MeV})^3$
$\langle GG \rangle$	$0.012^{+0.006}_{-0.012}\ \text{GeV}^4$	[44]	—
m_0^2	$0.8 \pm 0.2\,\text{GeV}^2$	[44]	—
r_{vac}	0.55 ± 0.45	[44]	—

Table 1. QCD parameters used in the LCSRs and two-point sum rules.

AK, B. Melić, Y. M. Wang and Y. B. Wei, [arXiv:2011.11275 [hep-ph]].

□ $B_{(s)}$ and $D_{(s)}$ decay constants, sum rules vs lattice QCD

Decay constant	Lattice QCD (FLAG 2019)*	QCD sum rules **
f_B[MeV]	190.0±1.3	207^{+17}_{-9}
f_{B_s}[MeV]	230.3±1.3	242^{+17}_{-12}
f_{B_s}/f_B	1.209± 0.005	$1.17^{+0.04}_{-0.03}$
f_D[MeV]	212.0± 0.7	201^{+12}_{-13}
f_{D_s}[MeV]	249.9± 0.5	238^{+13}_{-23}
f_{D_s}/f_D	1.1783±0.0016	$1.15^{+0.04}_{-0.05}$

$*$ $N_f = 2 + 1 + 1$

$**$ P.Gelhausen, AK, A.A.Pivovarov, D.Rosenthal, 1305.5432 hep/ph]

□ Universality of the method

- $\bar{q}Q$ currents with various flavour and J^P in the correlation functions $\Rightarrow$ sum rules for decay constants of B_s, D, D_s, π ρ, K, K^*, also baryonic, gluonic currents

(any Lorentz-covariant and colour-invariant local operator)

- the coefficients in the OPE depend on the currents, inputs are universal (quark masses, α_s , condensates)

- QCD (SVZ) sum rules address the question: why are the hadrons not alike ?

- $SU(3)_{flavour}$ and heavy-quark symmetry violations can be estimated (finite quark masses, strange/nonstrange condensates)

Summary of part 1

- QCD sum rule, the three key elements

Correlation function of quark-antiquark currents

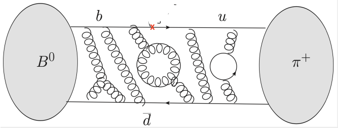
⇓ ⇓

Operator Product Expansion in terms of quark-gluon diagrams and universal QCD parameters	=	Dispersion Relation, a sum over hadronic amplitudes

- 2-point correlation functions of quark currents allow to relate QCD with hadronic observables, e.g. f_B or f_D
- flexible quantum numbers (flavour and spin-parity)
- QCD sum rules: analytical calculation in terms of diagrams, duality approximation for excited states
  $\sim$ 10% accuracy is probably the limit
- future goal: to better assess OPE/input/duality uncertainties

Part 2:
QCD Light-cone sum rules

□ $B \to \pi$ transition form factors



- hadronic matrix element is reduced to two form factors:

  functions of the momentum transfer squared q^2

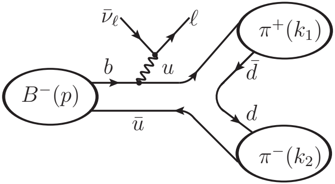
$$\langle \pi^+(p)|\bar{u}\gamma_\mu b|B(p+q)\rangle = f^+_{B\pi}(q^2)\left[2p_\mu + \left(1 - \frac{m_B^2 - m_\pi^2}{q^2}\right)q_\mu\right] + f^0_{B\pi}(q^2)\frac{m_B^2 - m_\pi^2}{q^2}q_\mu,$$

- this decomposition follows from symmetry considerations only,
- observable: differential width

$$\frac{d\Gamma(\bar{B}^0 \to \pi^+ l^- \nu)}{dq^2} = \frac{G_F^2|V_{ub}|^2}{24\pi^3}p_\pi^3|f^+_{B\pi}(q^2)|^2 + O(m_l^2)$$

$0 < q^2 < (m_B - m_\pi)^2 \sim 26$ GeV^2, p_π -kinematical factor

□ $B \to \pi\pi$ form factors



- semileptonic $B \to \pi\pi\ell\nu_\ell$ decay
- expansion of $B \to \pi\pi$ matrix element:

$$
\begin{aligned}
& i\langle \pi^+(k_1)\pi^0(k_2)|\bar{u}\gamma^\mu(1-\gamma_5)b|\bar{B}^0(p)\rangle \\
&= -F_\perp(q^2,k^2,\zeta)\,\frac{4}{\sqrt{k^2\lambda_B}}\, i\epsilon^{\mu\alpha\beta\gamma}\, q_\alpha\, k_{1\beta}\, k_{2\gamma} \\
&+ F_t(q^2,k^2,\zeta)\,\frac{q^\mu}{\sqrt{q^2}} + F_0(q^2,k^2,\zeta)\,\frac{2\sqrt{q^2}}{\sqrt{\lambda_B}}\left(k^\mu - \frac{k\cdot q}{q^2}q^\mu\right) \\
&+ F_\parallel(q^2,k^2,\zeta)\,\frac{1}{\sqrt{k^2}}\left(\bar{k}^\mu - \frac{4(q\cdot k)(q\cdot\bar{k})}{\lambda_B}\,k^\mu + \frac{4k^2(q\cdot\bar{k})}{\lambda_B}\,q^\mu\right),
\end{aligned}
$$

