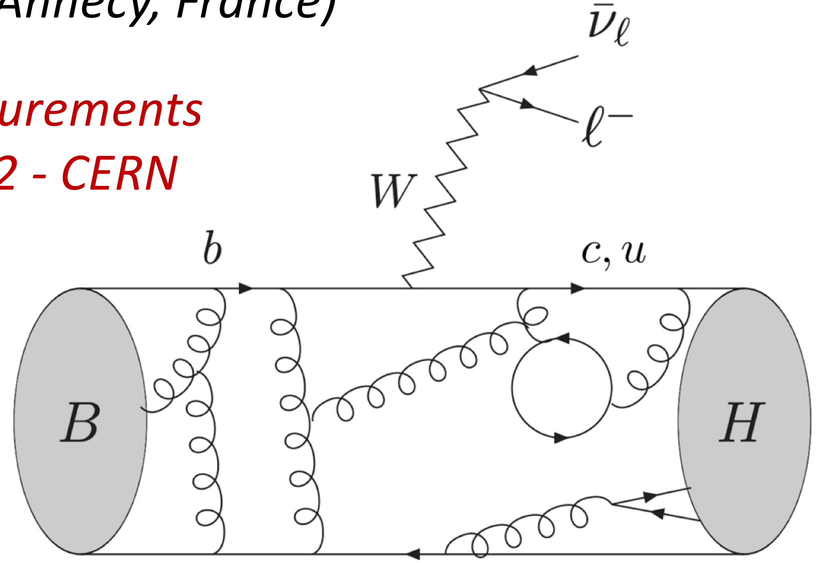
The Dispersive Matrix approach and exclusive $|V_{ub}|$

Work in collaboration with G. Martinelli and S. Simula

[PRD ’21 (2105.02497), JHEP ‘22 (2202.10285), …]

Ludovico Vittorio (LAPTh & CNRS, Annecy, France)

Implications of LHCb measurements and future prospects 2022 - CERN



(from J.Phys.G 46 (2019) 2, 023001)

State-of-the-art of the semileptonic heavy-to-light B decays

EXCLUSIVE *INCLUSIVE*

- V_{ub} puzzle: $$|V_{ub}| \times 10^3 = 3.74(17)$$ ***VS***

FLAG Review 2021 [EPJC '22 (2111.09849)]

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VS

Lot of averaged values:

$$|V_{ub}|_{incl} \cdot 10^3 = 4.19(12)\left(^{+0.11}_{-0.12}\right)$$

HFLAV Coll. [arXiv:2206.07501]

$$|V_{ub}|_{incl} \cdot 10^3 = 4.32\,(29)$$

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$\sim 1.5 - 2\,\sigma$ difference

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Although the difference is only about 1.5 - 2σ, in view of what happens in the case of V_{cb} it is important to address the problem of an accurate determination of V_{ub} from the relevant exclusive channels

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To this end, a central role is played by the hadronic Form Factors (FFs), which enter in the differential decay widths:

$$\frac{d\Gamma(B_{(s)} \to \pi(K)\ell\nu_\ell)}{dq^2} = \frac{G_F^2|V_{ub}|^2}{24\pi^3}\left(1-\frac{m_\ell^2}{q^2}\right)^2\left[|\vec{p}_{\pi(K)}|^3\left(1+\frac{m_\ell^2}{2q^2}\right)|f_+^{\pi(K)}(q^2)|^2\right.$$
$$\left. + \, m_{B_{(s)}}^2|\vec{p}_{\pi(K)}|\left(1-r_{\pi(K)}^2\right)^2\frac{3m_\ell^2}{8q^2}|f_0^{\pi(K)}(q^2)|^2\right],$$

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Lattice QCD (LQCD) simulations can determine the FFs ONLY at high values of momentum transfer…

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

- Pioneering works from S. Okubo [PRD, 3 (1971); PRD, 4 (1971)], C. Bourrely et al [NPB, 189 (1981)] and L. Lellouch [NPB, 479 (1996)]
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The resulting description of the FFs

- **is entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **is independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

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How does it work?

The DM method

Let us focus on a generic FF *f*: we can define

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \ldots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \ldots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \ldots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \ldots & \frac{1}{1-z_2 z_N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \ldots & \frac{1}{1-z_N^2} \end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \ (\text{with } i = 1, 2, \ldots N)$$

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_{B(s)} \pm m_{\pi(K)})^2$$

t: momentum transfer

The DM method

Non-perturbative values of the susceptibilities from the dispersion relations (see PRD '21 (2105.07851) and JHEP '22 (2202.10285))

Estimates of the FFs, computed on the lattice

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \dots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \dots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \dots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \dots & \frac{1}{1-z_2 z_N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \dots & \frac{1}{1-z_N^2} \end{pmatrix}$$

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One can show that

$$\det \mathbf{M} \geq 0$$

$$\Downarrow$$

$$f_{\rm lo}(z) \leq f(z) \leq f_{\rm up}(z)$$

Values of the momentum transfer @ which FFs are computed on the lattice

A sketch of the calculation of the susceptibilities

In Appendix A of JHEP '22 (2202.10285), we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to u$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

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How are they defined? The starting point is the HVP tensor:

$$\begin{aligned}\Pi_V^{\mu\nu}(Q) &= \int d^4x\, e^{-iQ\cdot x} \, \langle 0| \, T\{\bar{b}(x)\gamma_\mu^E u(x)\bar{u}(0)\gamma_\nu^E b(0)\} \, |0\rangle \\ &= (\delta^{\mu\nu}Q^2 - Q^\mu Q^\nu)\, \Pi_{1^-}(Q^2) - Q^\mu Q^\nu \, \Pi_{0^+}(Q^2)\,,\end{aligned}$$

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To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;,$$

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$$C_{0^+}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0 u(x)\;\bar{u}(0)\gamma_0 b(0)\right]|0\rangle\;,$$

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A sketch of the calculation of the susceptibilities

The possibility to compute the χs on the lattice allows us to choose *whatever value of Q^2 !* (i.e. near the region of production of the resonances)

NOT POSSIBLE IN PERTURBATION THEORY since

$$(m_b + m_u)\Lambda_{QCD} \ll (m_b + m_u)^2 + Q^2$$

POSSIBLE IMPROVEMENT IN THE STUDY OF THE FFs through our method

Work in progress...

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Non-perturbative computation of the susceptibilities

Let us choose for the moment zero Q^2:

$$\chi_{0^+}(Q^2=0) = \int_0^\infty dt\ t^2\ C_{0^+}(t)\ ,$$

$$\chi_{1^-}(Q^2=0) = \frac{1}{12}\int_0^\infty dt\ t^4\ C_{1^-}(t)\ ,$$

$$\chi_{0^-}(Q^2=0) = \int_0^\infty dt\ t^2\ C_{0^-}(t)\ ,$$

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$$\chi_{0^+}(Q^2=0) = \frac{1}{12}(m_b-m_u)^2\int_0^\infty dt\ t^4\ C_S(t)$$

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$$C_{0^+}(t) = \boxed{\widetilde{Z}_V^2}\int d^3x \langle 0|T\left[\bar{q}_1(x)\gamma_0 q_2(x)\ \bar{q}_2(0)\gamma_0 q_1(0)\right]|0\rangle\ ,$$

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We are working in twisted mass LQCD: the Wilson parameter *r* can be equal or opposite for the two quarks in the currents

→ Two possible **independent** combinations of (r_1, r_2)!

Z: appropriate renormalization constants

N. Carrasco et al. [ETM Coll.], NPB 887 (2014) [arXiv:1403.4504]

Non-perturbative computation of the susceptibilities

Following set of masses:

$$m_h(n) = \lambda^{n-1}\ m_c^{phys} \qquad \text{for } n = 1, 2, \ldots$$

$$m_h = a\mu_h/(Z_P a)$$

$$\lambda \equiv [m_b^{phys}/m_c^{phys}]^{1/10} = [5.198/1.176]^{1/10} \simeq 1.1602$$

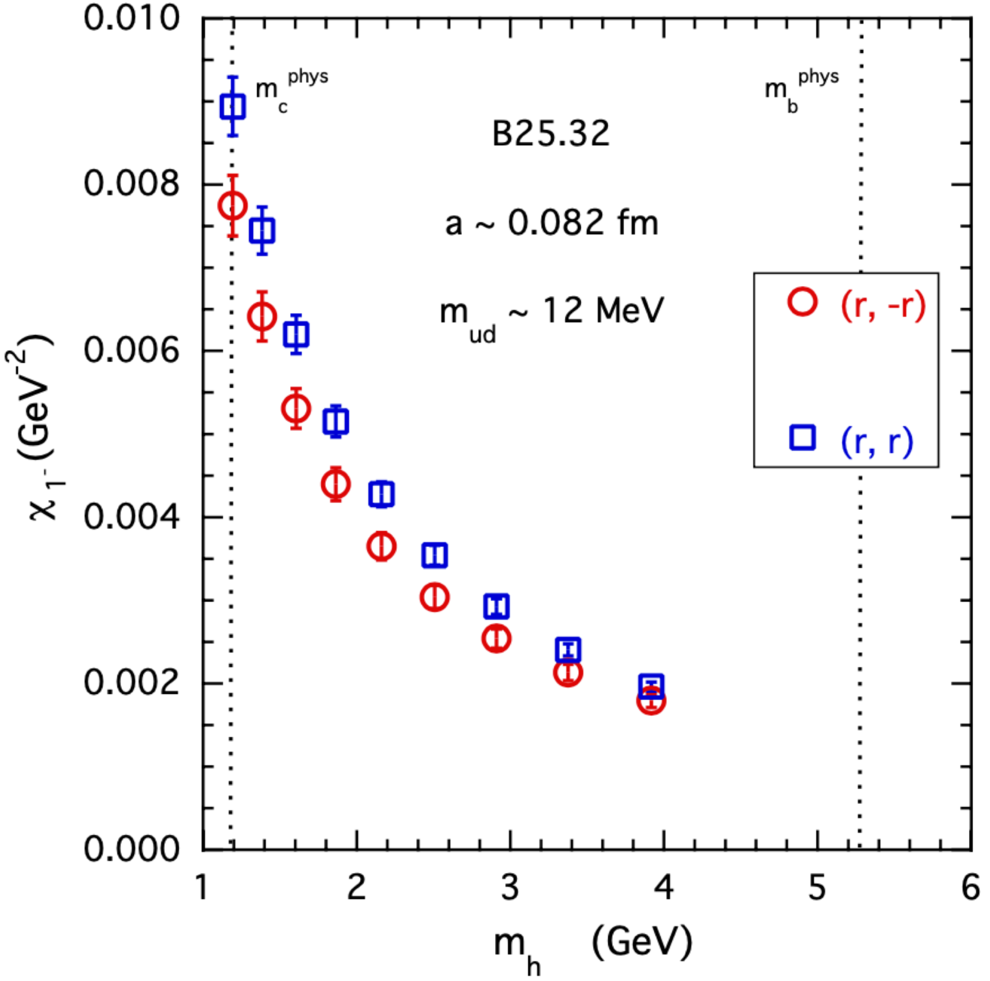
Nine masses values!

$$m_h(1) = m_c^{phys}$$

$$m_h(9) \simeq 3.9\ \text{GeV} \simeq 0.75\ m_b^{phys}$$

***r*: Wilson parameter**

Non-perturbative computation of the susceptibilities



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Nine masses values!

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$$m_h(9) \simeq 3.9 \text{ GeV} \simeq 0.75\, m_b^{phys}$$

***r*: Wilson parameter**

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method in ***JHEP '10 [0909.3187]:***

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \, \frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}$$

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to ensure that $\lim_{n\to\infty} R_j(n) = 1$

$$\rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1 \ ,$$

$$\rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2$$

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All the details are deeply discussed in ***PRD '21 (2105.07851)*** and ***JHEP '22 (2202.10285)***. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-light transition current densities:**

b → u

	Non-perturbative	With subtraction
$\chi_{V_L}\,[10^{-2}]$	2.04(20)	—
$\chi_{A_L}\,[10^{-2}]$	2.34(13)	—
$\chi_{V_T}\,[10^{-4}\ \mathrm{GeV}^{-2}]$	4.88(1.16)	4.45(1.16)
$\chi_{A_T}\,[10^{-4}\ \mathrm{GeV}^{-2}]$	4.65(1.02)	—

Consistency with the estimate using perturbative QCD (with small contributions from quark and gluon condensates):

$$\chi_{1^-}(m_b^{phys}) = 5.01 \cdot 10^{-4}\ \mathbf{GeV}^{-2}$$

Bourrely, Caprini and Lellouch, PRD '09 [0807.2722]

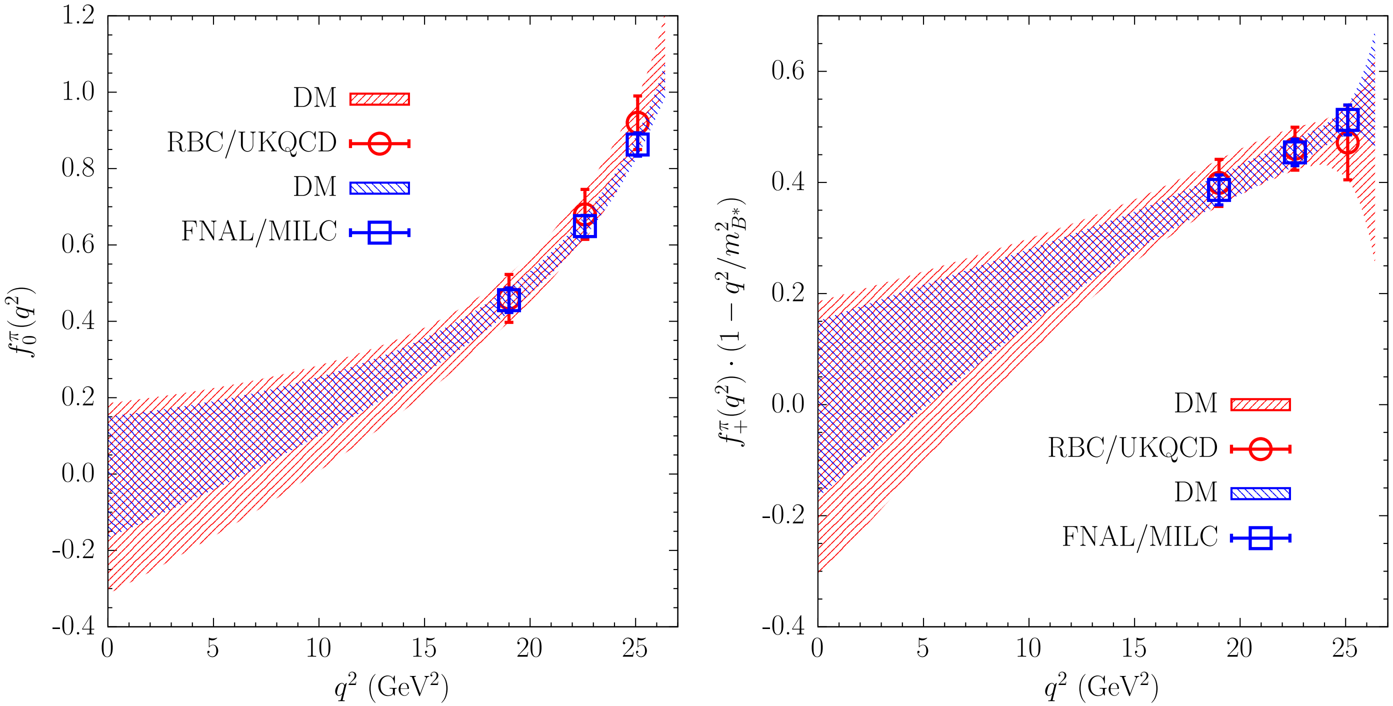
All this machinery can also be applied to heavy-to-heavy transition current densities…

DM applied to semileptonic B → π decays

Two LQCD inputs have been used for our DM method (**JHEP '22 [arXiv:2202.10285]**):

- **3 RBC/UKQCD synthetic data (points) [PRD '15 (1501.05363)]**
- **3 FNAL/MILC data (squares) from their fits [PRD '15 (1503.07839)]**

One KC: $f_0(0) = f_+(0)$



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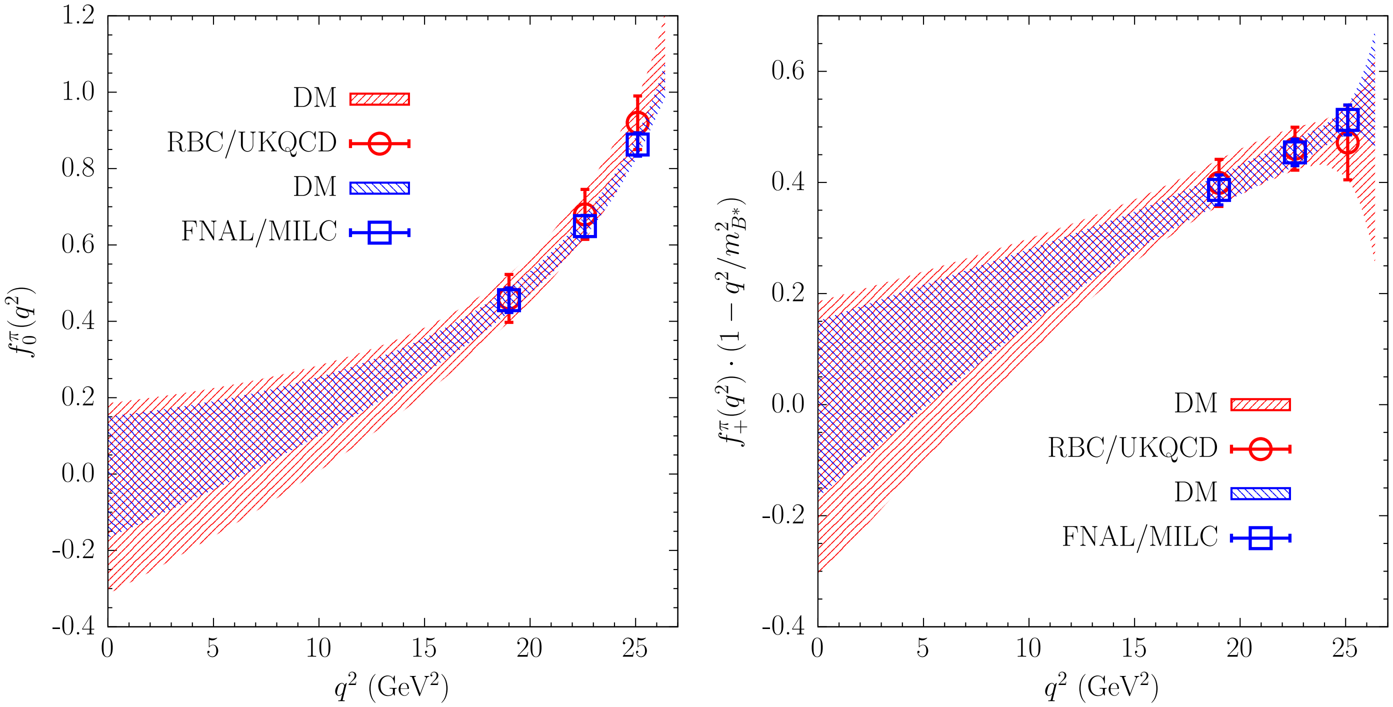
$$f^\pi(q^2=0)|_{\rm RBC/UKQCD} = -0.06 \pm 0.25$$

$$f^\pi(q^2=0)|_{\rm FNAL/MILC} = -0.01 \pm 0.16$$

$$f^\pi(q^2=0)|_{\rm combined} = -0.04 \pm 0.22$$

Peculiarity of B → π decays: LONG extrapolation in q^2



It seems that the mean value and the uncertainty are not stable under variation of the truncation order of a series expansion of the FFs in *z* *[see back-up slides]*…

The DM approach is independent of this issue!!!