$k(\bar{k}) = k_1 + (-)k_2$

- dipion state with $J^P = 0^+, 1^-, 2^+, \ldots$, a rich set of observables
  (decay width distributions, asymmetries etc.)
- final-state pions interact strongly and form resonances,
  $\rho(770)$ with $J^P = 1^-$, $B \to \rho\ell\nu_\ell$ is a model-dependent part of the $B \to 2\pi\ell\nu_\ell$

□ The $B \to K\ell^+\ell^-$, $B \to K\pi\ell^+\ell^-$ FCNC decays

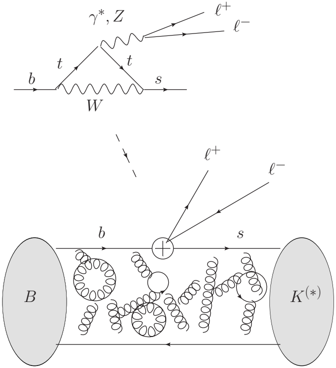
- the $b \to s\ell^+\ell^-$ loop diagrams reduced to effective local operators,
  since t, Z, W are much heavier than b

$$H_{eff} = -\frac{G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{10} C_i O_i \,,$$

$O_{9(10)} = \frac{\alpha_{em}}{2\pi} [\bar{s}\gamma_\mu(1-\gamma_5)b]\ell\gamma^\mu(\gamma_5)\ell,$

$C_9 \simeq 4.4,\ C_{10}(m_b) = -4.7,$

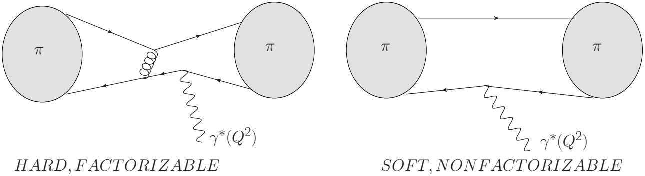
O_{1-8} play a secondary role



- in $B \to K\ell^+\ell^-$ the hadronic part for $O_{9,10}$ is again reduced to the two $B \to K$ form factors

- $B \to K\pi\ell^+\ell^-$, four form-factors , $K\pi$ state form the $K^*(892)$ and other $J^P = 0^+, 1^-, 2^+$ resonances

□ Hadron form factors in QCD



- pion e.m. form factor, QCD asympotitics, a convolution:

$$F_\pi(Q^2)^{asympt} = \frac{8\pi\alpha_s f_\pi^2}{9Q^2}\left(\int_0^1 du \frac{\varphi_\pi(u,\mu)}{\bar{u}}\right)^2\Bigg|_{\mu\sim Q},$$

- universal pion distribution amplitude :
  vacuum-pion matrix element expanded near $x^2 = 0$

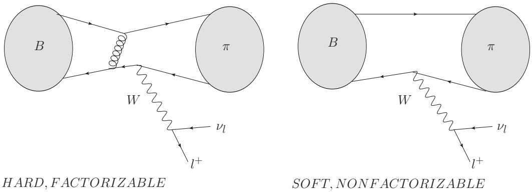
$$\langle\pi(p)|\bar{u}(x)[x,0]\gamma_\mu\gamma_5 d(0)|0\rangle_{x^2=0} = -ip_\mu f_\pi \int_0^1 du\, e^{iup\cdot x}\varphi_\pi(u)$$

[Chernyak, Zhitnisky; Efremov,Radyushkin; Brodsky-Lepage (1977-1980)]

- how large is the “soft” part ? $\sim 1/Q^4$

□ Heavy-to-light form factors in QCD

- use of effective theories obtained from QCD in a certain limit:
  heavy-quark limit $\to$ HQET, large recoil limit $\to$ SCET
- factorization theorems in $m_b \to \infty$, (originally for $B \to \pi\pi$)
  [M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda (1999)]
- for $B \to \pi$ form factor at large recoil: $(E_\pi \sim m_B/2,\, q^2 \to 0)$
  [M. Beneke, Th. Feldmann (2001)]



$$f_{B\pi}(q^2) \sim \alpha_s(\mu)\int d\omega du\, \phi_B^+(\omega,\mu) T_h(q^2,\omega,u,\mu)\varphi_\pi(u,\mu) + f_{B\pi}^{soft}(q^2)$$

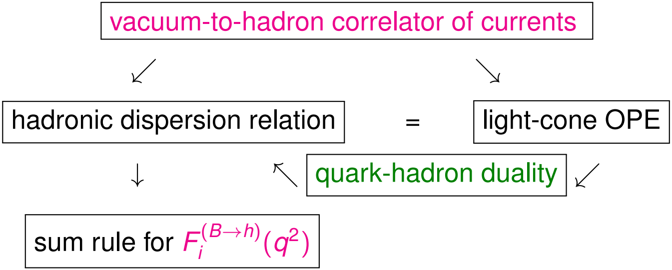
$$\mu = \sqrt{m_b \Lambda}$$

- the main challenge: calculate the soft (overlap) part of the form factor

□ The method of QCD light-cone sum rules (LCSRs)

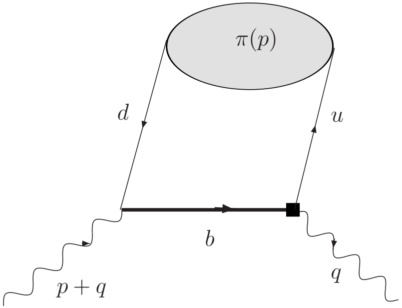
[I.I.Balitsky, V.M.Braun, A.V. Kolesnichenko (1986); V.L.Chernyak, I.Zhitnisky (1990)]

- a hybrid of asymptotic formulas a'la ERBL and SVZ sum rules:
  factorization predetermined
- outline of the method:



- the method is valid at $q^2 \ll (m_B - m_h)^2$ (large recoil of h)
- two different versions of LCSRs for $B \to h$ are used:
  - with vacuum $\to h$ correlator
  - with $B \to$ vacuum correlator (HQET)

□ The correlation function used for $B \to \pi$ form factor



$q^2, (p+q)^2 \ll m_b^2,$

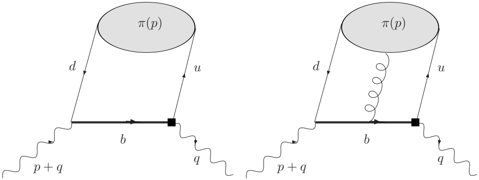
b-quark highly virtual $\Rightarrow$ $x^2 \sim 0$

$$F_\lambda(q,p) = i \int d^4x \, e^{iqx} \langle \pi(p) \mid T\{\bar{u}(x)\gamma_\lambda b(x), \bar{b}(0) i\gamma_5 d(0)\} \mid 0 \rangle$$

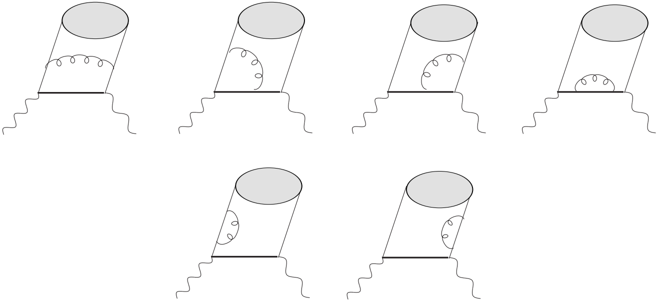
$$= F((p+q)^2, q^2) p_\lambda + \widetilde{F}((p+q)^2, q^2) q_\lambda$$

□ Diagrams

- leading order in α_s (LO) including soft, i.e. low-virtuality gluon



- NLO, $O(\alpha_s)$ contributions



□ Operator Product Expansion near the light-cone

- the correlation function expressed in a factorized form:

$$F((p+q)^2, q^2) = i \int d^4x\, e^{iqx} \Big\{ \left[S_0(x^2, m_b^2, \mu) + \alpha_s S_1(x^2, m_b^2, \mu) \right] \otimes \langle \pi(p) \mid \bar{u}(x) \Gamma d(0) \mid 0 \rangle|_\mu$$

$$+ \int_0^1 dv\, \tilde{S}(x^2, m_b^2, \mu, v) \otimes \langle \pi(p) \mid \bar{u}(x) G(vx) \tilde{\Gamma} d(0) \} \mid 0 \rangle|_\mu \Big\} + ...$$

- $S_{0,1}$, $\tilde{S}$ - perturbative amplitudes, (b-quark propagators)
- vacuum-pion matrix elements - expanded near $x^2 = 0$
  $\Longrightarrow$ universal pion light-cone distribution amplitudes (DAs) :

$$\langle \pi(p) | \bar{u}(x) [x, 0] \gamma_\mu \gamma_5 d(0) | 0 \rangle_{x^2=0} = -ip_\mu f_\pi \int_0^1 du\, e^{iup\cdot x} \varphi_\pi(u) + O(x^2)\,.$$

- the expansion near $x^2 = 0$ goes over twists ($t \geq 2$) of DAs
- terms $\sim \tilde{S}$ suppressed by powers of $1/\sqrt{m_b \Lambda}$;

□ The OPE result

$$F((p+q)^2,q^2)=\sum_{t=2,3,4,..}\int du\; T^{(t)}(q^2,(p+q)^2,m_b^2,\alpha_s,u,\mu)\,\varphi_\pi^{(t)}(u,\mu)$$

hard scattering amplitudes $\otimes$ pion light-cone DA

- LO twist 2,3,4 $q\bar{q}$ and $\bar{q}qG$ terms:

  [V.Belyaev, A.K., R.Rückl (1993); V.Braun, V.Belyaev, A.K., R.Rückl (1996)]

- LO twist 5,6 $q\bar{q}DG$ terms in factorizable approximation:

  [A.Rusov (2017)]

-NLO $O(\alpha_s)$ twist 2, (collinear factorization)

  [A.K., R.Rückl, S.Weinzierl, O. Yakovlev (1997); E.Bagan, P.Ball, V.Braun (1997);]

-NLO $O(\alpha_s)$ twist 3 (coll.factorization for asympt. DA)