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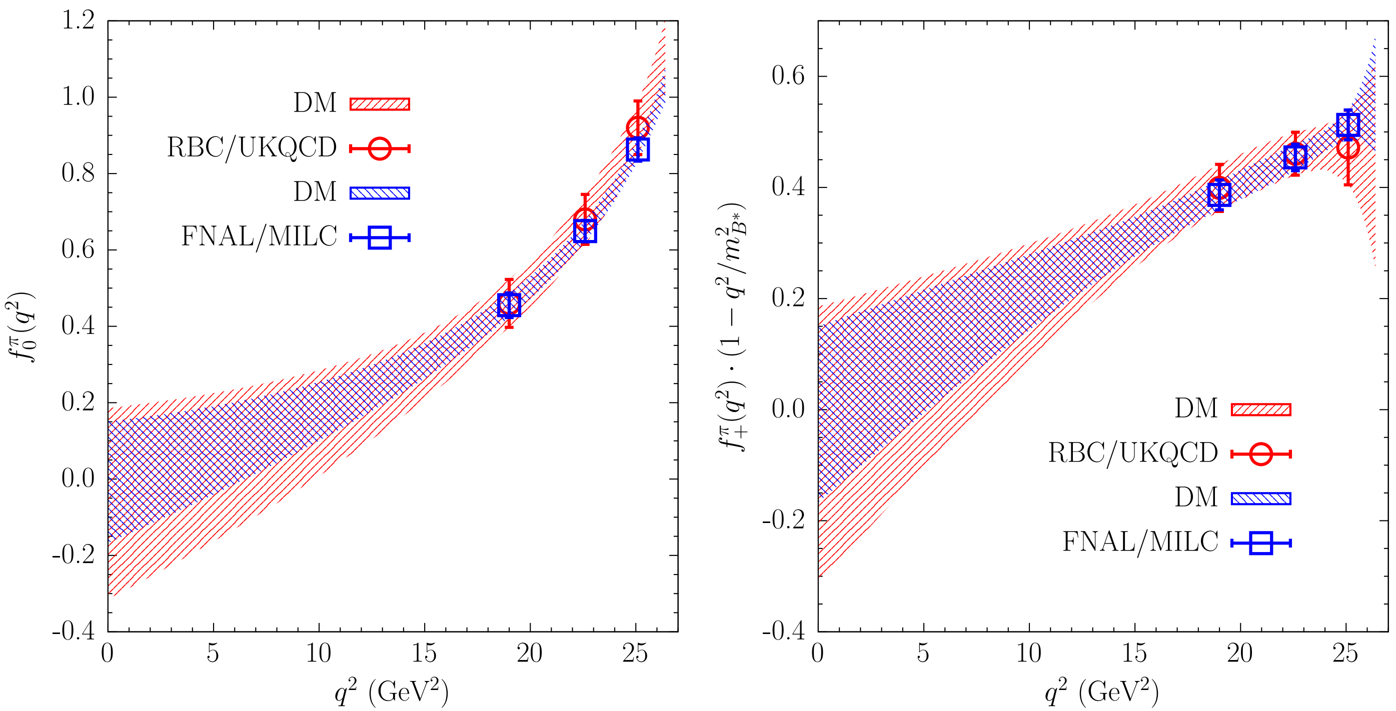

Important issue: *the DM method equivalent to the results of **all** possible (BCL) fits which satisfy unitarity and at the same time reproduce exactly the input data*

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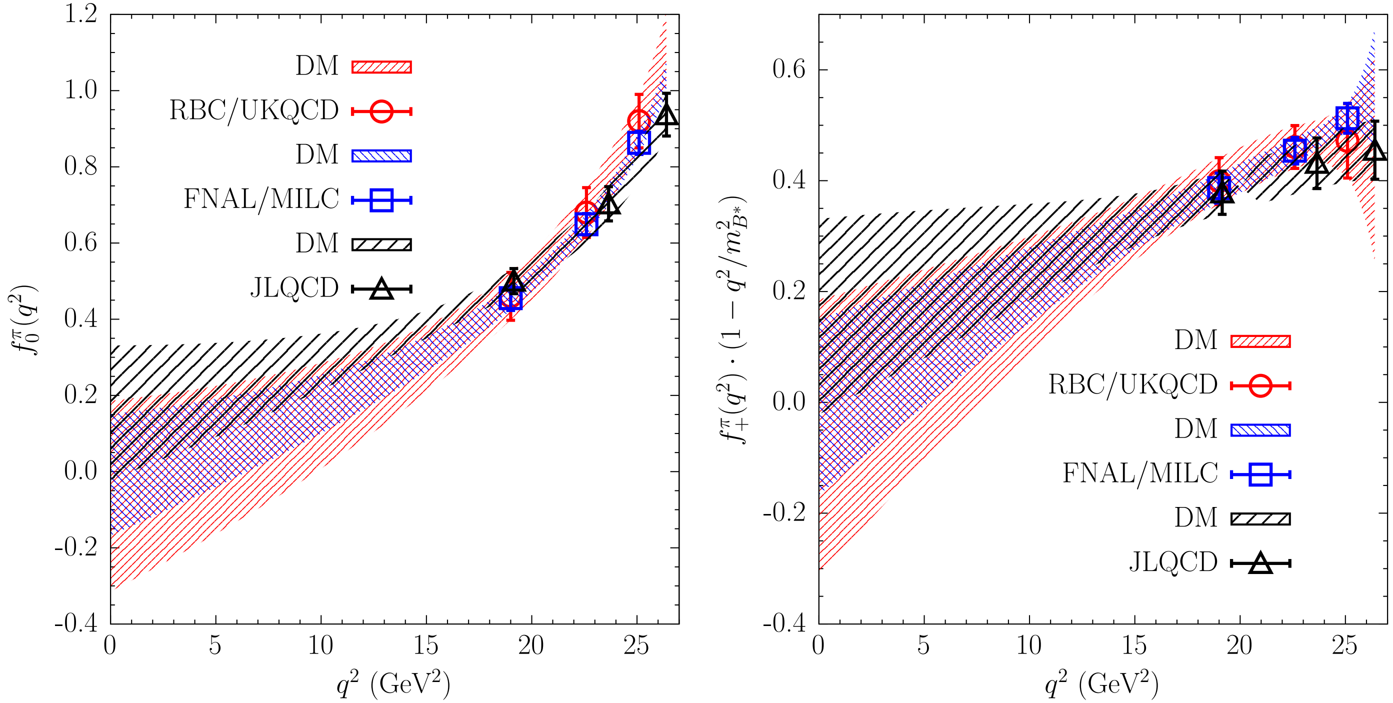
IMPORTANT: new LQCD computations published by JLQCD Collaboration (PRD '22 [2203.04938])!



Some differences in the slopes with respect to the RBC/UKQCD and the FNAL/MILC cases, although the extrapolations at zero momentum transfer are compatible to each other:

$$f^{\pi}(q^2=0)|_{\rm JLQCD} = 0.155 \pm 0.176$$

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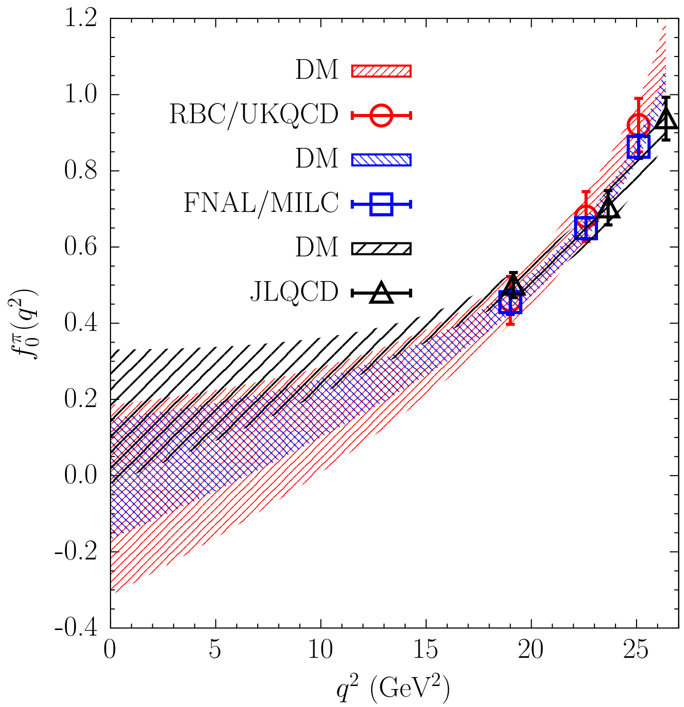

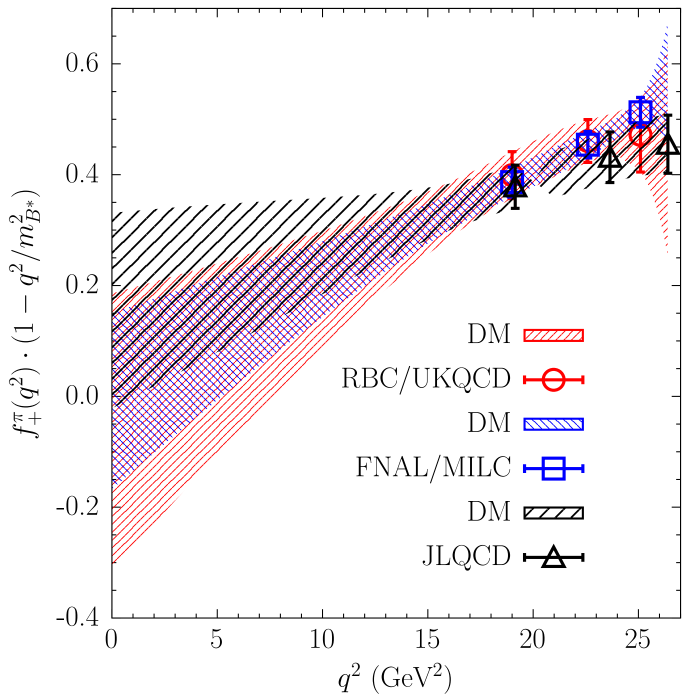

*We are going to update our analysis together with the **new** LQCD computation of the FFs by **RBC/UKQCD Collaboration**, see for instance **PoS LATTICE2021 (2022) 306, [2112.10580]***

LFU in semileptonic B → π decays

The extrapolation of the FFs at zero momentum transfer is of capital importance to test LFU:

$$R_\pi^{\tau/\mu} \equiv \frac{\Gamma(B \to \pi\tau\nu_\tau)}{\Gamma(B \to \pi\mu\nu_\mu)}$$

THEORY with DM method

Input	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767(145)	0.838(75)	0.793(118)

EXPERIMENT

$$R_\pi^{\tau/\mu}|_{exp} = 1.05 \pm 0.51$$

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Expected improved precision @ Belle II (PTEP '19 (1808.10567))

$$\delta R_\pi^{\tau/\mu} \simeq 0.09$$

~80% reduction of the error!

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Expected improved precision in LQCD computations of the FFs @ high momentum transfer

Input	RBC/UKQCD	FNAL/MILC	combined
$\delta R_\pi^{\tau/\mu}$	0.73	0.38	0.59

Hypothetical 50% reduction of the error...

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For further investigation of possible NP effects in the future, it is fundamental to extrapolate appropriately the FFs behaviour in the whole kinematical range

$|V_{ub}|$ from semileptonic B → π decays

Six sets of data from Belle and BaBar collaborations:

BaBar 2011, 1 channel [PRD ‘11 (1005.3288)]

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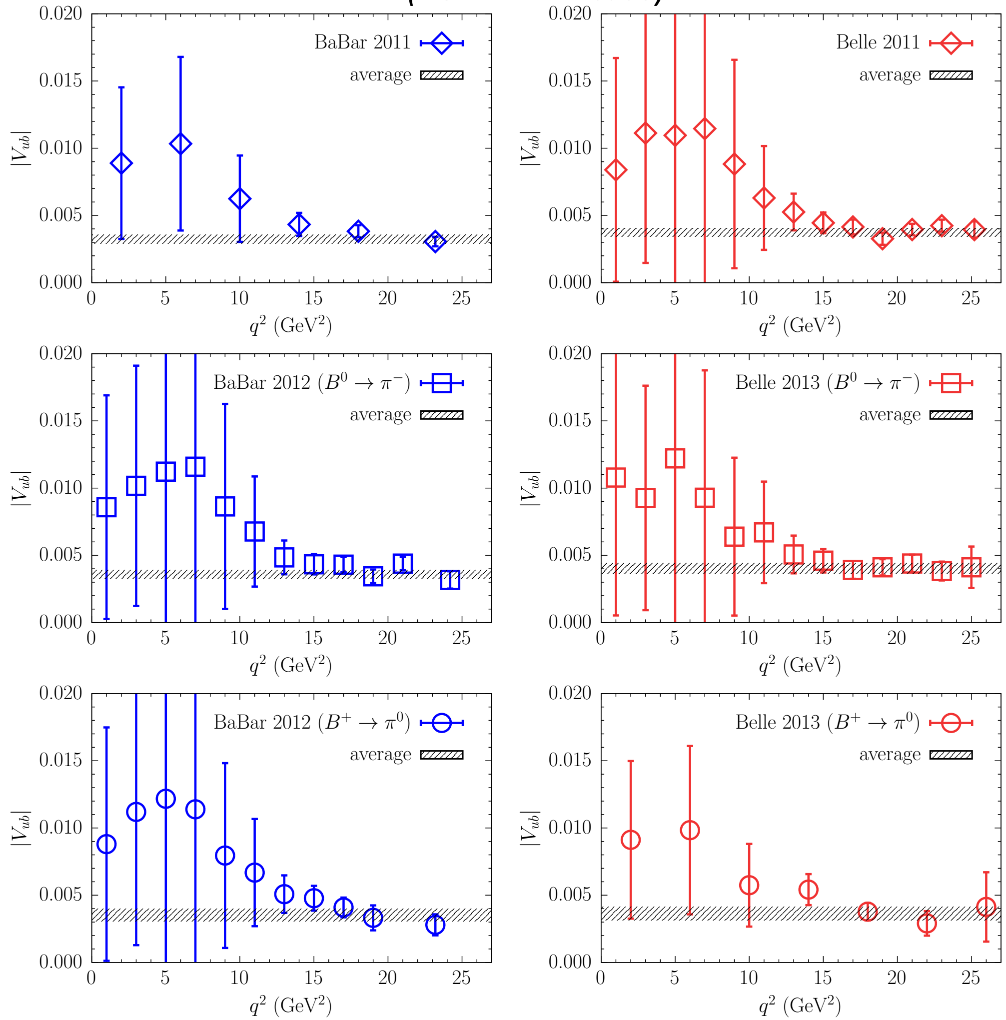
Isospin factor — Exper. data

$$|V_{ub}|_i = \sqrt{\frac{C_v}{\tau_{B^v}} \cdot \frac{\Delta\mathcal{B}|_i^{exp}}{\Delta\zeta_i}}$$

B0/B+ meson lifetime — Theor. decay width

(COMBINED case)



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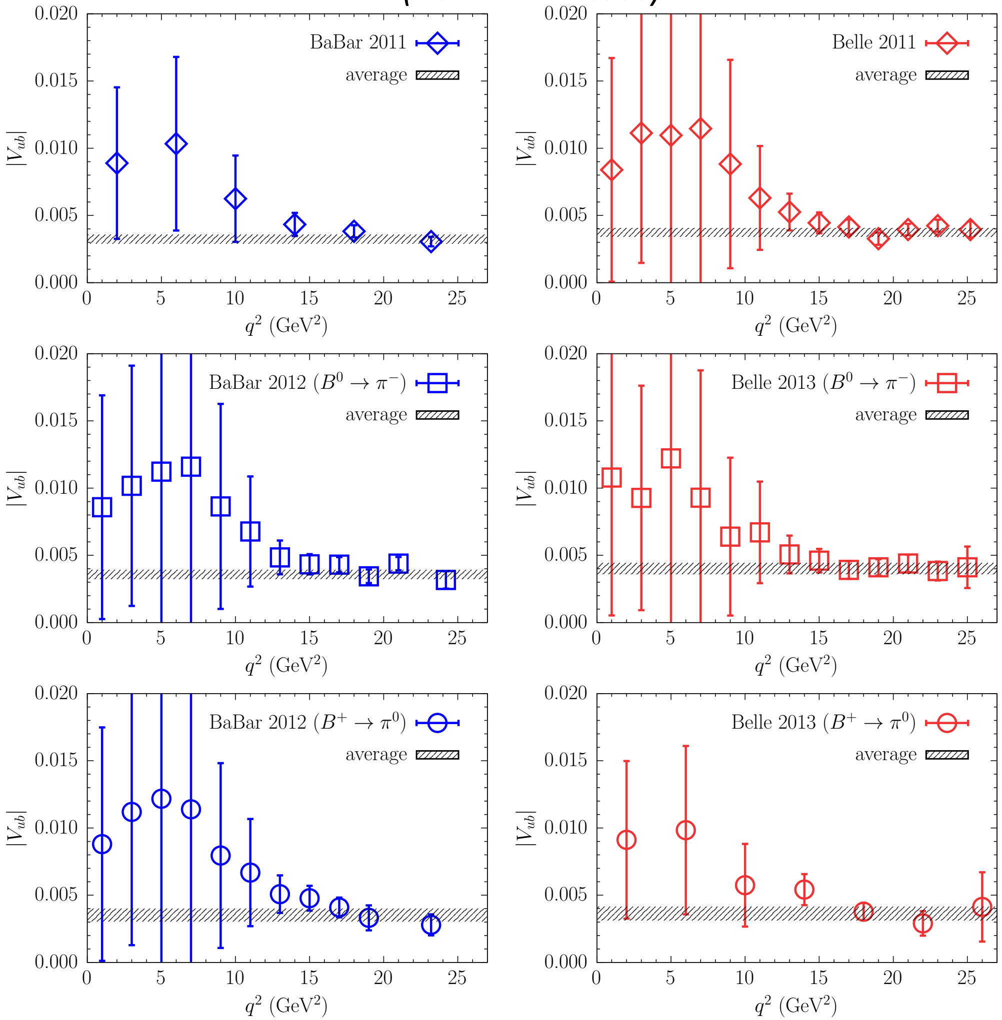