  [P. Ball, R. Zwicky (2001); G.Duplancic, A.K., B.Melic, Th.Mannel, N.Offen (2007)]

-NNLO $O(\alpha_s^2\beta_0)$ correction [A.Bharucha, (2012)]

□ Basics of the pion DA's

- twist 2 DA: normalized with f_π, expansion in Gegenbauer polynomials

$$\varphi_\pi(u,\mu) = 6u(1-u)\left[1 + \sum_{n=2,4,..} a_n^\pi(\mu) C_n^{3/2}(2u-1)\right],$$

$$a_{2n}^\pi(\mu) \sim [Log(\mu/\Lambda_{QCD})]^{-\gamma_{2n}} \to 0 \quad \text{at} \;\; \mu \to \infty$$

[Efremov-Radyushkin-Brodsky-Lepage evolution]

- essential parameters: $a_{2,4}^\pi(\mu_0)$, determined from:
  - matching measured pion form factors to LCSRs,
  - two-point QCD sum rules,
  - lattice QCD

TABLE V. Comparison of the second and fourth Gegenbauer moments obtained with various methods.

Method	a_2(1 GeV)	a_4(1 GeV)	Reference
Lattice QCD	0.135 ± 0.032	...	[1]
QCD sum rule	0.28 ± 0.08	...	[9]
QCD sum rule with nonlocal condensate	$0.203^{+0.069}_{-0.057}$	$-0.143^{+0.094}_{-0.087}$	[18,45]
LCSR fitted to Jlab data	0.17 ± 0.08	0.06 ± 0.10	[20]
LCSR fitted to dispersion relation	0.22–0.33	0.12–0.25	this work

[S.Cheng, AK, A. Rusov (2020)]

- recent determination vs older results
- remaining minor input parameters: normalization constants and moments of twist 3,4 DAs, determined mainly from two-point sum rules

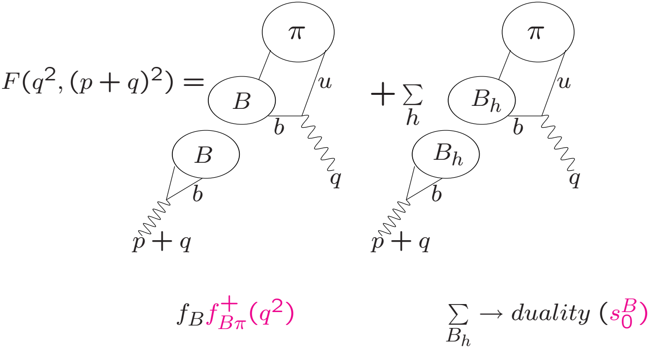
[P. Ball, V.Braun, A.Lenz (2006)]

Hadronic dispersion relation

- Analytical continuation of the correlation function in the complex variable $(p+q)^2$ at fixed q^2 $\Rightarrow$ Cauchy theorem,

$$F((p+q)^2, q^2) = \int\limits_{s_{min}}^{\infty} ds \frac{\mathrm{Im} F(s, q^2)}{s - (p+q)^2 - i\epsilon}$$

- replacing the Im part by the sum over all possible hadronic states with B-meson quantum numbers, located at $s_{min} = m_B^2$ and above



□ Derivation of LCSR

- matching OPE with disp. relation at $q^2, (p+q)^2 \ll m_b^2$

$$[F((p+q)^2, q^2)]_{OPE} = \frac{m_B^2 f_B f^+_{B\pi}(q^2)}{m_B^2 - (p+q)^2} + \int\limits_{(m_{B^*}+m_\pi)^2}^{\infty} ds \, \frac{\mathrm{Im} F(s, q^2)}{s - (p+q)^2}$$

- quark-hadron duality approximation

(based on the $s \to \infty$ limit: $F(s) \to F_{OPE}(s)$)

$$\int\limits_{(m_{B^*}+m_\pi)^2}^{\infty} ds \, \frac{\mathrm{Im} F(s, q^2)}{s - (p+q)^2} = \int\limits_{s_0^B}^{\infty} ds \, \frac{[\mathrm{Im} F(s, q^2)]_{OPE}}{s - (p+q)^2}$$

- subtraction and Borel transform, $\Longrightarrow$ LCSR

$$m_B^2 f_B f^+_{B\pi}(q^2) e^{-m_B^2/M^2} = \int\limits_{m_b^2}^{s_0^B} ds \, e^{-s/M^2} [\mathrm{Im} F(s, q^2)]_{OPE}$$

- fixing s_0^B : acting with $-d/d(1/M^2)$ over both parts and dividing by the same LCSR $\Longrightarrow$ the ratio equals to m_B^2

□ Obtaining the $B \to \pi$ form factors from LCSRs

- the second form factor $f_0(q^2)$ is obtained using the LCSR from the second invariant amplitude $\widetilde{F}$
- universal inputs: $\overline{m}_b$, α_s, $\varphi_\pi^{(t)}(u)$, t=2,3,4; f_B - from two-point (SVZ) sum rule;
- specific inputs: optimal interval of M^2, μ
- uncertainties due to:
  - variation of input parameters,
  - quark-hadron duality

  (suppressed with Borel transformation, controlled by the m_B calculation)
- LCSRs predict *both* “soft-overlap” (dominant !) and “hard-scattering” contributions to the form factors
- the method uses finite m_b , yields $1/m_b$ expansion
- $B_{(s)} \to K$ form factors , including $m_s \neq 0$.