The bands are the results of correlated weigthed averages:

$$|V_{ub}|_n = \frac{\sum_{i,j}(\mathbf{C}^{-1})_{ij}|V_{ub}|_j}{\sum_{i,j}(\mathbf{C}^{-1})_{ij}}, \qquad \sigma^2_{|V_{ub}|_n} = \frac{1}{\sum_{i,j}(\mathbf{C}^{-1})_{ij}}$$

$|V_{ub}|$ from semileptonic $B \to \pi$ decays

DM results: RBC/UKQCD, FNAL/MILC, Combined

Input	$\lvert V_{ub}\rvert \times 10^3$
RBC/UKQCD	**3.52(49)**
FNAL/MILC	**3.76(41)**
Combined	**3.62(47)**
PDG exclusive [PTEP 2020, 083C01]	3.70(16)
FLAG '21 exclusive [EPJC '22 (2111.09849)]	3.74(17)
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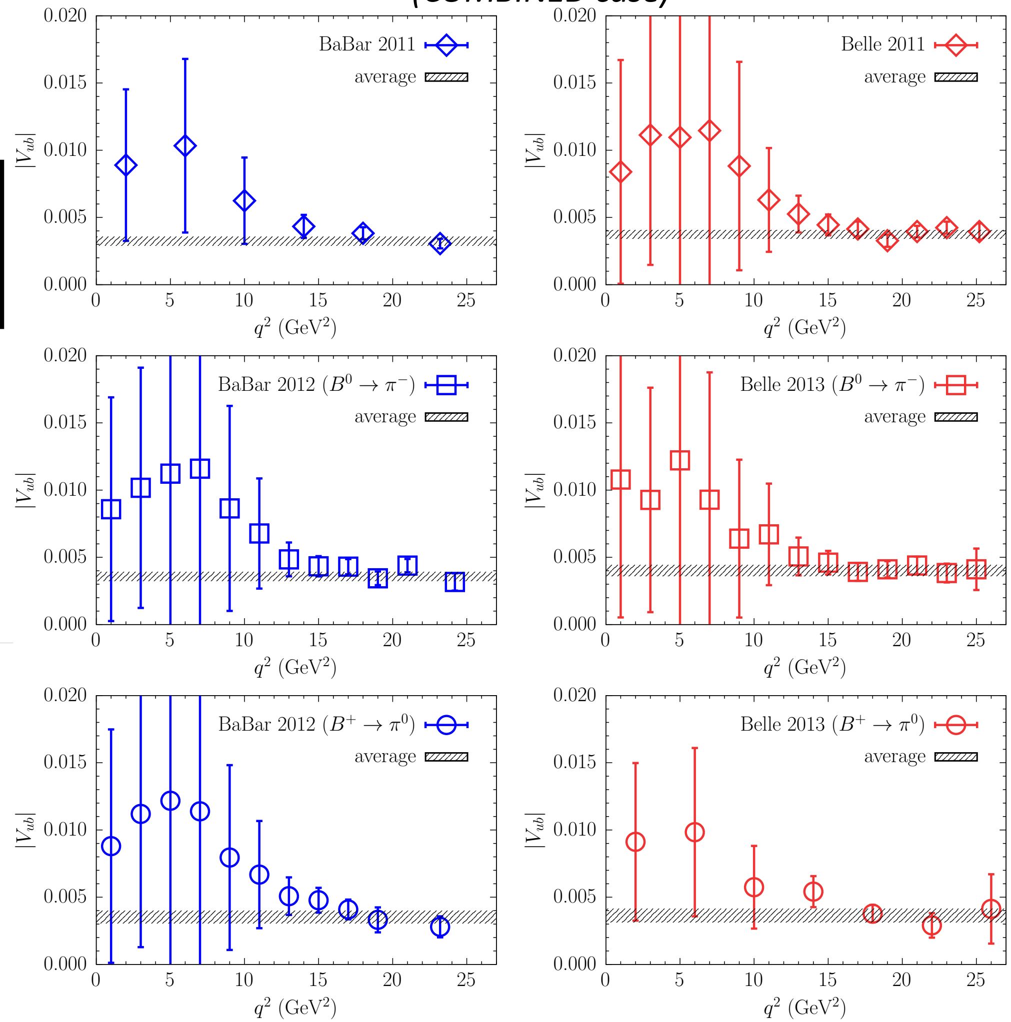

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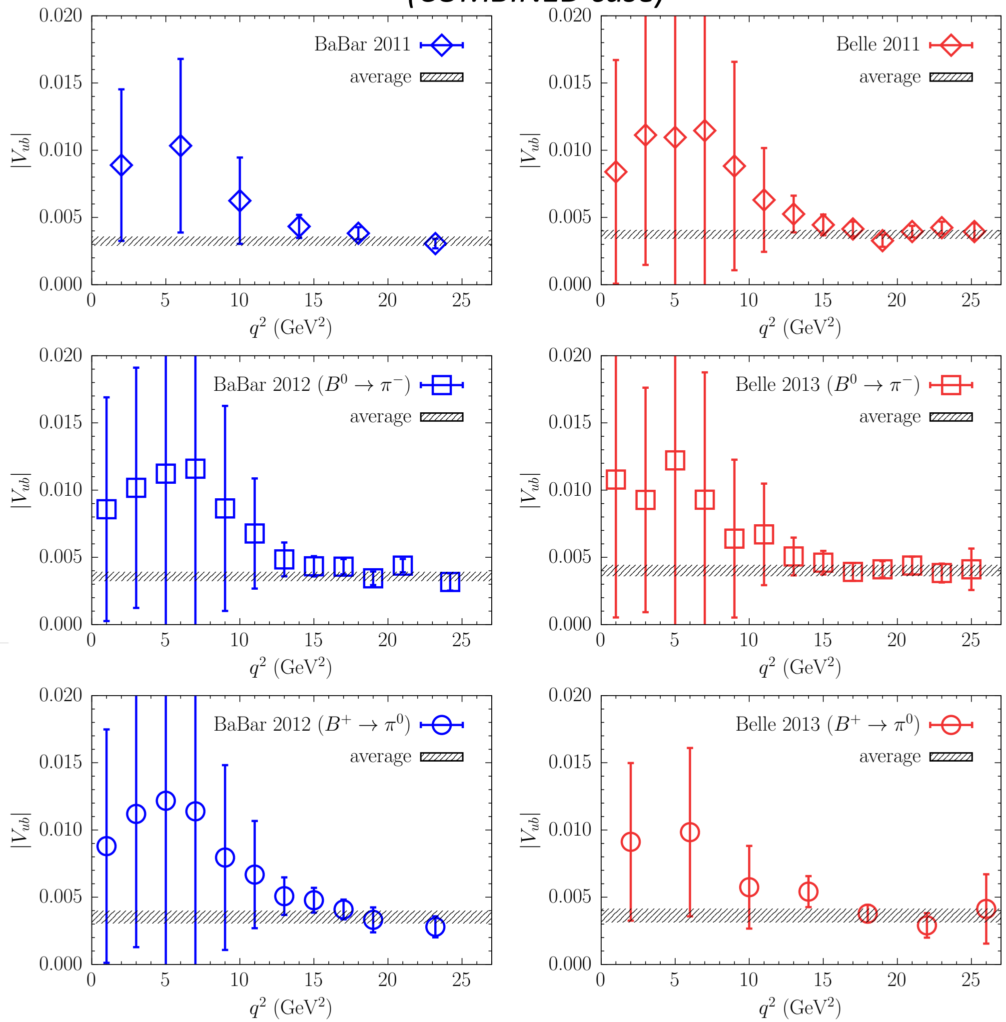

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*We are going to update our analysis with the **new** measurements of the differential decay widths by **Belle II Collaboration**, see for instance **arXiv:2210.04224***

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DM applied to semileptonic $B_s \to K$ decays & phenomenology

Three LQCD inputs have been used (**JHEP '22 [arXiv:2202.10285]**):

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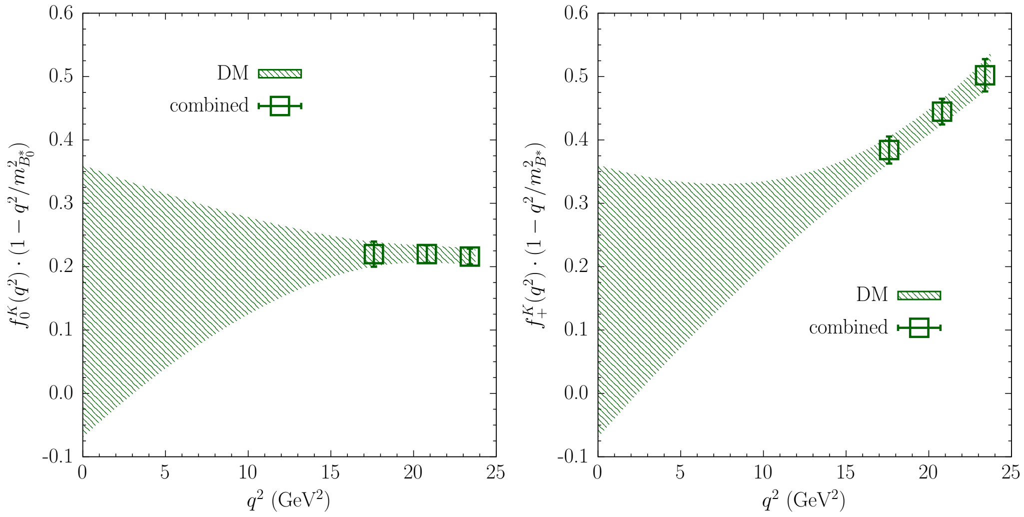

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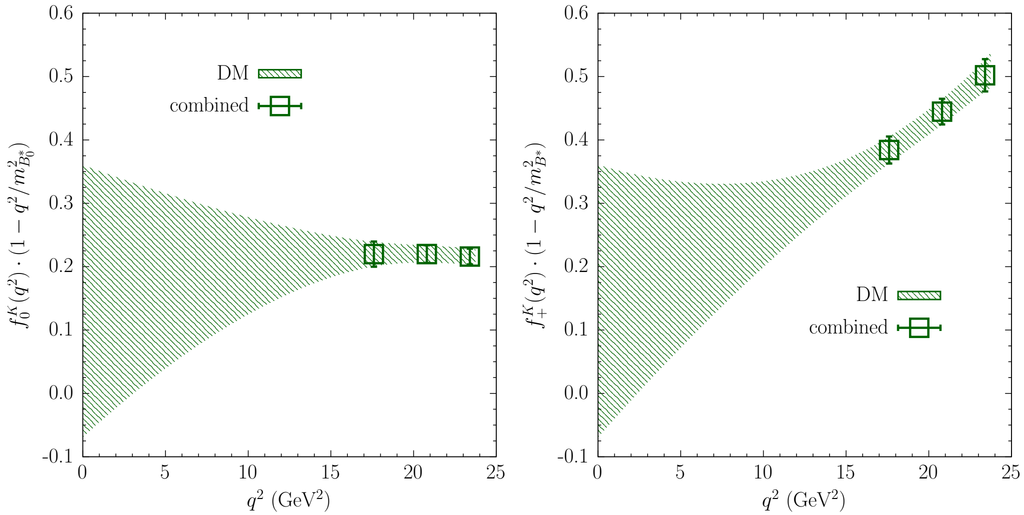

*We can firstly investigate **LFU**:* $R_K^{\tau/\mu} = 0.755 \pm 0.138$

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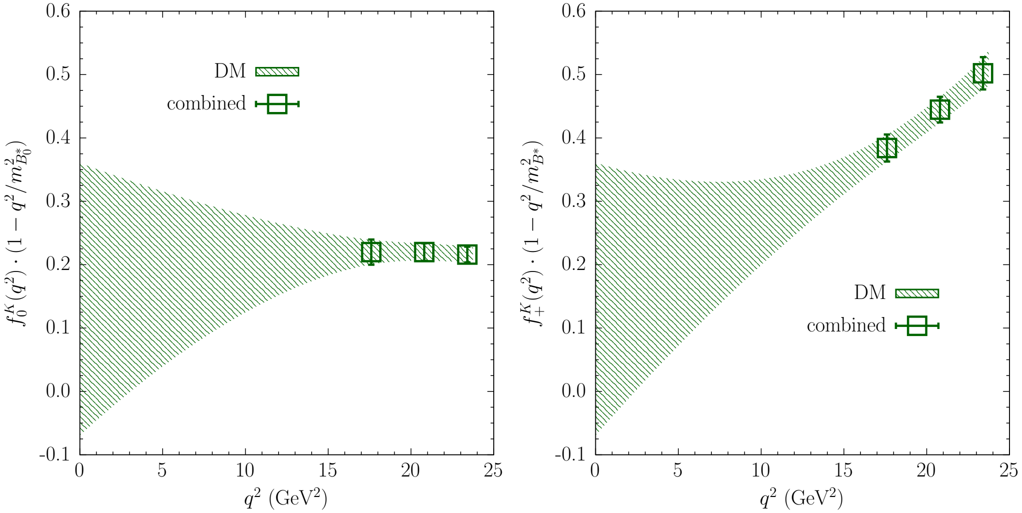

|Vub|: LHCb Coll. has measured for the first time

$$R_{BF} \equiv \frac{\mathcal{B}(B_s^0 \to K^- \mu^+ \nu_\mu)}{\mathcal{B}(B_s^0 \to D_s^- \mu^+ \nu_\mu)}$$

Low-q²: $q^2 \leq 7\,\mathrm{GeV}^2$

High-q²: $q^2 \geq 7\,\mathrm{GeV}^2$

LHCb Collaboration, PRL '21 [2012.05143]

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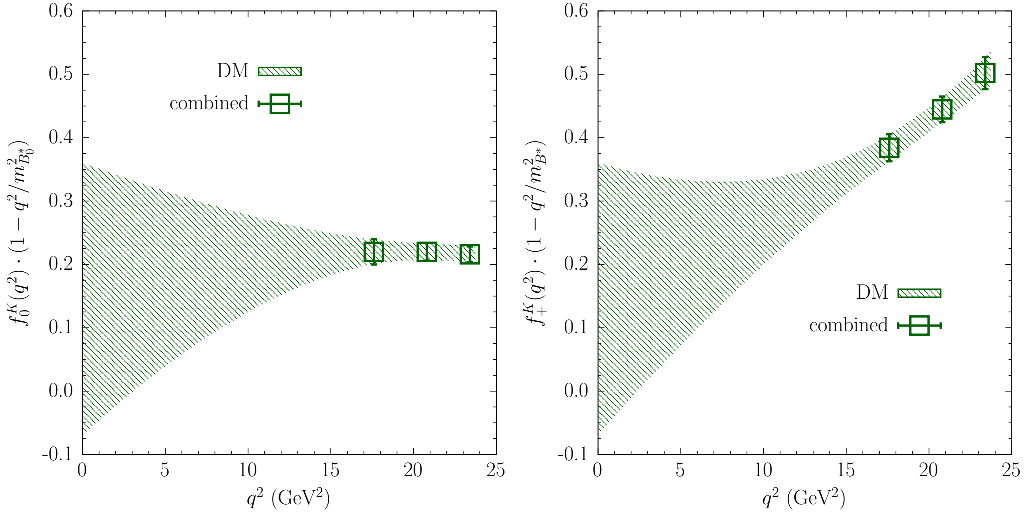

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*by using the exp. value of the BR @ denominator from **LHCb Collaboration, PRD '20 [2001.03225].***

q^2-bin	RBC/UKQCD	FNAL/MILC	HPQCD	combined
low	6.70 ± 3.26	6.43 ± 2.03	3.57 ± 1.94	5.31 ± 3.02
high	4.20 ± 0.56	4.10 ± 0.38	3.54 ± 0.43	3.94 ± 0.59
average	3.93 ± 0.46	3.93 ± 0.35	3.54 ± 0.35	3.77 ± 0.48

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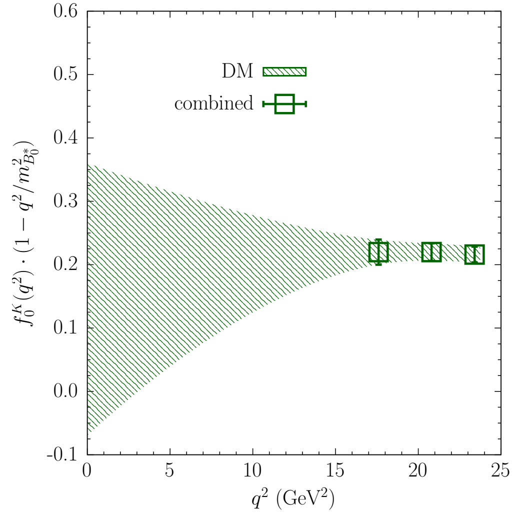

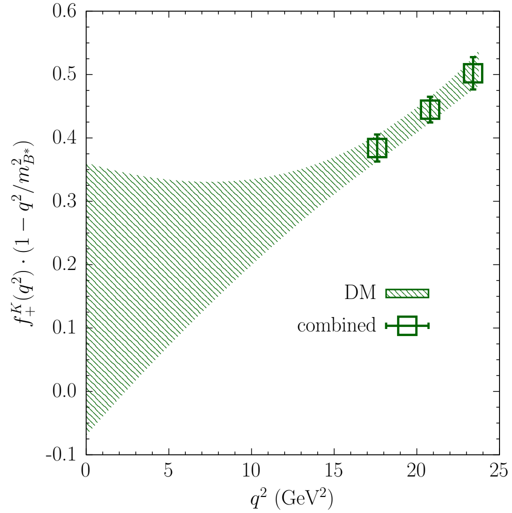

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Final improved determination of $|V_{ub}|$ from the DM method

$$|V_{ub} f_+(q_i^2)| = \sqrt{\frac{d\Gamma}{dq_i^2} \frac{1}{z_i}}$$

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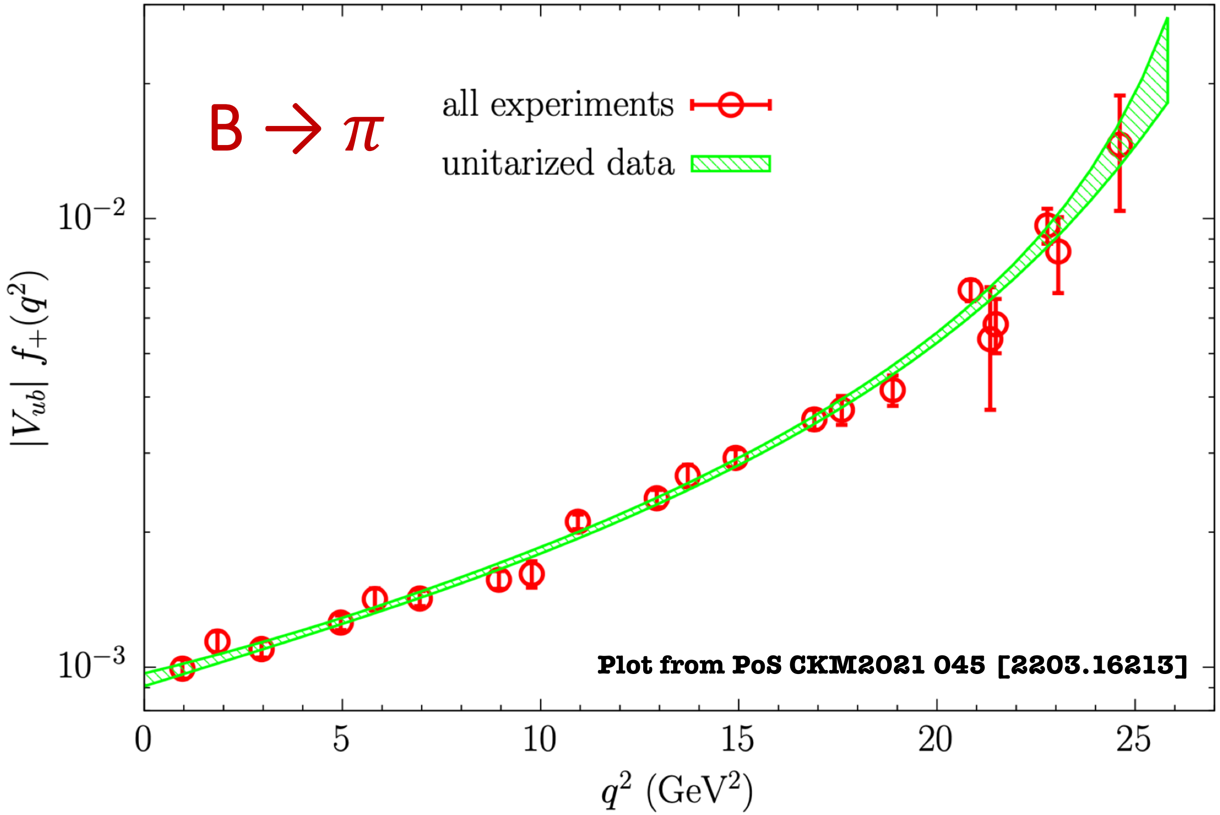

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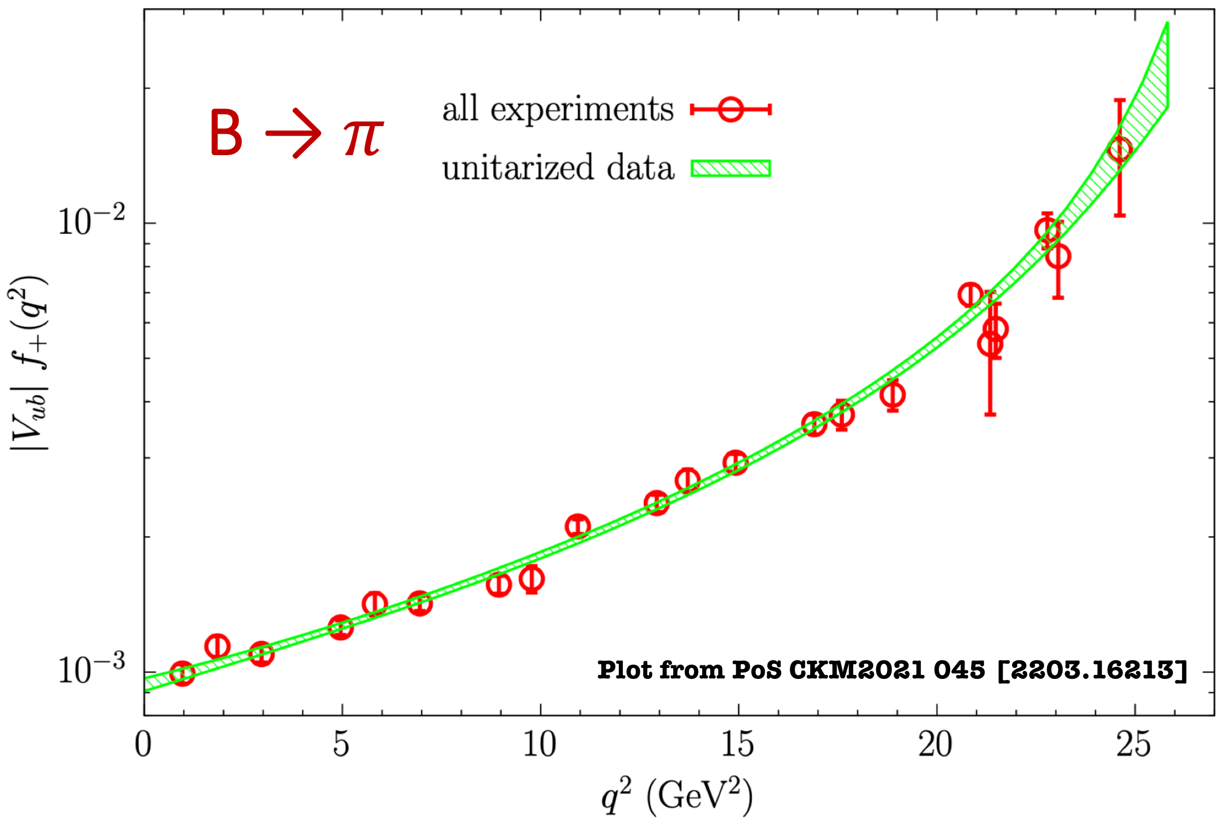

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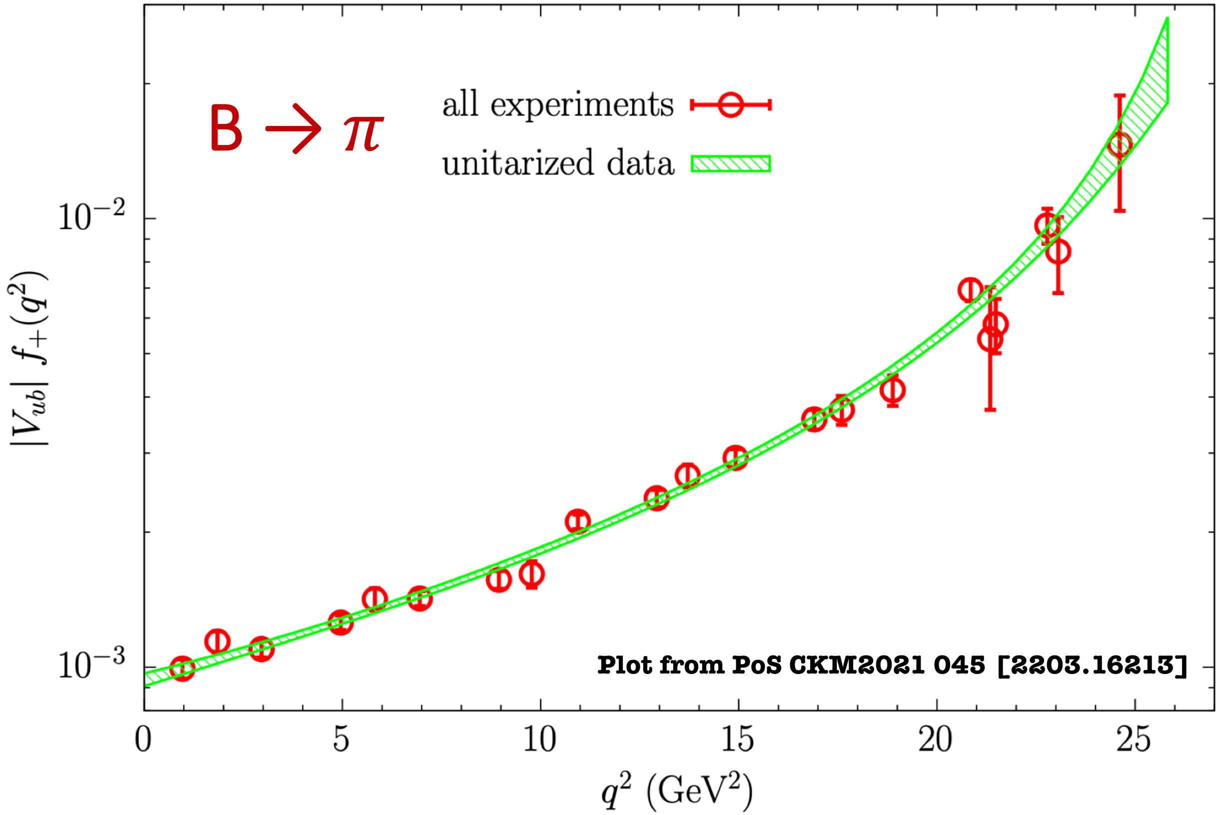

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Final DM Vub value:

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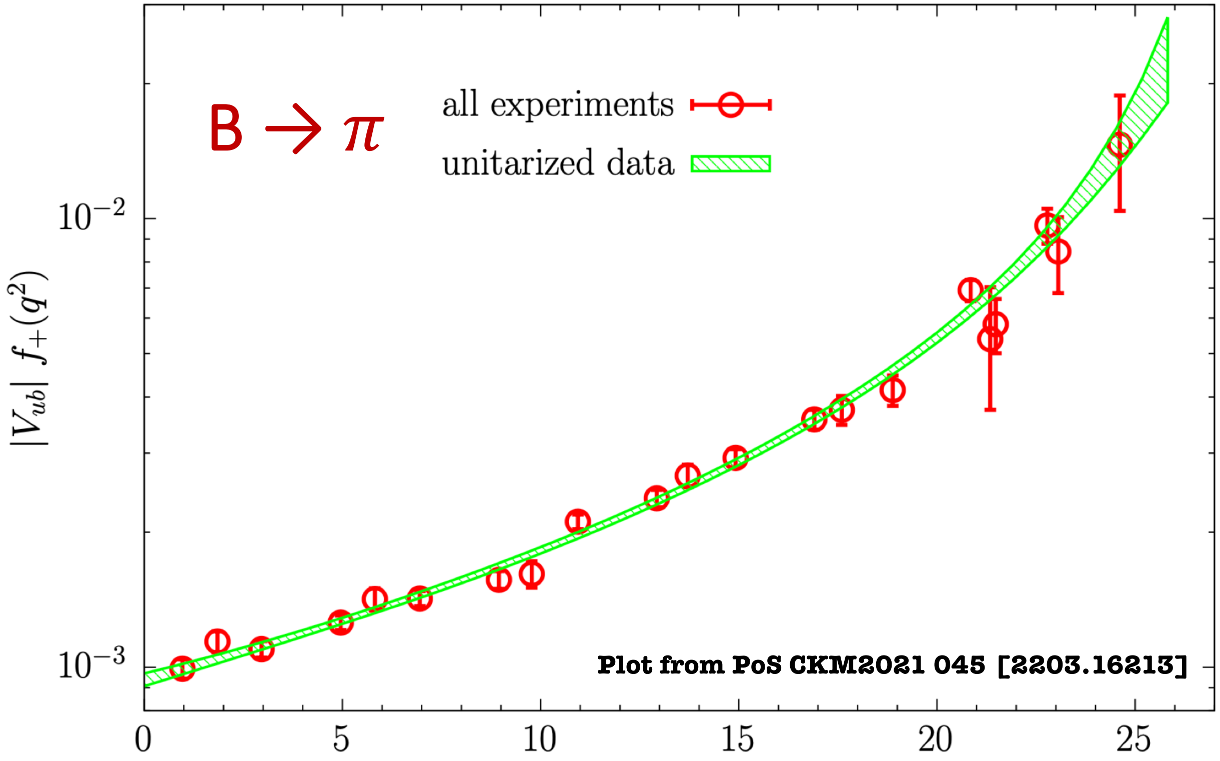

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Important: *we still **keep separate the theoretical calculations and the experimental data** for describing the shape of the FFs!*

Other exclusive determinations of Vub in literature

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(LATEST) EXCLUSIVE

$$|V_{ub}| \cdot 10^3 = 3.77(15)$$

D. Leljak, B. Melic and D. van Dyk, JHEP '21 [2102.07233]

$$|V_{ub}| \cdot 10^3 = 3.68(5)$$

S. Gonzalez-Solis, P. Masjuan and C. Rojas, PRD '21 [2110.06153]

$$|V_{ub}| \cdot 10^3 = 3.87(13)$$

A. Biswas, S. Nandi, S.K. Patra and I. Ray, JHEP '21 [2103.01809]
(see also the recent study of b→{u,d} quark transition in arXiv:2208.14463)

INCLUSIVE

$$|V_{ub}|_{incl} \cdot 10^3 = 4.19(12)\left({}^{+0.11}_{-0.12}\right)$$

HFLAV Coll. [arXiv:2206.07501]

$$|V_{ub}|_{incl} \cdot 10^3 = 4.32\,(29)$$

FLAG Review 2021 [EPJC '22 (2111.09849)]

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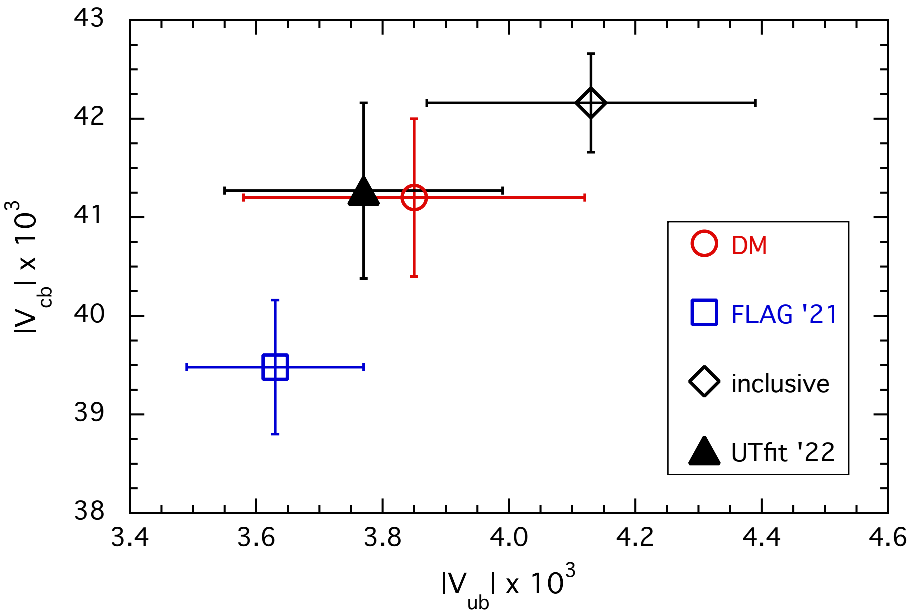
Nice consistency of the DM result with both the other exclusive and the inclusive determinations

Summary plots/tables



	decays	DM	FLAG '21	inclusive
$\|V_{cb}\| \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)
$\|V_{ub}\| \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)

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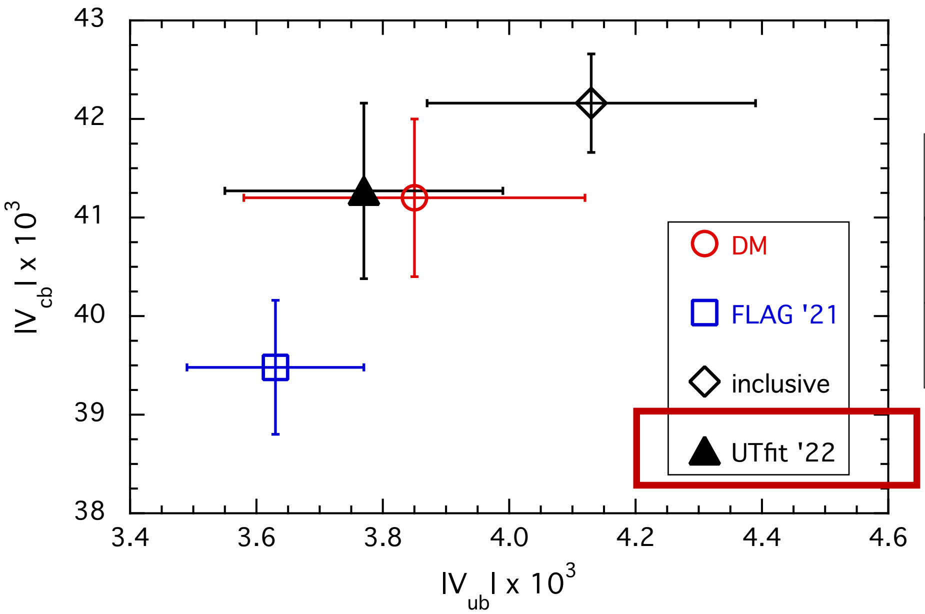

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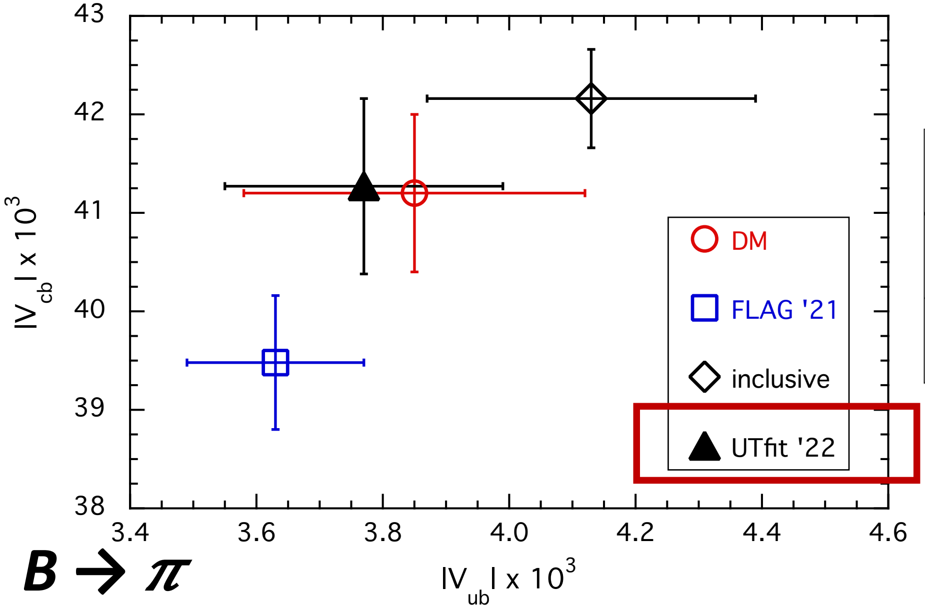

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$B \to \pi$

	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767(145)	0.838(75)	0.793(118)
$\bar{\mathcal{A}}_{FB}^{\mu,\pi}$	0.0043(39)	0.0018(14)	0.0034(31)
$\bar{\mathcal{A}}_{FB}^{\tau,\pi}$	0.219(25)	0.221(19)	0.220(24)
$\bar{\mathcal{A}}_{polar}^{\mu,\pi}$	0.985(11)	0.991(4)	0.988(9)
$\bar{\mathcal{A}}_{polar}^{\tau,\pi}$	0.294(87)	0.309(82)	0.301(86)

$B_s \to K$

	RBC/UKQCD	FNAL/MILC	HPQCD	combined
$R_K^{\tau/\mu}$	0.845(122)	0.816(64)	0.680(134)	0.755(138)
$\bar{\mathcal{A}}_{FB}^{\mu,K}$	0.0032(18)	0.0024(12)	0.0059(29)	0.0046(28)
$\bar{\mathcal{A}}_{FB}^{\tau,K}$	0.257(14)	0.246(14)	0.278(19)	0.262(23)
$\bar{\mathcal{A}}_{polar}^{\mu,K}$	0.990(5)	0.992(4)	0.982(8)	0.986(7)
$\bar{\mathcal{A}}_{polar}^{\tau,K}$	0.172(54)	0.254(64)	0.112(79)	0.172(91)

THANKS FOR

YOUR ATTENTION!