☐ Results for $B_{(s)} \to \pi, K$ form factors

[AK, A.Rusov, 1703.04765]

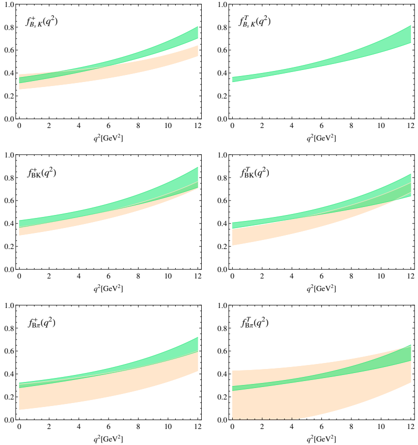


Figure 1. The vector (tensor) form factors of $B_s \to K$, $B \to K$ and $B \to \pi$ transitions calculated from LCSRs including estimated parametrical uncertainties are shown on the upper, middle and lower left (right) panels, respectively, with the dark-shaded (green) bands. Extrapolations of the lattice QCD results for $B_s \to K$ [Fermilab-MILC (2014)], $B \to K$ [HPQCD] and $B \to \pi$ [Fermilab=MILC (2015)] form factors are shown with the light-shaded (orange) bands.

□ LCSR results on $D \to \pi$ form factor

[Ch. Klein, A.K., Th. Mannel, N. Offen (2009)]

- simply replacing *b* quark to *c* quark in the correlation function
- $c \to d$ flavour-changing transitions

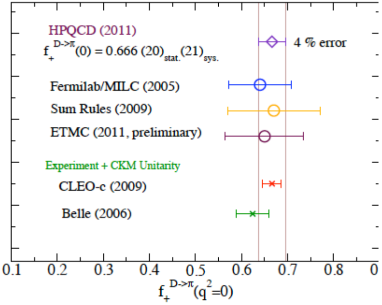


FIG. 6: The $D \to \pi$ form factor $f_+^{D\to\pi}(0)$ from this work an comparisons with other determinations [12, 13, 23–25].

from HPQCD (2011)

□ LCSRs with B-meson distribution amplitudes (DAs)

[A.K., N. Offen, Th. Mannel (2006)]

"SCET sum rules", [F. De Fazio, Th. Feldmann, T.Hurth (2006)]

- vacuum-to-B-correlation function:

B on-shell state

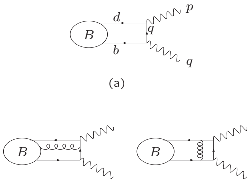
$$F_{ab}(p,q) = i\int d^4x\, e^{ip\cdot x}\langle 0|\mathrm{T}\{\bar{d}(x)\Gamma_b q(x), \bar{q}(0)\Gamma_a b(0)\}|\bar{B}^0(q+p)\rangle = L_{ab}(p,q)F(p^2,q^2)$$

h-interpolating current — transition current

- OPE in terms of B-meson DA's

defined in HQET,

- valid at $0 < q^2 \ll m_B^2$
- dispersion relation:



(a) (b) (c)

$$F(p^2,q^2) = \frac{1}{\pi}\int_{m_h^2}^{\infty}\frac{ds}{s-p^2}\mathrm{Im}F(s,q^2) = F_{OPE}(p^2,q^2)$$

$$\mathrm{Im}F_{ab} = \langle 0|\bar{d}\Gamma_b q|h\rangle\langle h|\bar{q}\Gamma_a b|\bar{B}^0(q+p)\rangle\pi\delta(m_h^2-s) + \ldots = L_{ab}\mathrm{Im}F(s,q^2)$$

$$\boxed{\frac{f_h}{m_h^2-p^2}F^{(B\to h)}(q^2) = \frac{1}{\pi}\int_{m_h^2}^{s_0}\frac{ds}{s-p^2}\mathrm{Im}F_{OPE}(s,q^2)}$$

☐ B-meson DAs

- definition of two-particle DA in HQET:

$$\langle 0|\bar{q}_{2\alpha}(x)[x,0]h_{v\beta}(0)|\bar{B}_v\rangle$$
$$= -\frac{if_B m_B}{4}\int\limits_0^\infty d\omega e^{-i\omega v\cdot x}\left[(1+\not{v})\left\{\phi_+^B(\omega) - \frac{\phi_+^B(\omega)-\phi_-^B(\omega)}{2v\cdot x}\not{x}\right\}\gamma_5\right]_{\beta\alpha}$$

$\oplus$ higher twists

- key input parameter: the inverse moment

$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty d\omega \frac{\phi_+^B(\omega,\mu)}{\omega}$$

  - possible to extract λ_B from $B\to\gamma\ell\nu_\ell$ using QCDF$\oplus$LCSR

    [Y.-M. Wang (2016) , M.Beneke, V.M. Braun, Y.Ji, Y.-B. Wei (2018)],

  - QCD sum rules in HQET: $\lambda_B(1\text{ GeV}) = 380 \pm 150$ MeV

    [V.Braun, D.Ivanov, G.Korchemsky (2004); AK,R.Mandal, Th.Mannel (2021)]