BACK-UP SLIDES

A methodological break: comparison with BGL/BCL

What is the **main improvement** with respect to BGL/BCL parametrization?

Boyd, Grinstein and Lebed, Phys. Lett. B353, 306 (1995)
Boyd, Grinstein and Lebed, Nucl. Phys. B461, 493 (1996)
Boyd, Grinstein and Lebed, Phys. Rev. D 56, 6895 (1997)

Basics of BGL: the hadronic FFs corresponding to definite spin-parity can be represented as an expansion, originating from unitarity, analyticity and crossing symmetry, in terms of the conformal variable *z*, for instance

$$g(z) = \frac{1}{\sqrt{\chi_{1^-}(q_0^2)}} \frac{1}{\phi_g(z, q_0^2)\, P_{1^-}(z)} \sum_{n=0}^{\infty} a_n\, z^n$$

Unitarity:

$$\sum_{n=0}^{\infty} a_n^2 \leq 1$$

Basics of BCL: similar to BGL, the expansion series has a simpler form, for instance

$$f_+(z) = \frac{1}{1 - q^2/m_{B^*}^2} \sum_{n=0}^{N_z - 1} a_k \left[z^n - (-1)^{n-N_z} \frac{n}{N_z} z^{N_z} \right],$$

$$f_0(z) = \sum_{n=0}^{N_z - 1} b_k z^k.$$

Bourrely, Caprini and Lellouch, Phys. Rev. D 79, 013008 (2009)

Unitarity:

$$\sum_{i,j=0}^{N_z} B^{+}_{mn} a_m a_n \leq 1, \qquad \sum_{i,j=0}^{N_z} B^{0}_{mn} b_m b_n \leq 1$$

LFU in semileptonic B → π decays

Fit	$N_z = 3$	$N_z = 4$	$N_z = 5$
χ^2/dof	2.5	0.64	0.73
dof	6	4	2
p	0.02	0.63	0.48
$\sum B^+_{mn} b^+_m b^+_n$	0.11(2)	0.016(5)	1.0(2.3)
$\sum B^0_{mn} b^0_m b^0_n$	0.33(8)	2.8(1.7)	8(19)
$f(0)$	0.00(4)	0.20(14)	0.36(27)
b^+_0	0.395(15)	0.407(15)	0.408(15)
b^+_1	−0.93(11)	−0.65(16)	−0.60(21)
b^+_2	−1.6(1)	−0.5(9)	−0.2(1.4)
b^+_3		0.4(1.3)	3(4)
b^+_4			5(5)
b^0_0	0.515(19)	0.507(22)	0.511(24)
b^0_1	−1.84(10)	−1.77(18)	−1.69(22)
b^0_2	−0.14(25)	1.3(8)	2(1)
b^0_3		4(1)	7(5)
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Table XIII
*of **arXiv:1503.07839***
(FNAL/MILC Coll.)

$$f^\pi(q^2=0)|_{\text{RBC/UKQCD}} = -0.06 \pm 0.25$$

$$f^\pi(q^2=0)|_{\text{FNAL/MILC}} = -0.01 \pm 0.16$$

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The DM approach is independent of this issue!!!

LFU in semileptonic B → π decays

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Table XIX
*of **arXiv:1501.05363***
(RBC/UKQCD Coll.)

		$f_+^{B\pi}$						$f_0^{B\pi}$						
K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	$f(q^2=0)$	χ^2/dof	p
1	0.447(36)				0.00394(63)							0.447(36)	4.02	2%
2	0.410(39)	-1.30(52)			0.0120(59)							0.241(83)	0.30	58%
3	0.420(43)	-1.46(59)	-4.7(7.2)		0.15(42)							0.07(32)		
						1	0.460(61)				0.0225(60)	0.460(61)	90.1	0%
						2	0.516(61)	-4.09(55)			0.408(63)	-0.074(73)	0.03	87%
						3	0.516(61)	-3.94(97)	0.7(3.8)		0.32(41)	-0.02(28)		
2	0.366(37)	-2.79(54)			0.0337(85)	2	0.587(58)	-3.33(38)			0.346(55)	0.040(65)	6.18	0%
3	0.427(40)	-1.62(46)	-7.7(1.5)		0.38(15)	2	0.521(60)	-4.03(52)			0.404(62)	-0.066(70)	0.10	91%
2	0.410(39)	-1.24(51)			0.0113(56)	3	0.520(60)	-3.12(42)	4.5(1.3)		0.41(17)	0.248(82)	0.58	56%
3	0.424(41)	-1.50(57)	-6.0(5.0)		0.24(38)	3	0.519(60)	-3.81(81)	1.2(3.4)		0.27(25)	0.01(24)	0.07	79%

L. Vittorio (LAPTh & CNRS, Annecy)

LFU in semileptonic **B** $\rightarrow \pi$ decays

Same considerations developed for the FNAL/MILC case…

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Important issue: *the DM method equivalent to the results of **all** possible fits which satisfy unitarity and at the same time reproduce exactly the input data*

How to build up the ***combined*** case

FFs with mean values $x_i^{(k)}$ and uncertainties $\sigma_i^{(k)}$ $(k = 1, \cdots, N)$

Mean values and uncertainties of the ***new combined*** values

$$x_i = \sum_{k=1}^{N} \omega^{(k)} x_i^{(k)},$$

Weights

$$\sigma_i^2 = \sum_{k=1}^{N} \omega^{(k)} (\sigma_i^{(k)})^2 + \sum_{k=1}^{N} \omega^{(k)} (x_i^{(k)} - x_i)^2$$

$$\left[\sum_{k=1}^{N} \omega^{(k)} = 1 \right]$$

$$\omega^{(k)} = 1/N$$

Conservative choice in arXiv:2202.10285

Cov. Matrices of the k-th LQCD computation

Covariance matrix of the ***new combined*** values

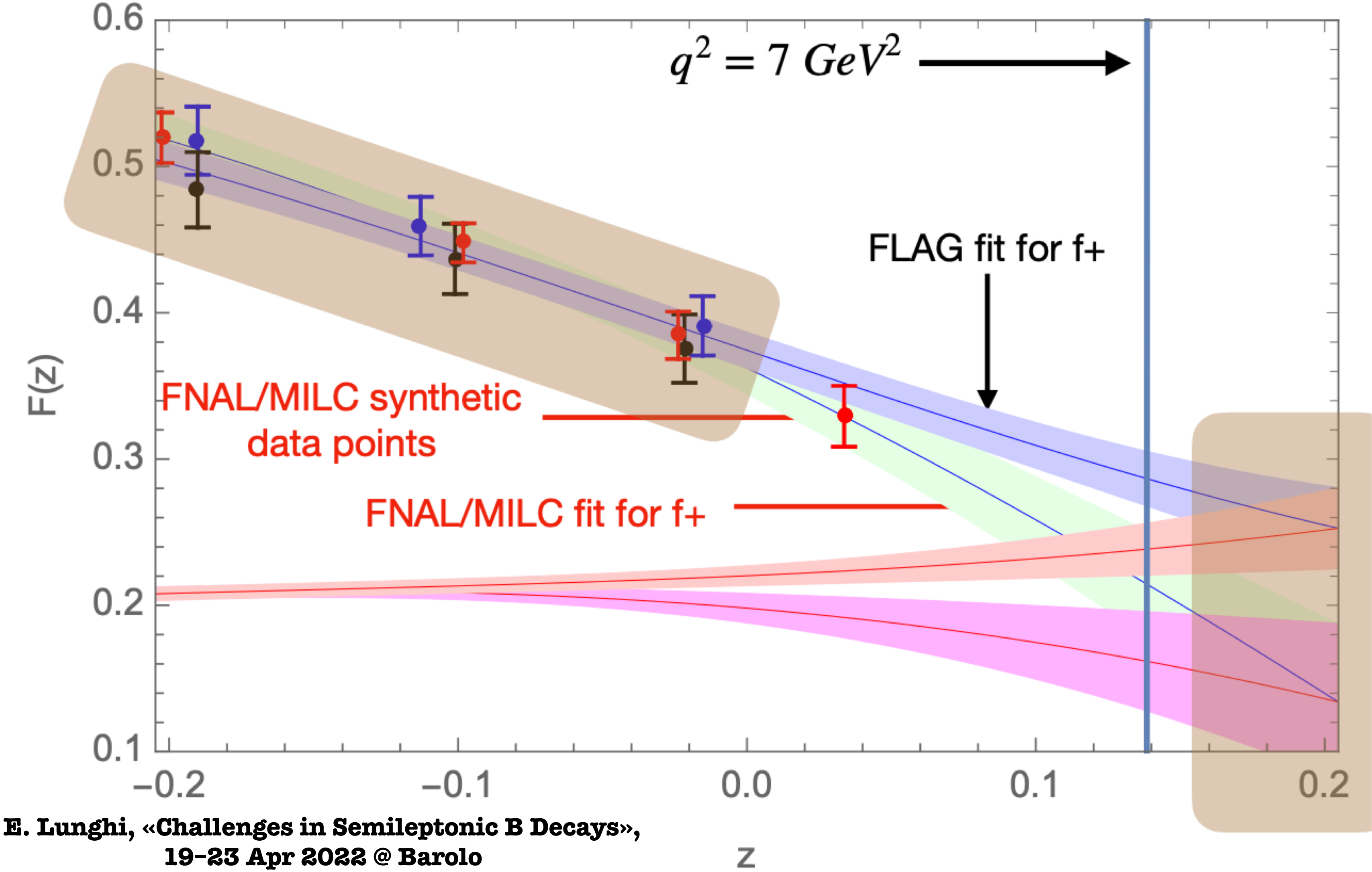
$$C_{ij} \equiv \frac{1}{N} \sum_{k=1}^{N} C_{ij}^{(k)} + \frac{1}{N} \sum_{k=1}^{N} (x_i^{(k)} - x_i)(x_j^{(k)} - x_j)$$

How to build up the ***combined*** case

	RBC/UKQCD	HPQCD	FNAL/MILC	Combined
$f_+^K(17.6\ \mathrm{GeV}^2)$	0.99(4)(5)	1.04(5)	1.01(4)	1.01(6)
$f_+^K(20.8\ \mathrm{GeV}^2)$	1.64(6)(7)	1.68(7)	1.68(5)	1.67(8)
$f_+^K(23.4\ \mathrm{GeV}^2)$	2.77(9)(11)	2.94(13)	2.91(9)	2.87(15)
$f_0^K(17.6\ \mathrm{GeV}^2)$	0.48(2)(3)	0.53(3)	0.44(2)	0.48(4)
$f_0^K(20.8\ \mathrm{GeV}^2)$	0.63(2)(4)	0.64(3)	0.59(1)	0.62(4)
$f_0^K(23.4\ \mathrm{GeV}^2)$	0.81(2)(5)	0.79(4)	0.76(2)	0.79(5)

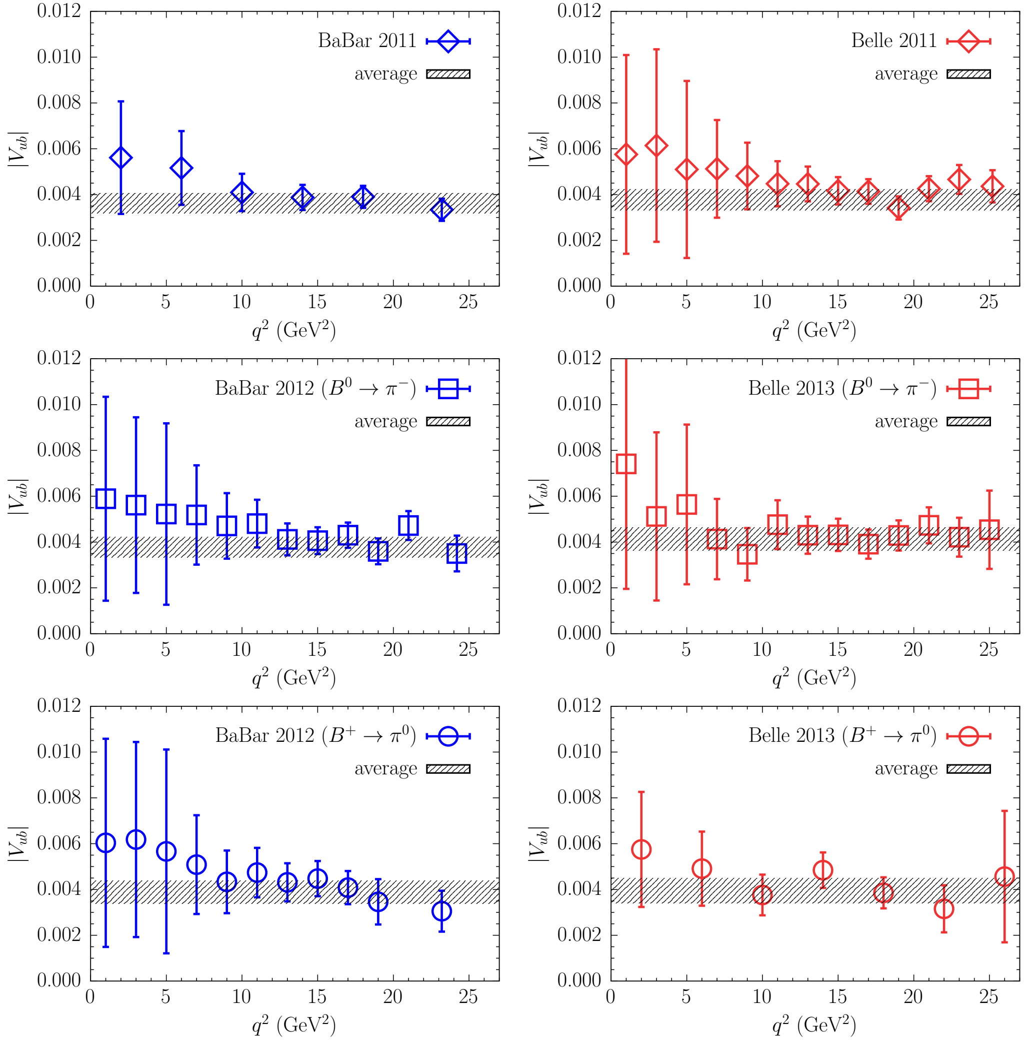
Table 2. *Mean values and uncertainties of the LQCD computations of the FFs* $f_{+,0}^K(q^2)$ *obtained at three selected values of* q^2 *from the results of the RBC/UKQCD [20], HPQCD [22] and FNAL/MILC [23] Collaborations. For the RBC/UKQCD computations the first error is statistical while the second one is systematic. The last column contains the results of the combination procedure given in Eqs. (3.1)-(3.2) with* $\omega^{(k)} = 1/N$.

How to build up the *combined* case



E. Lunghi, «Challenges in Semileptonic B Decays», 19-23 Apr 2022 @ Barolo

Bin-per-bin |Vub| with new JLQCD data



The bands are the results of correlated weigthed averages:

$$|V_{ub}|_n = \frac{\sum_{i,j}(\mathbf{C}^{-1})_{ij}|V_{ub}|_j}{\sum_{i,j}(\mathbf{C}^{-1})_{ij}}, \qquad \sigma^2_{|V_{ub}|_n} = \frac{1}{\sum_{i,j}(\mathbf{C}^{-1})_{ij}}$$

FINAL VALUE OF the CKM matrix element:

$$|V_{ub}|_{\mathrm{JLQCD}} \times 10^3 = 3.85(51)$$

Other observables for phenomenology

Starting point:

$$\frac{d^2\Gamma(B_{(s)}\to\pi(K)\ell\nu_\ell)}{dq^2 d\cos\theta_\ell} = \frac{G_F^2|V_{ub}|^2}{128\pi^3 m_{B_{(s)}}^2}\left(1-\frac{m_\ell^2}{q^2}\right)^2$$

$$\cdot\Big\{4m_{B_{(s)}}^2|\vec{p}_{\pi(K)}|^3\left(\sin^2\theta_\ell+\frac{m_\ell^2}{2q^2}\cos^2\theta_\ell\right)|f_+^{\pi(K)}(q^2)|^2$$

$$+\frac{4m_\ell^2}{q^2}(m_{B_{(s)}}^2-m_{\pi(K)}^2)m_{B_{(s)}}|\vec{p}_{\pi(K)}|^2\cos\theta_\ell\,\Re\left(f_+^{\pi(K)}(q^2)f_0^{*\pi(K)}(q^2)\right)$$

$$+\frac{m_\ell^2}{q^2}(m_{B_{(s)}}^2-m_{\pi(K)}^2)^2|\vec{p}_{\pi(K)}||f_0^{\pi(K)}(q^2)|^2\Big\}\ ,$$

θ_l is the angle between the final charged lepton and the $B_{(s)}$-meson momenta in the rest frame of the final state leptons

- Forward-backward asymmetry:

$$\mathcal{A}_{FB}^{\ell,\pi(K)}(q^2)\equiv\int_0^1\frac{d^2\Gamma}{dq^2 d\cos\theta_l}d\cos\theta_l-\int_{-1}^0\frac{d^2\Gamma}{dq^2 d\cos\theta_l}d\cos\theta_l$$

$$\longrightarrow\quad \bar{\mathcal{A}}_{FB}^{\ell,\pi(K)}\equiv\frac{\int dq^2\,\mathcal{A}_{FB}^{\ell,\pi(K)}(q^2)}{\int dq^2\,d\Gamma^{\pi(K)}/dq^2}$$

- Lepton polarization asymmetry:

$$\mathcal{A}_{polar}^{\ell,\pi(K)}(q^2)\equiv\frac{d\Gamma_-^{\pi(K)}}{dq^2}-\frac{d\Gamma_+^{\pi(K)}}{dq^2}$$

$$\longrightarrow\quad \bar{\mathcal{A}}_{polar}^{\ell,\pi(K)}\equiv\frac{\int dq^2\,\mathcal{A}_{polar}^{\ell,\pi(K)}(q^2)}{\int dq^2\,d\Gamma^{\pi(K)}/dq^2}$$

U. G. Meißner and W. Wang, JHEP '14 [1311.5420]

Pole heavy-quark mass

How to compute the pole heavy-quark mass?

- Start from the heavy mass computed in $\overline{MS}(2\ \mathrm{GeV})$ scheme
- Scale evolution from μ = 2 GeV to the value μ = m_h using N^3LO perturbation theory
- Finally:

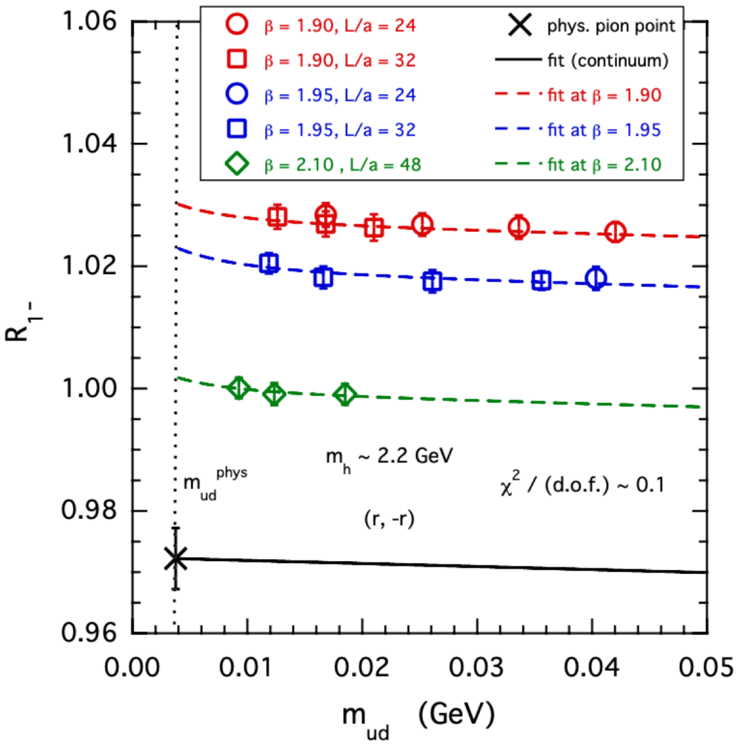
$$m_h^{pole} = m_h(m_h)\left\{1 + \frac{4}{3}\frac{\alpha_s(m_h)}{\pi} + \left(\frac{\alpha_s(m_h)}{\pi}\right)^2 \cdot \left[\frac{\beta_0}{24}(8\pi^2 + 71) + \frac{35}{24} + \frac{\pi^2}{9}\ln(2) - \frac{7\pi^2}{12} - \frac{\zeta_3}{6}\right] + \mathcal{O}(\alpha_s^3)\right\}$$

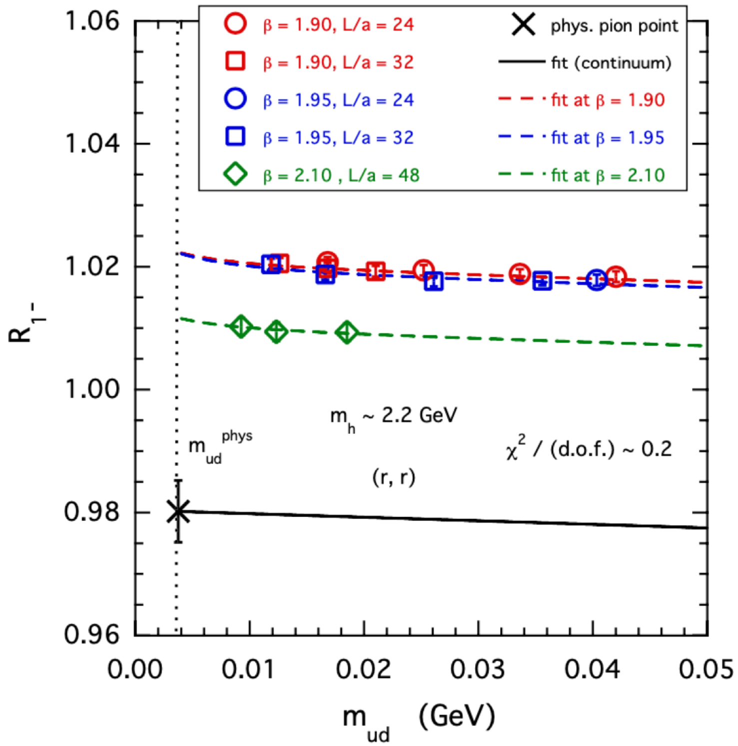
where $$\beta_0 = (33 - 2n_\ell)/12 \text{ and } \zeta_3 \simeq 1.20206$$

Fit to lattice data

$$R_j(n; a^2, m_{ud}) = R_j(n) \left[1 + A_1 \left(m_{ud} - m_{ud}^{phys} \right) + D_1 \frac{a^2}{r_0^2} + D_2 \frac{a^4}{r_0^4} \right]$$

$$\cdot \left(1 + F_1 \frac{\overline{M}^2}{(4\pi f)^2} \frac{e^{-\overline{M}L}}{(\overline{M}L)^p} \right) .$$



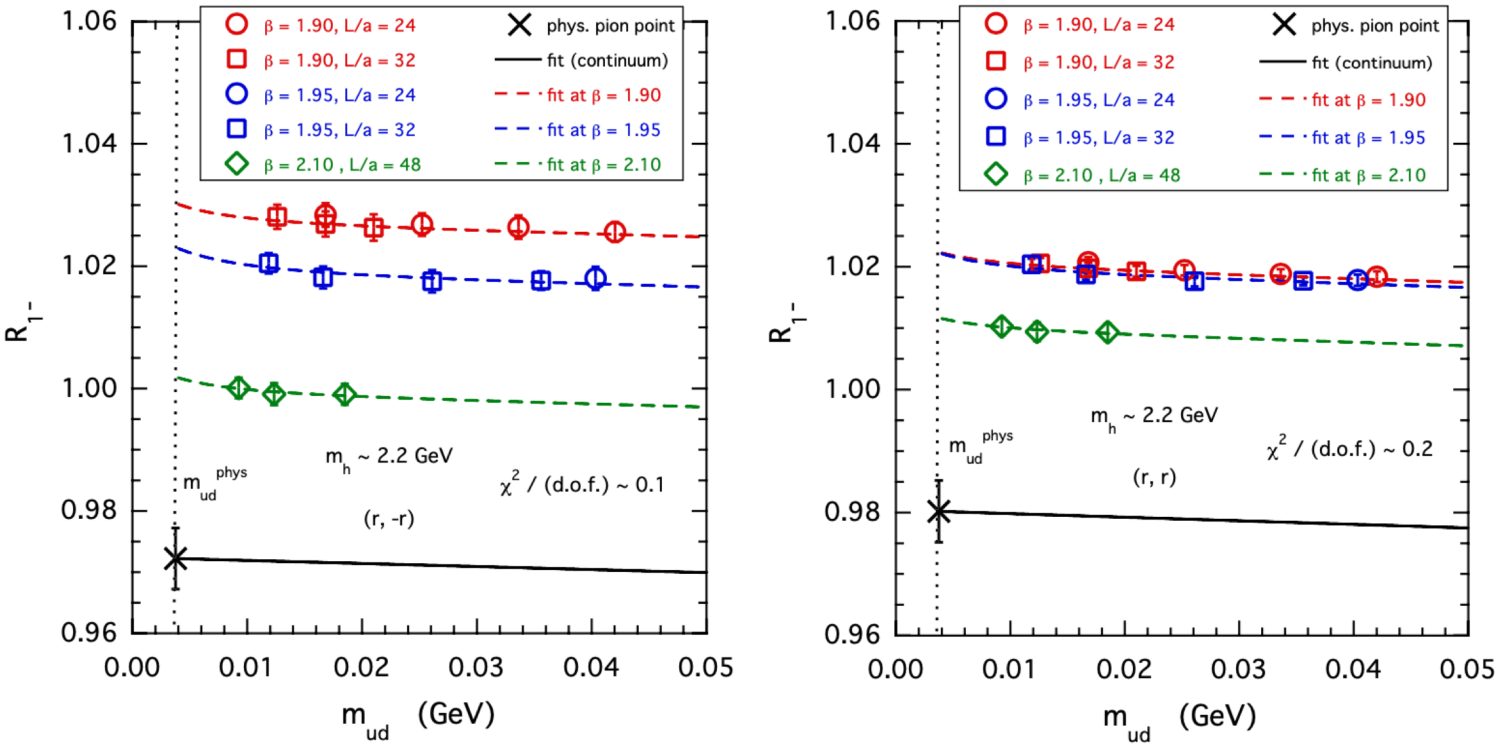


Fit to lattice data

For the numerical values of the lattice parameters see *NPB '14 [1403.4504]!*

$$R_j(n; a^2, m_{ud}) = R_j(n)\left[1 + A_1\left(m_{ud} - m_{ud}^{phys}\right) + D_1\frac{a^2}{r_0^2} + D_2\frac{a^4}{r_0^4}\right] \cdot \left(1 + F_1\frac{\overline{M}^2}{(4\pi f)^2}\frac{e^{-\overline{M}L}}{(\overline{M}L)^p}\right).$$

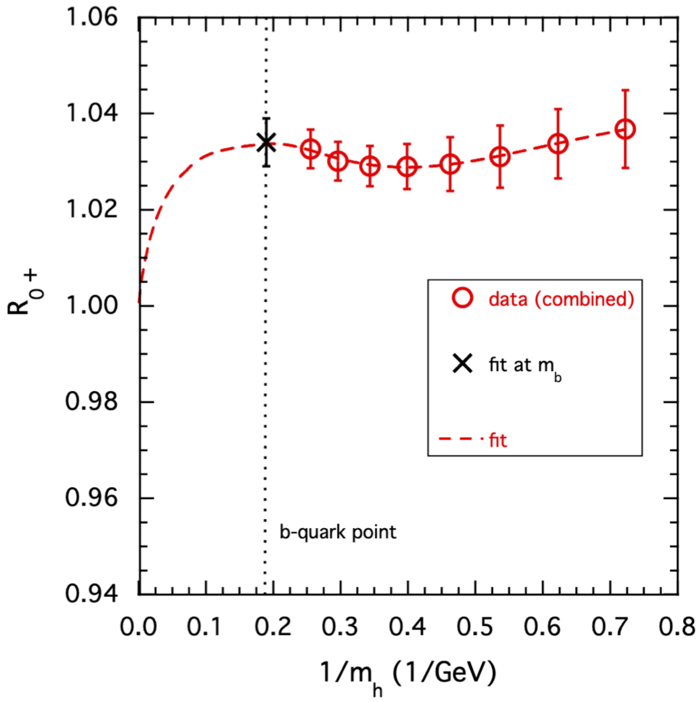
Finite volume effects

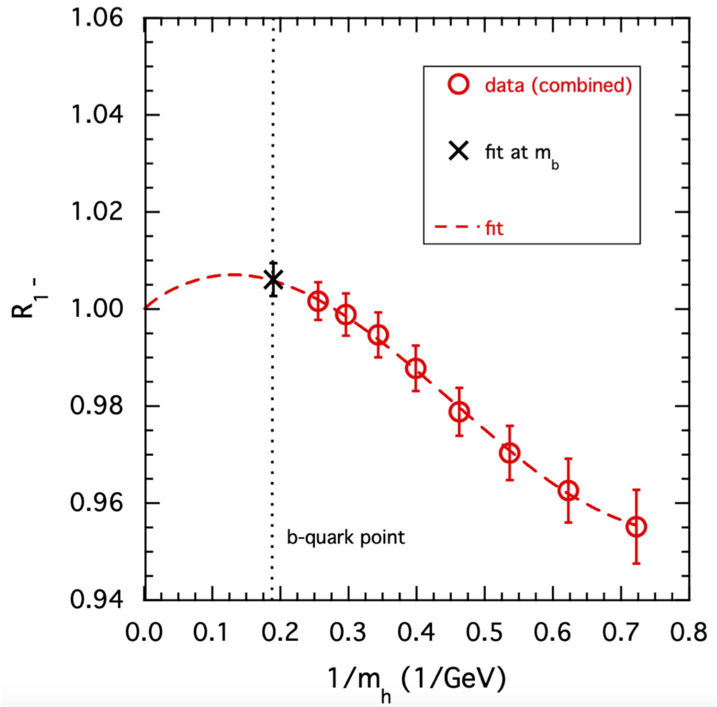


Final extrapolation at the *physical b-quark point*

For the final extrapolation at the physical b-quark point:

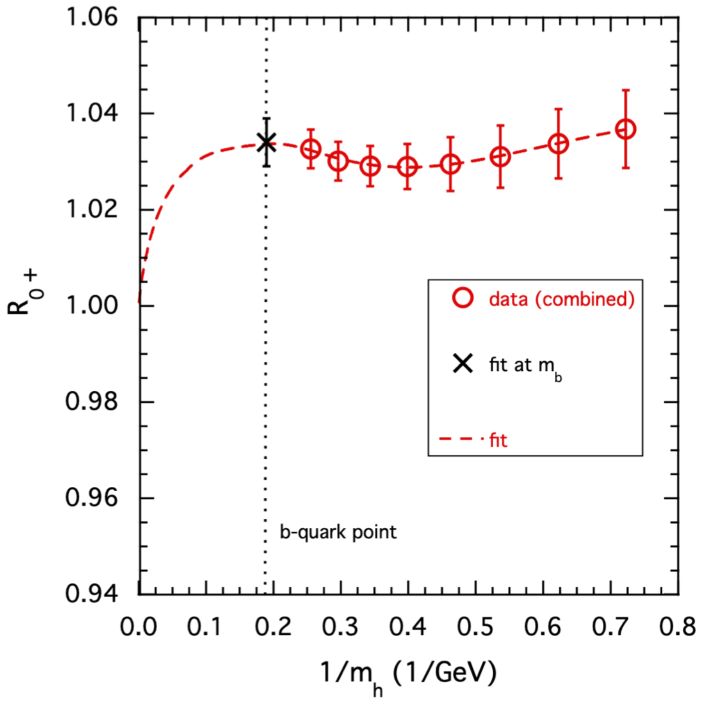
$$R_j(n) = 1 + \sum_{k=1}^{M} \left[A_k + A_k^s \frac{\alpha_s(m_h(n))}{\pi} \right] \left(\frac{1}{m_h(n)} \right)^k \qquad \left\{ M = 3 \right\}$$

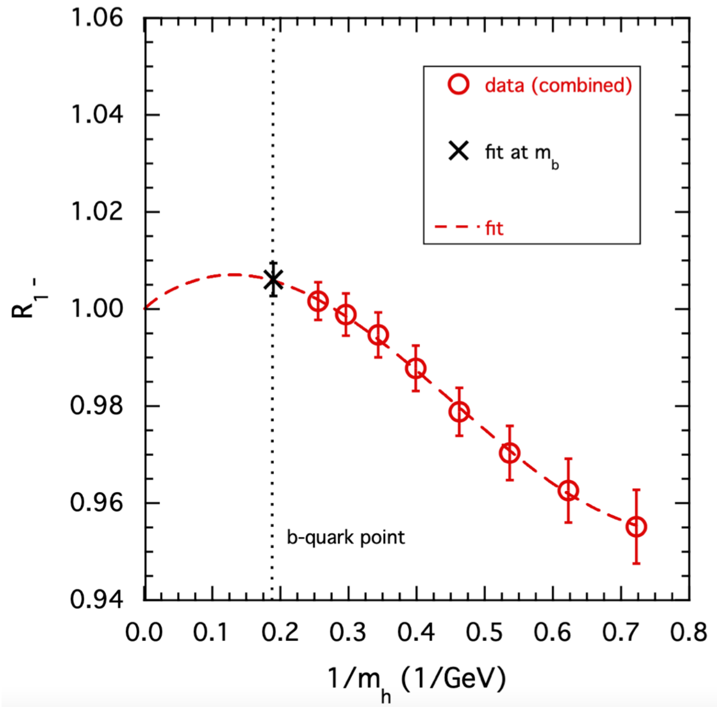




Final extrapolation at the *physical b-quark point*

$$\chi_j(m_b^{phys}) = \chi_j(m_c^{phys}) \cdot \frac{\rho_j(m_c^{phys})}{\rho_j(m_b^{phys})} \cdot \prod_{n=2}^{11} R_j(n)$$





Subtraction of bound-state contributions

channel j	$\chi_j(m_c^{phys})$	$\chi_j(m_b^{phys})$
0^+	$(1.50 \pm 0.13) \cdot 10^{-2}$	$(2.04 \pm 0.20) \cdot 10^{-2}$
1^-	$(4.81 \pm 1.14) \cdot 10^{-3}\ \text{GeV}^{-2}$	$(4.88 \pm 1.16) \cdot 10^{-4}\ \text{GeV}^{-2}$
0^-	$(2.36 \pm 0.15) \cdot 10^{-2}$	$(2.34 \pm 0.13) \cdot 10^{-2}$
1^+	$(3.61 \pm 0.81) \cdot 10^{-3}\ \text{GeV}^{-2}$	$(4.65 \pm 1.02) \cdot 10^{-4}\ \text{GeV}^{-2}$

Subtraction of bound-state contributions

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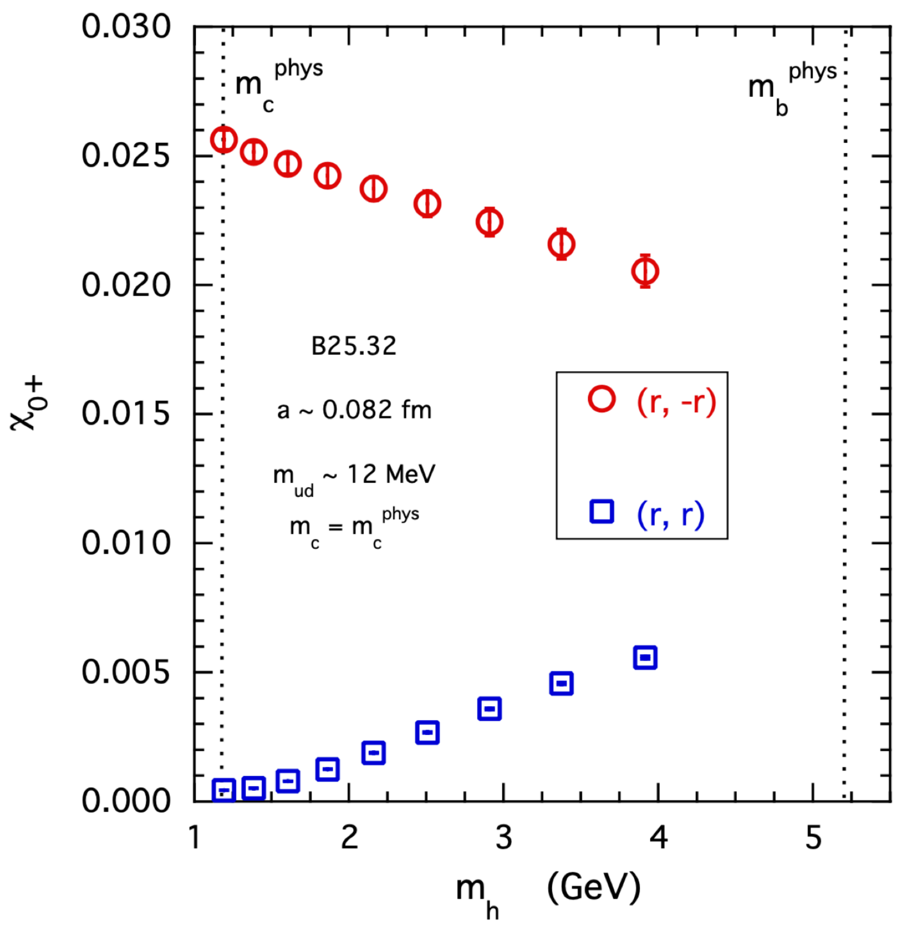
Ground state?