- higher twists DAs [V. Braun, Y. Ji, A. Manashov (2017)]

Uses of LCSRs with B meson DA's

- adjusting the interpolating current to the h state

- $B \to \pi, K, K^*, \rho$ [A.K., T.Mannel, N.Offen (2006)]
- $B \to D, D^*$ [S.Faller,A.K., C.Klein,T.Mannel (2009)]
- NLO corrections to $B \to \pi$ FFs [Y.-M. Wang, Y.-L.Shen (2015)]
- NLO corrections to $B \to D$ FFs [C.-D.Lü, Y.-L. Shen,.Y-M. Wang,Y.-B. Wei (2017)]
- higher twists in OPE, $B \to \pi, K$ [C.-D.Lü,Y.-L. Shen,.Y-M. Wang,Y.-B. Wei (2018)]
- all $B \to \pi, K, D, \rho, K^*, D^*$ form factors in LO
  (higher twists; uncertainties - Bayesian analysis) [N.Gubernari, A.Kokulu, D. van Dyk, (2018)]
- $B \to D^{**}(1^+)$ form factors [N.Gubernari, AK, R.Mandal, Th.Mannel (2022)]

□ Results and comparison ($B \to \pi, K, D$)

from [N.Gubernari, A.Kokulu, D. van Dyk, (2018)]

form factor at $q^2 = 0$	result	literature	DAs	[Ref.]
$f_+^{B\to\pi}$	0.21 ± 0.07	0.258 ± 0.031	π	[Ball,Zwicky 05']
		0.25 ± 0.05	B	[AK,Mannel,Offen 06']
		0.28 ± 0.05	B	[AK,Mannel,Offen,Wang 11
		0.31 ± 0.02	π	[Imsong,AK,Mannel,vanDyk
		0.281 ± 0.038	B	[Wang,Shen 15']
		0.301 ± 0.023	π	[AK, Rusov 17']
$f_T^{B\to\pi}$	0.19 ± 0.06	0.253 ± 0.028	π	[Ball,Zwicky 05']
		0.21 ± 0.04	B	[AK,Mannel,Offen 06']
		0.273 ± 0.021	π	[AK, Rusov 17']
		0.26 ± 0.06	B	[Lü,Shen,Wang,Wei18']
$f_+^{B\to K}$	0.27 ± 0.08	0.331 ± 0.041	K	[DAs, Ball,Zwicky 05']
		0.31 ± 0.04	B	[AK,Mannel,Offen 06']
		0.395 ± 0.033	K	[AK, Rusov 17']
		0.364 ± 0.05	B	[Lü,Shen,Wang,Wei18']
$f_T^{B\to K}$	0.25 ± 0.07	0.358 ± 0.037	K	Ball,Zwicky 05']
		0.27 ± 0.04	B	[AK,Mannel,Offen 06']
		0.381 ± 0.027	K	[AK, Rusov 17']
		0.363 ± 0.08	B	[Lü,Shen,Wang,Wei18']
$f_+^{B\to D}$	0.65 ± 0.08	0.69 ± 0.2	B	[Faller,AK,Klein,Mannel, 08
		0.673 ± 0.063	B	[Wang,Wei, Lü, Shen,17']
$f_T^{B\to D}$	0.57 ± 0.05	—	B	

□ Semileptonic transitions to di-mesons

- a practical problem: to assess "nonresonant" background in $B \to \pi\pi\ell\nu_\ell$ or $B \to K\pi\ell\ell$
- in the theory language:
  - use general $B \to \pi\pi$ form factors:

$$\langle \pi^+(k_1)\pi^0(k_2)|\bar{u}\gamma^\mu(1-\gamma_5)b|\bar{B}^0(p)\rangle = -F_\perp(q^2,k^2,\zeta)\,\frac{4}{\sqrt{k^2\lambda_B}}\,i\epsilon^{\mu\alpha\beta\gamma}\,q_\alpha\,k_{1\beta}\,k_{2\gamma} + ...$$

$$(2\zeta-1) = (1-4m_\pi^2/k^2)^{1/2}cos\theta_\pi\,,\ \text{in dipion c.m.}$$

  - expand in partial waves, isolate dipion P-wave $F_\perp(q^2,k^2,\zeta) \Rightarrow F_\perp^{(\ell=1)}(q^2,k^2)$
  - hadronic dispersion relation in dipion invariant mass

☐ Dispersion relation for the $B \to \pi\pi$ vector FF

- three-resonance ansatz:

$$\frac{\sqrt{3}F_{\perp}^{(\ell=1)}(q^2,k^2)}{\sqrt{k^2}\sqrt{\lambda_B}} = \frac{g_{\rho\pi\pi}}{m_\rho^2 - k^2 - i m_\rho \Gamma_\rho(k^2)} \frac{V^{B\to\rho}(q^2)}{m_B + m_\rho}$$
$$+ \frac{g_{\rho'\pi\pi}}{m_{\rho'}^2 - k^2 - i m_{\rho'} \Gamma_{\rho'}(k^2)} \frac{V^{B\to\rho'}(q^2)}{m_B + m_{\rho'}} +$$
$$+ \frac{g_{\rho''\pi\pi}}{m_{\rho''}^2 - k^2 - i m_{\rho''} \Gamma_{\rho''}(k^2)} \frac{V^{B\to\rho''}(q^2)}{m_B + m_{\rho''}} + \ldots$$