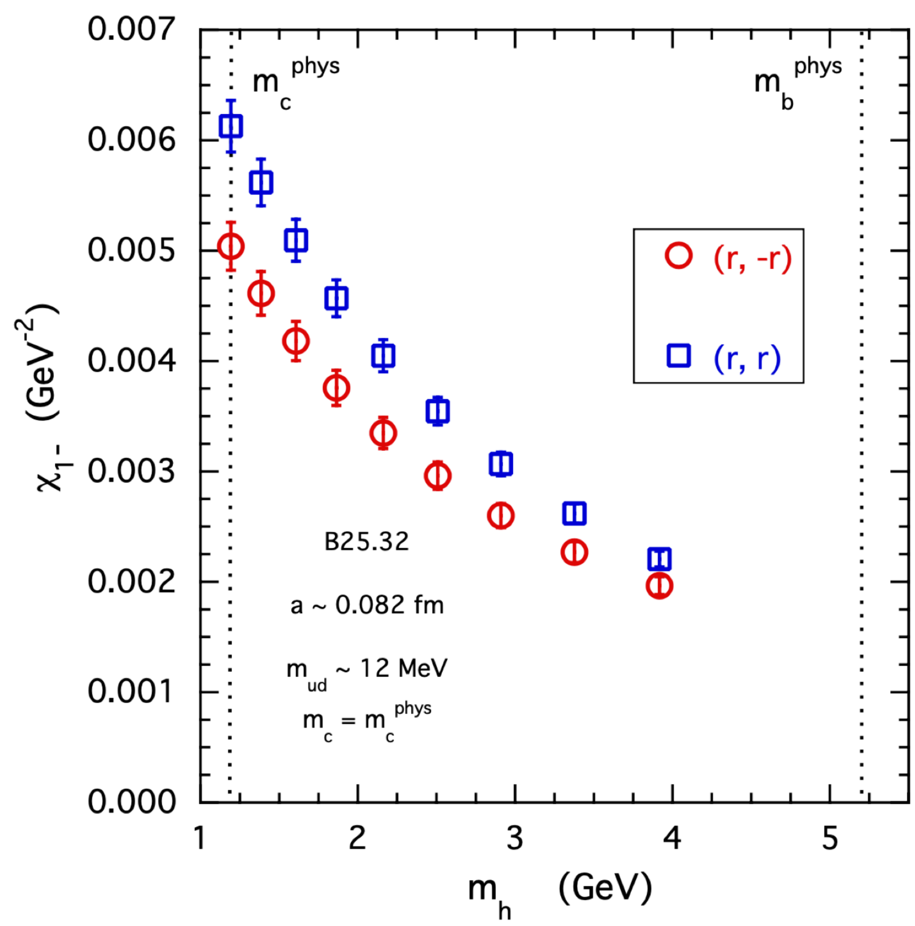
The previous estimates can be improved by removing the contributions of the **bound states lying below the pair production threshold:**

$$\chi_{1^-}^{(gs)}(m_b^{phys}) = \frac{f_{B^*}^2}{M_{B^*}^4} \Rightarrow \chi_{1^-}^{(gs)}(m_b^{phys}) = (0.431 \pm 0.033) \cdot 10^{-4}\ \text{GeV}^{-2}$$

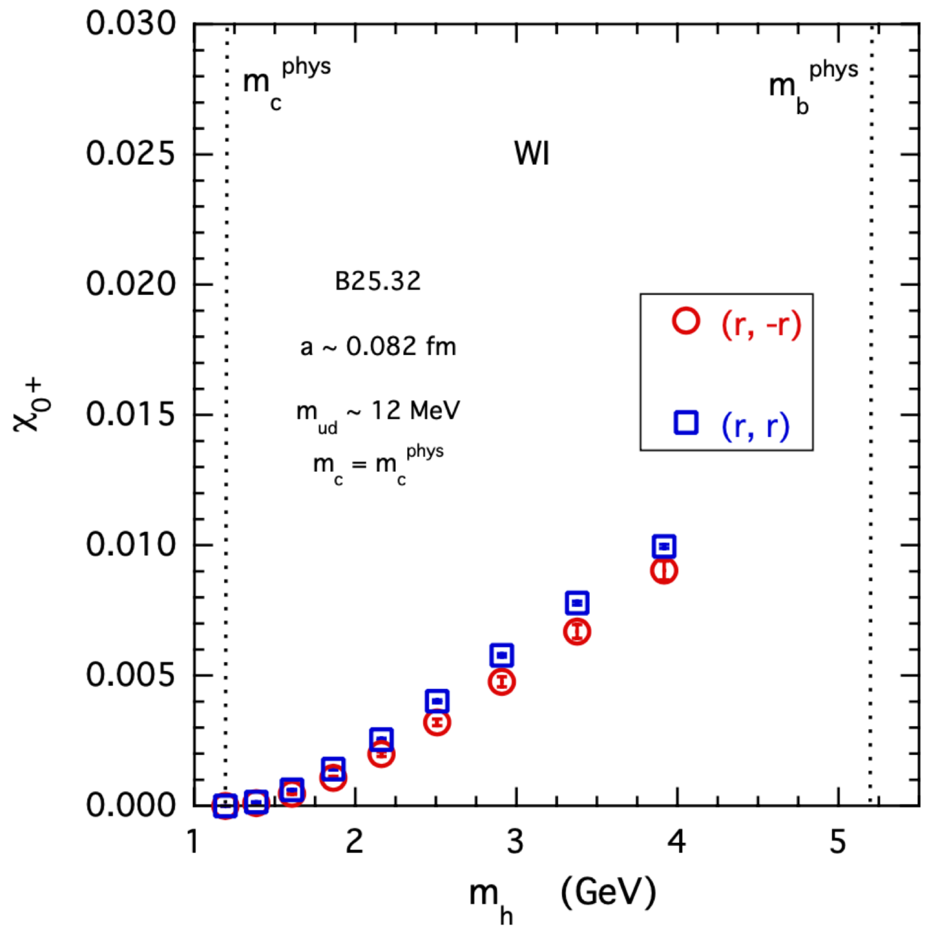
$$\chi_{1^-}(m_b^{phys}) = (4.45 \pm 1.16) \cdot 10^{-4}\ \text{GeV}^{-2}$$

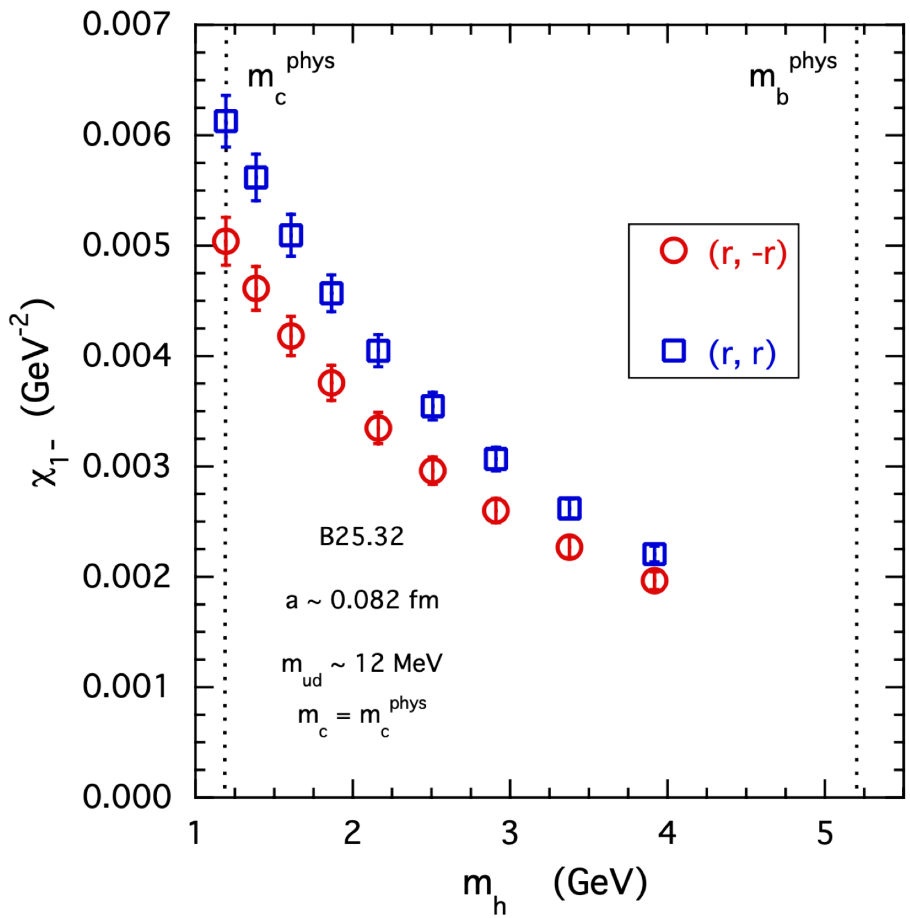
Contact terms & perturbative subtraction





Contact terms & perturbative subtraction





WHY?

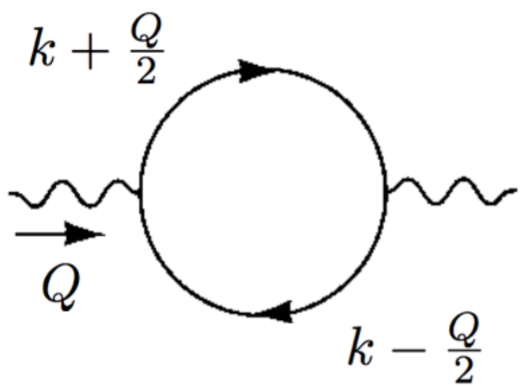
Contact terms & perturbative subtraction

In **twisted mass LQCD:**

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\Big[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\Big],$$

$$G_i(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + \mathcal{M}_i(p) - i r_i \mu_{q,i}\gamma_5}{\mathring{p}_\mu^2 + \mathcal{M}_i^2(p) + \mu_{q,i}^2}$$

$$\mathring{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu), \qquad \mathcal{M}_i(p) \equiv m_i + \frac{r_i}{2} a\hat{p}_\mu^2, \qquad \hat{p} \equiv \frac{2}{a}\sin\Big(\frac{ap_\mu}{2}\Big).$$



$$\begin{aligned}\Pi_V^{\alpha\beta} = {} & a^{-2}(Z_1^I + (r_1^2 - r_2^2)Z_2^I + (r_1^2 - r_2^2)(r_1^2 + r_2^2)Z_3^I)g^{\alpha\beta} \\ & + (\mu_1^2 Z^{\mu_1^2} + \mu_2^2 Z^{\mu_2^2} + \mu_1\mu_2 Z^{\mu_1\mu_2})g^{\alpha\beta} + (Z_1^{Q^2} + (r_1^2 - r_2^2)Z_2^{Q^2})Q\cdot Q g^{\alpha\beta} \\ & + (Z_1^{Q^\alpha Q^\beta} + (r_1^2 - r_2^2)Z_2^{Q^\alpha Q^\beta})Q^\alpha Q^\beta + r_1 r_2(a^{-2}Z_1^{r_1r_2}g^{\alpha\beta} + (Z_2^{r_1r_2} + (r_1^2 + r_2^2)Z_3^{r_1r_2} \\ & + (r_1^4 + r_2^4)Z_4^{r_1r_2})Q\cdot Q g^{\alpha\beta} + (\mu_1^2 Z_5^{r_1r_2} + \mu_2^2 Z_6^{r_1r_2})g^{\alpha\beta}) + O(a^2),\end{aligned}$$

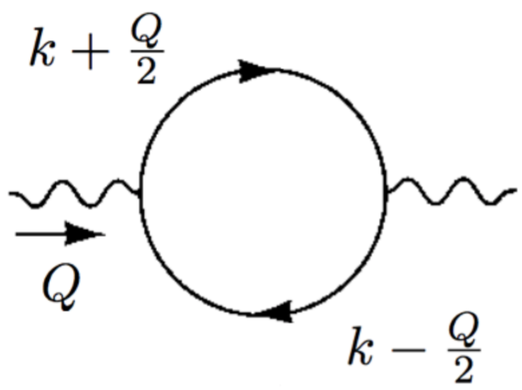
CONTACT TERMS!!!

F. Burger et al., ETM Coll., JHEP '15 [arXiv:1412.0546]

Contact terms & perturbative subtraction

In **twisted mass LQCD** (tmLQCD):

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\right],$$



Thus, by separating the *longitudinal* and the *transverse* contributions, we can compute the susceptibilities for all the spin-parity quantum numbers **in the free theory on the lattice**, *i.e.* at order $\mathcal{O}(\alpha_s^0)$ using twisted-mass fermions!

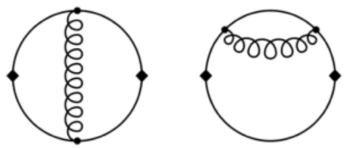
$$\chi_j^{free} = \boxed{\chi_j^{LO}} + \boxed{\chi_j^{discr}}$$

LO term of PT @ $\mathcal{O}(\alpha_s^0)$ contact terms and discretization effects @ $\mathcal{O}(\alpha_s^0 a^m)$ with $m \geq 0$

Perturbative subtraction:

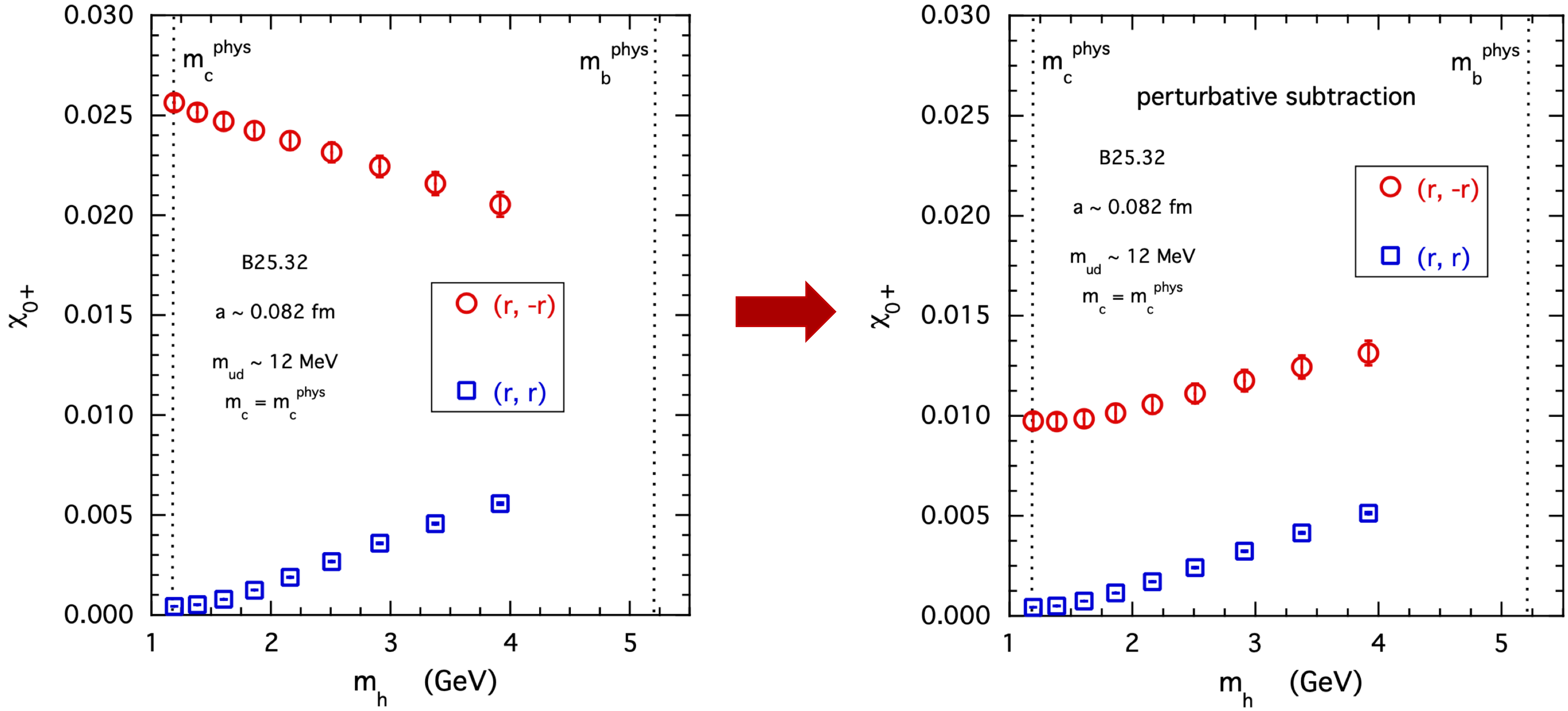
$$\chi_j \to \chi_j - \left[\chi_j^{free} - \chi_j^{LO}\right]$$

Higher order corrections?



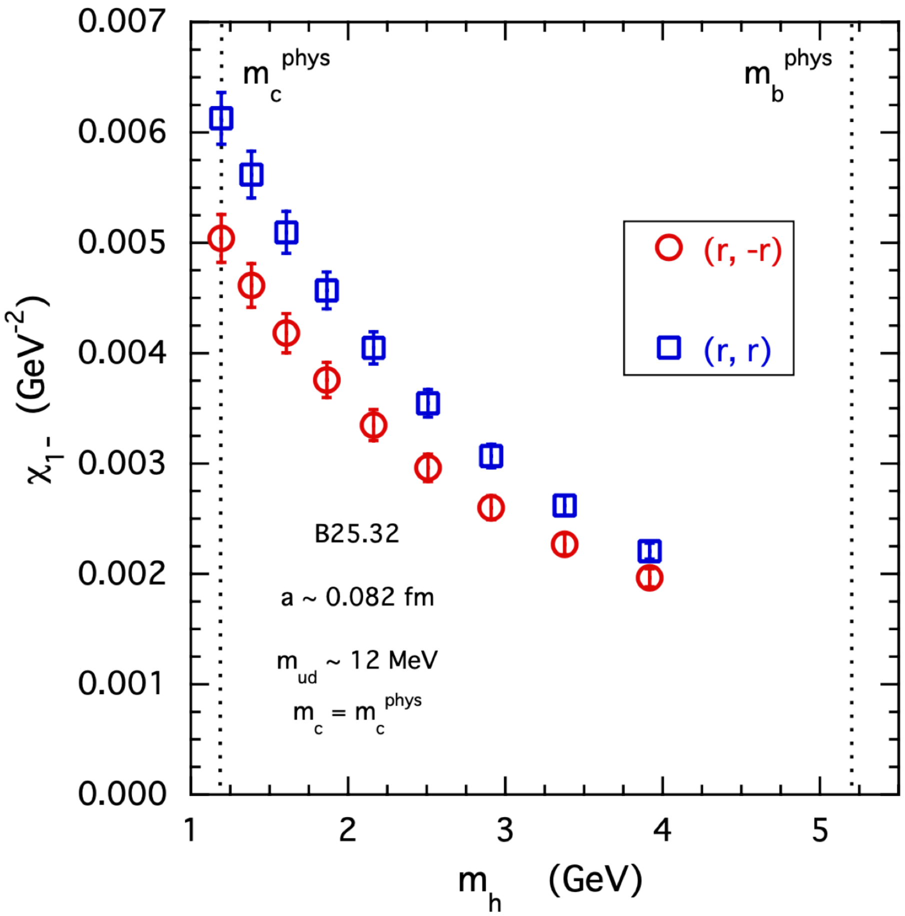
Work in progress...