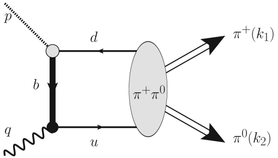
- inspired by the timelike pion e.m. form factor in $e^+e^- \to \pi^+\pi^-$ or in $\tau \to \pi^-\pi^0\nu_\tau$: modelled at $\sqrt{k^2} \lesssim 1.5$ GeV to a sum of ρ,$\rho'(1450)$, $\rho''(1750)$

- calculate $B \to \pi\pi$ or $B \to K\pi$ form factors with QCD methods $\rho, \rho', \ldots$ or K^*,... have to be "embedded" in this calculation

- model-dependence of the input is unavoidable

□ Use of LCSRs with dipion distribution amplitudes

[Ch. Hambrock, AK, (2015)]

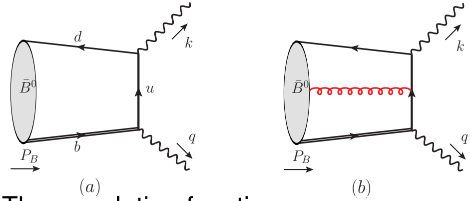


- consider $\bar{B}^0 \to \pi^+\pi^0\ell^-\nu_\ell$, isospin 1, $L = 1, 3, , ..$
- vacuum $\to$ dipion correlation function
- nonperturbative input: dipion distribution amplitudes (DAs)
- introduced and developed for $\gamma^*\gamma \to 2\pi$ processes
  [M. Diehl, T. Gousset, B. Pire and O. Teryaev, (1998),
  D. Müller, D. Robaschik, B. Geyer, F.-M. Dittes and J. Horejsi, (1994),
  M. V. Polyakov, (1999)]
- only LO, twist-2 approximation for dipion DAs available
- DAs model available only at small $k^2 \sim 4m_\pi^2$
- problems addressed:
  - how important are $L > 1$ partial waves of 2π state in $B \to \pi\pi\ell\nu_\ell$?
  - comparison with $B \to \rho$ FFs calculated from LCSRs with narrow ρ DAs

□ Applying LCSRs with B-meson distribution amplitudes

[S.Cheng, AK, J.Virto, (2017)]

- LCSRs with B-meson DA and $\bar{u}\gamma_\mu d$ interpolating current



- The correlation function:

$$F_{\mu\nu}(k,q) = i\int d^4x e^{ik\cdot x}\langle 0|\mathrm{T}\{\bar{d}(x)\gamma_\mu u(x), \bar{u}(0)\gamma_\nu(1-\gamma_5)b(0)\}|\bar{B}^0(q+k)\rangle,$$

$$= \varepsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma F_{(\varepsilon)}(k^2,q^2) + ig_{\mu\nu}F_{(g)}(k^2,q^2) + iq_\mu k_\nu F_{(qk)}(k^2,q^2) + ...$$

□ Accessing $B \to \pi\pi$ form factors

- OPE diagrams ⇒ invariant amplitudes ⇒ dispersion form in k^2:

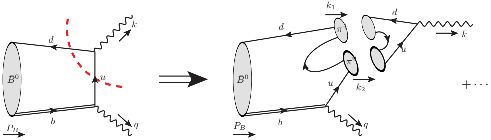
$$F^{\rm OPE}_{(\varepsilon)}(k^2,q^2) = f_B m_B \int_0^\infty d\sigma \, \frac{\phi_+^B(\sigma m_B)}{\bar\sigma(s-k^2)} + \{3-\textit{particle DAs}\}$$

$$s = s(\sigma,q^2) = \sigma m_B^2 - \sigma q^2/\bar\sigma \,, \qquad \bar\sigma \equiv 1-\sigma$$

- hadronic dispersion.relation and unitarity:

$$F_{(\varepsilon)}(k^2,q^2) = \frac{1}{\pi} \int\limits_{4m_\pi^2}^{\infty} ds \, \frac{{\rm Im} F_{(\varepsilon)}(s,q^2)}{s-k^2} \,.$$

$$2\,{\rm Im} F_{\mu\nu}(k,q) = \int d\tau_{2\pi} \underbrace{\langle 0|\bar d\gamma_\mu u\,|\pi^+\pi^0\rangle}_{F_\pi(s)} \underbrace{\langle \pi^+\pi^0|\bar u\gamma_\nu(1-\gamma_5)b|\bar B^0(q+k)\rangle}_{B\,\to\,2\pi\ (\ell\,=\,1)\ \text{form factors}} + \cdots \,,$$