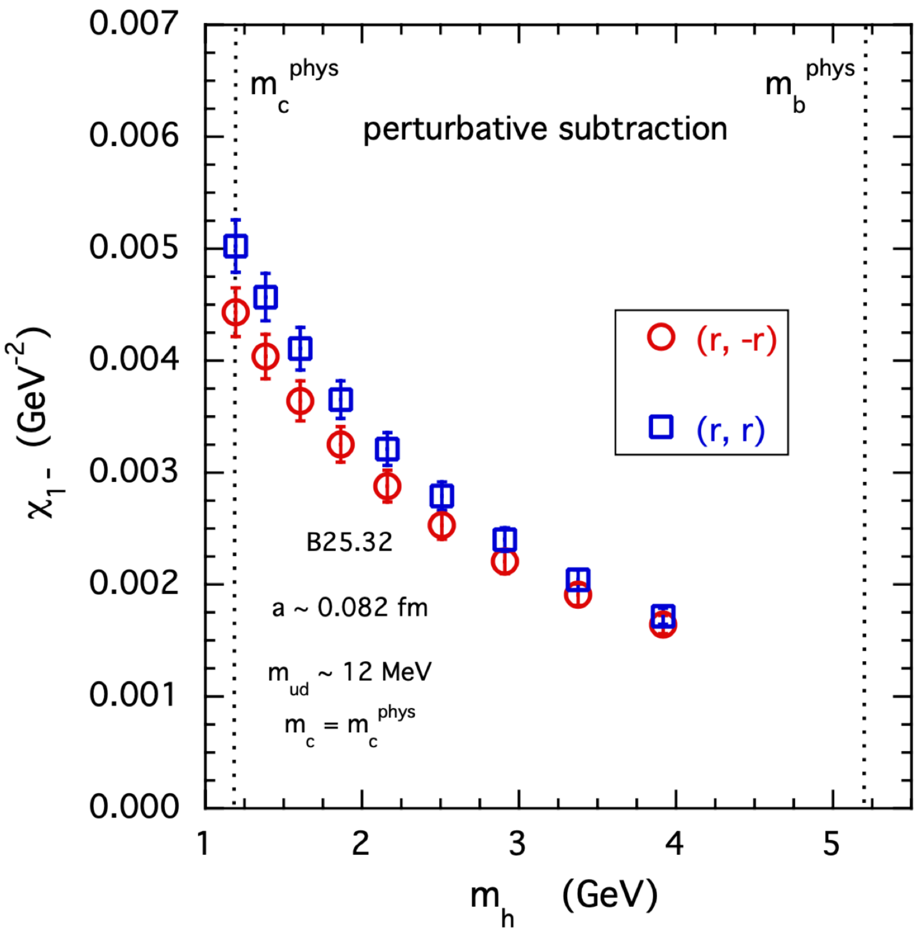
Contact terms & perturbative subtraction



NOT ENOUGH…

Contact terms & perturbative subtraction





OK

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \boxed{\frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}}$$

to ensure that $\lim_{n\to\infty} R_j(n) = 1$

$$\boxed{\begin{array}{l} \rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1 \;, \\ \rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2 \end{array}}$$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light, *in prep.*) transition current densities:**

b → c

	Perturbative	With subtraction	Non-perturbative	With subtraction
$\chi_{V_L}\,[10^{-3}]$	6.204(81)	—	7.58(59)	—
$\chi_{A_L}\,[10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)
$\chi_{V_T}\,[10^{-4}\text{ GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)
$\chi_{A_T}\,[10^{-4}\text{ GeV}^{-2}]$	3.894	—	4.69(30)	—

b → u

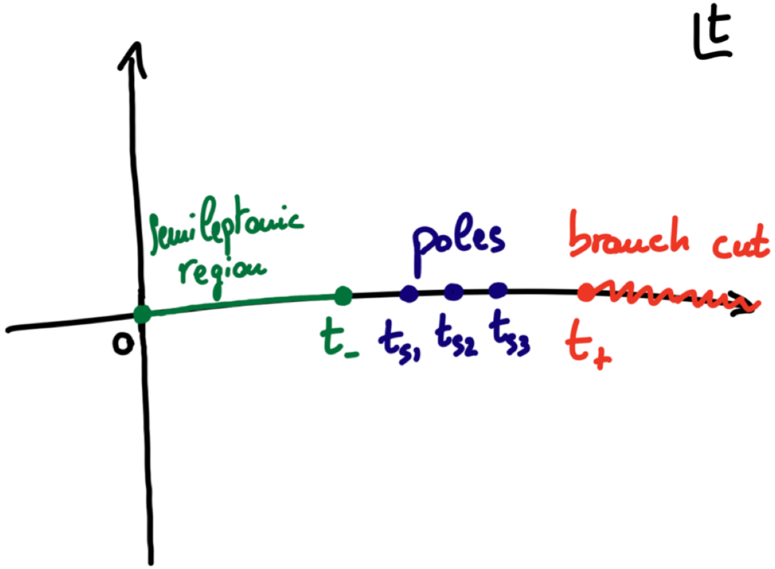
Non-perturbative	With subtraction
2.04(20)	—
2.34(13)	—
4.88(1.16)	4.45(1.16)
4.65(1.02)	—

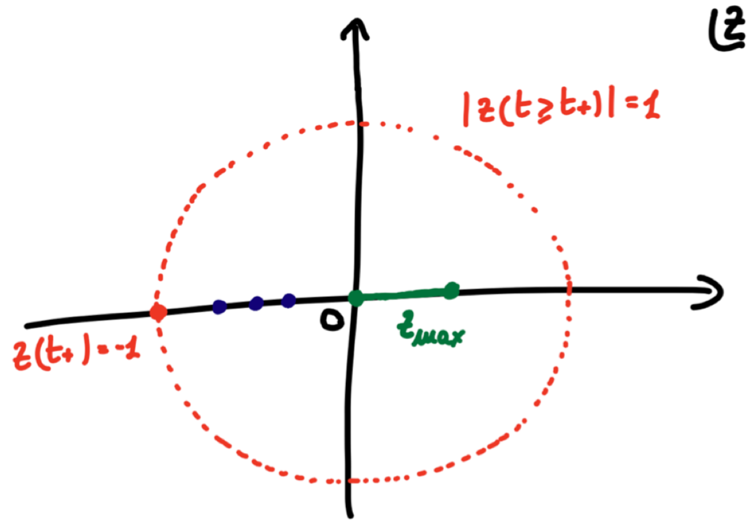
Differences with PT? **~4% for 1^-, ~7% for 0^-, ~20 % for 0^+ and 1^+**

Bigi, Gambino PRD '16
Bigi, Gambino, Schacht PLB '17
Bigi, Gambino, Schacht JHEP '17

Poles & branch cuts

$B \to \pi$

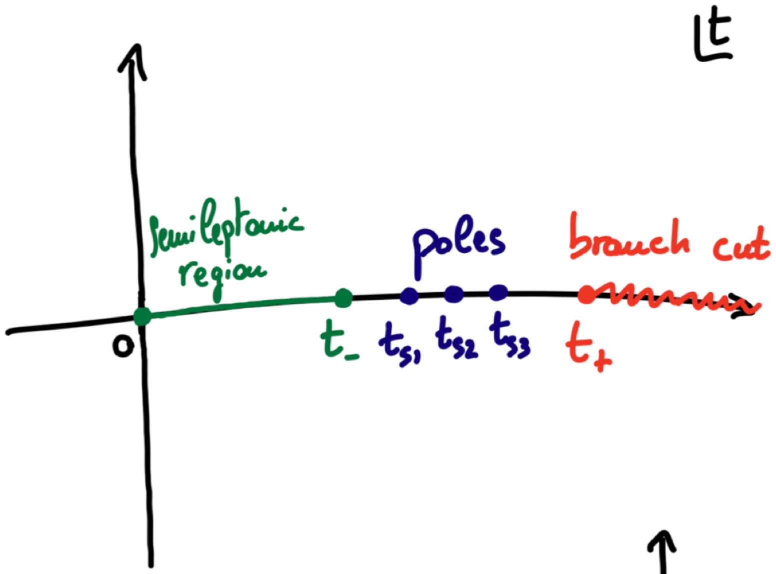


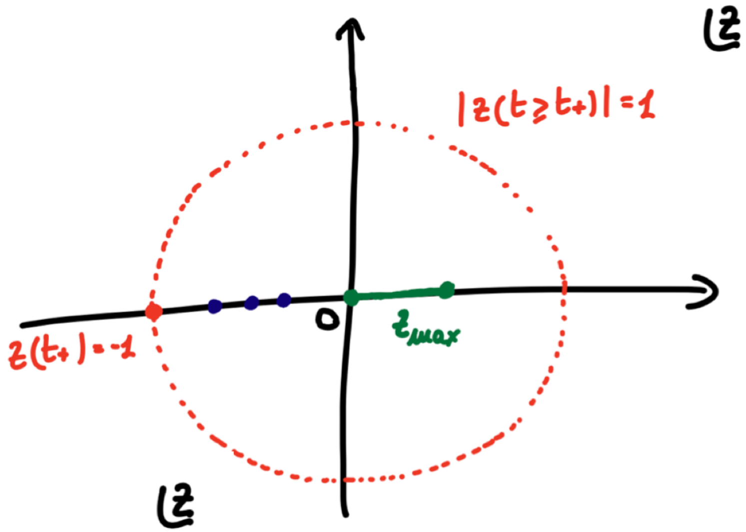


$$t_{\pm} \equiv (m_{B(s)} \pm m_{\pi(K)})^2$$

Poles & branch cuts

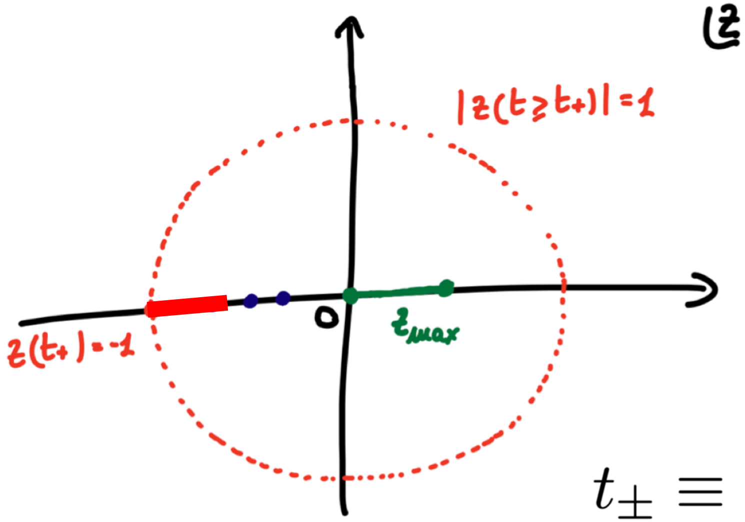
$B \to \pi$





$B_s \to K$

$(\Lambda_b \to p, \ldots)$



$$t_\pm \equiv (m_{B(s)} \pm m_{\pi(K)})^2$$

Poles & branch cuts

How to parametrize the effect of the branch cut?

C: coupling in diagrams connecting the (V − A) current to an external B-D or B-D∗ pair through non-resonant on-shell intermediate states.

$$\mathrm{Im}\, g(t) = C\left(\sqrt{t - M_b^2}\,\theta(t - M_b^2) - \sqrt{t - M_a^2}\,\theta(t - M_a^2)\right)$$

Boyd, Grinstein and Lebed, NPB '96 [arXiv:hep-ph/9508211]

$$M_a^2 = (m_B + m_\pi)^2$$

$$M_b^2 = (m_{B_s} + m_K)^2$$

$$g_{\mathrm{cut}}(z) = 4cM^{s-2}\sqrt{r}\left(\frac{\sqrt{(z - z_a)(1 - zz_a)}}{(1 - z)(1 - z_a)} - \frac{\sqrt{(z - z_b)(1 - zz_b)}}{(1 - z)(1 - z_b)}\right)$$

Poles & branch cuts

At the end of the day: if $f_{\rm cut} = g_{\rm cut}\phi P$, then we have guaranteed the analiticity (on the unit disc) of $\tilde{f}\phi P$, where

$$\tilde{f}(z) = f(z) - g_{\rm cut}(z)$$

How to describe then the unitarity constraint?

$$\left(\int_0^{2\pi} d\theta\, |\tilde{f}\phi|^2\right)^{1/2} \leq \left(\int_0^{2\pi} d\theta\, |f\phi|^2\right)^{1/2} + \left(\int_0^{2\pi} d\theta\, |f_{\rm cut}|^2\right)^{1/2} \leq \sqrt{2\pi}(1 + I_{\rm cut}^{1/2})$$

$$I_{\rm cut} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta\, |f_{\rm cut}|^2$$

In the B_s \to K case, we expect I_{cut} to be small… Moreover:

- We are far from the unitarity limit (practically the 100% of the generated bootstraps is accepted within the DM approach)
- The susceptibilities are affected by big uncertainties…

The Dispersive Matrix (DM) method

Let us examine the case of the production of a pseudoscalar meson (as for the $B \to D$ case). Supposing to have *n* LQCD data for the FFs at the quadratic momenta $\{t_1, \cdots, t_n\}$ (hereafter $t \equiv q^2$), we define

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

$$\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)$$

$$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}$$

CENTRAL REQUIREMENT:

$$\det \mathbf{M} \geq 0$$

The conformal variable z is related to the momentum transfer as:

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

Two advantages:

1. *z is real*
2. *1-to-1 correspondence:*

[0, t_{max}=t_-] ⇨ [z_{max}, 0]

A lot of work in the past:

L. Lellouch, NPB, 479 (1996), p. 353-391
C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp. 157 - 181
E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

We also have to define the kinematical functions

$$\phi_0(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{3t_+t_-}{4\pi}} \frac{1}{t_+ - t_-} \frac{1+z}{(1-z)^{5/2}} \left(\beta(0) + \frac{1+z}{1-z}\right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-2},$$

$$\phi_+(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{1}{\pi(t_+ - t_-)}} \frac{(1+z)^2}{(1-z)^{9/2}} \left(\beta(0) + \frac{1+z}{1-z}\right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-3}, \quad \beta(t) \equiv \sqrt{\frac{t_+ - t}{t_+ - t_-}}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ $\{t_1,\dots,t_n\}$: from Cauchy's theorem (for generic m)

$$\langle g_{t_m} | \phi f \rangle = \phi(t_m, Q^2) f(t_m)$$

LQCD data!

$$\langle g_{t_m} | g_{t_l} \rangle = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle \phi f | \phi f \rangle$$

Since the susceptibilities are computed on the lattice, we can in principle use whatever value of Q^2 !

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle\phi f|\phi f\rangle & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle \end{pmatrix}$$

In the presence of **poles @** $t_{P1}, t_{P2}, \cdots ..., t_{PN}$**:**

$$\phi(z, q^2) \to \phi_P(z, q^2) \equiv \phi(z, q^2) \times \frac{z - z(t_{P1})}{1 - \bar{z}(t_{P1})z} \times \cdots \times \frac{z - z(t_{PN})}{1 - \bar{z}(t_{PN})z}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, *@ {t_1,...,t_n}*: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m}|\phi f\rangle = \phi(t_m, Q^2) f(t_m) \qquad \langle g_{t_m}|g_{t_l}\rangle = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

LQCD data!

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle\phi f|\phi f\rangle$$

The DM method

The positivity of the original inner products guarantee that $\det \mathbf{M} \geq 0$: the solution of this inequality can be computed analitically, bringing to

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: unitarity is satisfied if γ is semipositive definite, namely if

$$\chi \geq \sum_{i,j=1} N f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

This is a **parametrization-independent unitarity test** of the LQCD input data

Kinematical Constraints (KCs)

REMINDER: after the unitarity filter we were left with *N_U < N survived events!!!*

Let us focus on the pseudoscalar case. Since by construction the following *kinematical constraint* holds

$$f_0(0) = f_+(0)$$

we will filter only the $N_{KC} < N_U$ events for which the two bands of the FFs intersect each other @ *t = 0*.
Namely, for each of these events we also define

$$\phi_{lo} = \max[F_{+,lo}(t=0), F_{0,lo}(t=0)]$$

$$\phi_{up} = \min[F_{+,up}(t=0), F_{0,up}(t=0)]$$

From WE theorem

$$\langle D(p_D)|\, V^\mu \,|B(p_B)\rangle \quad = \quad f_+(p_B+p_D)^\mu + f_-(p_B-p_D)^\mu$$

One then defines

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_B^2 - m_D^2} f_-(q^2)$$

$$\langle D(p_D)\,|\,V^\mu\,|\,B(p_B)\rangle = f^+(q^2)\left(p_B^\mu + p_D^\mu - \frac{m_B^2 - m_D^2}{q^2} q^\mu\right) + f^0(q^2)\frac{m_B^2 - m_D^2}{q^2} q^\mu$$

Kinematical Constraints (KCs)

We then consider a **modified matrix**

$$\mathbf{M_C} = \begin{pmatrix} \langle\phi f|\phi f\rangle & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle & \langle\phi f|g_{t_{n+1}}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle & \langle g_t|g_{t_{n+1}}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle & \langle g_{t_1}|g_{t_{n+1}}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle & \langle g_{t_n}|g_{t_{n+1}}\rangle \\ \langle g_{t_{n+1}}|\phi f\rangle & \langle g_{t_{n+1}}|g_t\rangle & \langle g_{t_{n+1}}|g_{t_1}\rangle & \cdots & \langle g_{t_{n+1}}|g_{t_n}\rangle & \langle g_{t_{n+1}}|g_{t_{n+1}}\rangle \end{pmatrix}$$

with $t_{n+1} = 0$. Hence, we compute the new lower and upper bounds of the FFs in this way. For each of the N_{KC} events, we extract $N_{KC,2}$ values of $f_0(0) = f_+(0) \equiv f(0)$ with uniform distribution defined in the range $[\phi_{lo}, \phi_{up}]$. Thus, for both the FFs and for each of the N_{KC} events we define

$$\begin{aligned} F_{lo}(t) &= \min[F_{lo}^1(t), F_{lo}^2(t), \cdots, F_{lo}^{N_{KC,2}}(t)], \\ F_{up}(t) &= \max[F_{up}^1(t), F_{up}^2(t), \cdots, F_{up}^{N_{KC,2}}(t)] \end{aligned}$$