□ Resulting sum rules

- e,g,. for the form factor $F_{\perp}^{(\ell=1)}$ of the vector current

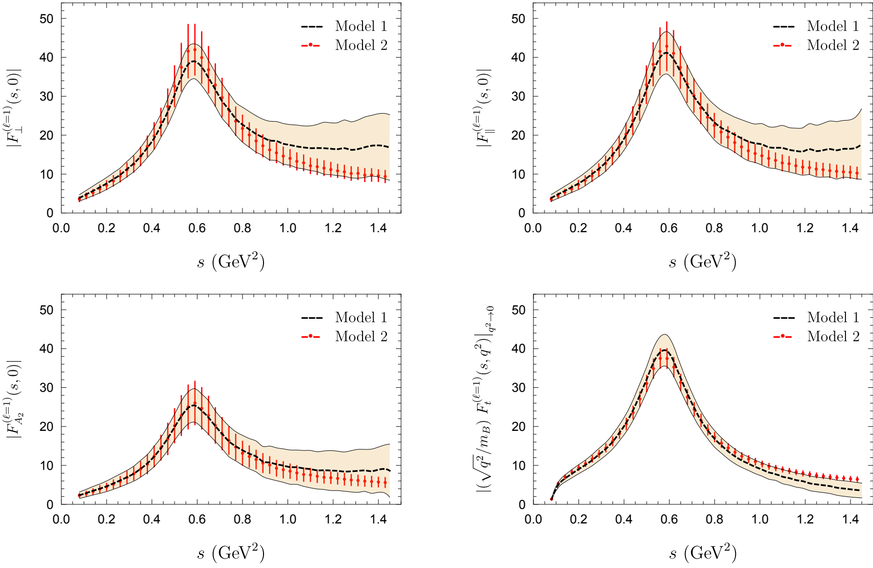
$$\int_{4m_\pi^2}^{s_0^{2\pi}} ds\, e^{-s/M^2} \frac{\sqrt{s}\,[\beta_\pi(s)]^3}{4\sqrt{6}\pi^2\sqrt{\lambda}}\, F_\pi^\star(s)\, F_\perp^{(\ell=1)}(s,q^2)$$

$$= \quad f_B m_B \left[\int_0^{\sigma_0^{2\pi}} d\sigma\; e^{-s(\sigma,q^2)/M^2}\, \frac{\phi_+^B(\sigma m_B)}{\bar{\sigma}} + m_B\, \Delta V^{BV}(q^2, \sigma_0^{2\pi}, M^2) \right],$$

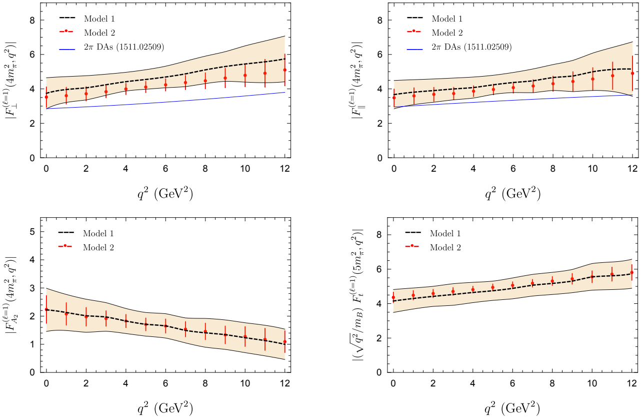
$\sigma_0^{2\pi}$ - the solution of $\sigma m_B^2 - \sigma q^2/\bar{\sigma} = s_0^{2\pi}$, three-particle DA contribution ΔV^{BV}

- similar sum rules for all other P-wave $B \to 2\pi$ form factors
- not a direct calculation, given the shape of the $B \to 2\pi$ form factors, these sum rules can provide normalization
- probing two different ρ-resonance models for the $B \to \pi\pi$ FF $\Rightarrow$ an appreciable contribution of ρ' (up to 20% of ρ in residue) is consistent with the LCSRs

□ $B \to 2\pi$ ($\ell = 1$) FFs: dipion mass dependence



□ $B \to 2\pi$ ($\ell = 1$) FFs: q^2-dependence at small k^2



- extension of the method to $B \to K\pi(J^P = 1^-, 0^+)$ form factors

[S.Descotes-Genon, AK, J.Virto, (2019)], [S.Descotes-Genon, AK, J.Virto, K.Vos, (2023)]

Summary

- two main versions of LCSRs for $B \to h$ form factors: with light-hadron DAs and with B-meson DAs.
  complement each other and results mutually agree within uncertainties
- LCSRs provide a variety of $B \to h$ form factors at large recoil of h,
  support lattice QCD extrapolation with independent estimates
- LCSRs provide probes of resonance models for the full $B \to \pi\pi, K\pi$ form factors,
- future perspectives:
  - the accuracy of lattice QCD calculation already in the nearest future cannot be achieved by QCD sum rules and LCSRs
  - but: there are hadronic matrix elements where even a 30-40% accuracy would be sufficient, and they are not yet accessible on the lattice

More details in these reviews:

- M. A. Shifman,
  Snapshots of hadrons or the story of how the vacuum medium determines the properties of the classical mesons which are produced, live and die in the QCD vacuum,
  arXiv:hep-ph/9802214 [hep-ph].
- V. M. Braun,
  QCD sum rules for heavy flavors,
  arXiv:hep-ph/9911206
- P. Colangelo and A. Khodjamirian,
  QCD sum rules, a modern perspective,
  arXiv:hep-ph/0010175
- A. Khodjamirian,
  Quantum chromodynamics and hadrons: An Elementary introduction,
  (lectures at European School on High Energy Physics (2003))
  arXiv:hep-ph/0403145.

even more details in this book: