Systematics of U-spin Amplitude Sum Rules

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Based on MG, Y. Grossman and S. Schacht
JHEP, 2022, 278 (2022) arXiv:2205.12975

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Cornell University

Challenge: who can do better with U-spin?

U-spin symmetry

- $SU(3)$ flavor is an approximate symmetry of light quarks u, d, s
- U-spin is an $SU(2)$ subgroup of $SU(3)$ flavor that relates d and s quarks
- Fundamental doublets under U-spin are:

$$\begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} |1/2, +1/2\rangle \\ |1/2, -1/2\rangle \end{bmatrix}, \qquad \begin{bmatrix} \bar{s} \\ -\bar{d} \end{bmatrix} = \begin{bmatrix} |1/2, +1/2\rangle \\ |1/2, -1/2\rangle \end{bmatrix}$$

- $SU(3)$ flavor is broken by $\epsilon = O(30\%)$
- The breaking comes from quark mass differences and electromagnetism

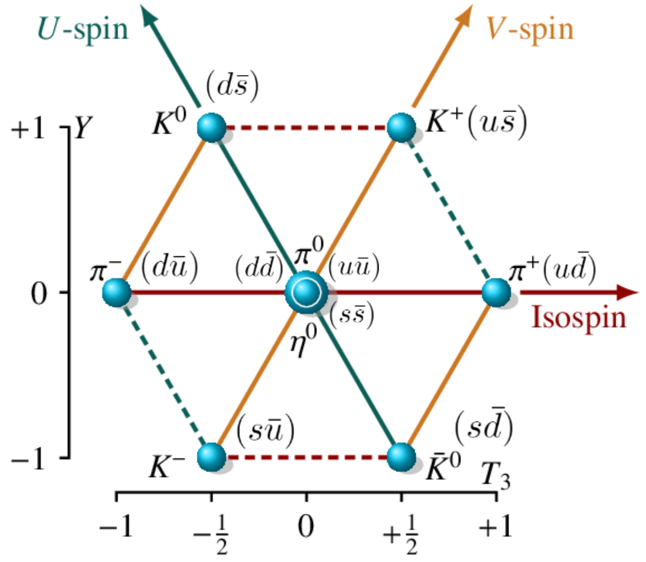

Image from arXiv:1502.07089

U-spin is a "simpler" symmetry since it is broken **only by quark masses**

Example: $D^0 \to P^+P^-$

Initial and final state multiplets:

$$D^0 = |c\bar{u}\rangle = |0,0\rangle, \qquad P^+ = \begin{bmatrix} K^+ \\ \pi^+ \end{bmatrix} = \begin{bmatrix} |u\bar{s}\rangle \\ -|u\bar{d}\rangle \end{bmatrix} = \begin{bmatrix} |1/2,+1/2\rangle \\ |1/2,-1/2\rangle \end{bmatrix}, \qquad P^- = \begin{bmatrix} \pi^- \\ K^- \end{bmatrix} = \begin{bmatrix} |d\bar{u}\rangle \\ |s\bar{u}\rangle \end{bmatrix} = \begin{bmatrix} |1/2,+1/2\rangle \\ |1/2,-1/2\rangle \end{bmatrix}$$

U-spin set of processes:

$$D^0 \to \pi^+K^-, \qquad D^0 \to K^+\pi^-, \qquad D^0 \to \pi^+\pi^-, \qquad D^0 \to K^+K^-$$

Example: $D^0 \to P^+P^-$

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$$D^0 \to \pi^+K^-, \qquad D^0 \to K^+\pi^-, \qquad D^0 \to \pi^+\pi^-, \qquad D^0 \to K^+K^-$$

One of the U-spin limit predictions:

$$\frac{\mathcal{B}(D^0 \to K^+K^-)}{\mathcal{B}(D^0 \to \pi^+\pi^-)} = 1$$

⟵ sum rule

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Data: $$\frac{\mathcal{B}(D^0 \to K^+K^-)}{\mathcal{B}(D^0 \to \pi^+\pi^-)} = 2.8 \pm 0.1$$

$\frac{(1+\epsilon)^2}{(1-\epsilon)^2} \sim 3$ is consistent with $\epsilon \sim 30\%$

R.L. Workman *et al.* (PDG), Prog. Theor. Exp. Phys. 2022, 083C01 (2022)

One message to take home:

Systematic expansion in U-spin breaking can be used for precision physics

↓

Multibody decays allow for precision theory predictions

In our work we performed a systematic study of U-spin Sum Rules. We have found beautiful mathematical structure and now we fully understand higher order U-spin sum rules at the amplitude level.

Outline

- Standard approach to U-spin sum rules
- Systematics of U-spin sum rules ← our work

Standard approach to U-spin sum rules

Standard approach to writing sum rules

- The standard approach to writing sum rules is based on the Wigner-Eckart theorem:

$$\langle u_2; m_2 | O(u,m) | u_1; m_1 \rangle = C^{u_2,m_2}_{\substack{u_1,m_1 \\ u,m}} \langle u_2 | O(u) | u_1 \rangle$$

- Then the amplitudes can be written as (under certain assumptions)

$$\mathcal{A}_j = f_{u,m} \sum_\alpha C_{j\alpha} X_\alpha$$

CKM

Reduced matrix element, α is a multiindex that contains information about u, u_1, u_2 (and b)

number of amplitudes in physical basis > number of RME → **Sum Rules**

$$\mathcal{A}_j = f_{u,m} \sum_\alpha C_{j\alpha} X_\alpha$$

X_α is a short notation for reduced matrix elements

Example: $C_b \to L_b P^+ P^-$

- Below is the matrix $C_{j\alpha}$ up to $b = 2$
- To find the **sum rules** one needs to find the **null space of the matrix $C^T_{j\alpha}$**

Decay amplitude	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
$A(\Lambda_c^+ \to \Sigma^+ K^- K^+)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\sqrt{2}}{3}$	0	0	0
$A(\Xi_c^+ \to p\pi^-\pi^+)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{\sqrt{2}}{3}$	0	0	0
$A(\Lambda_c^+ \to \Sigma^+\pi^-\pi^+)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A(\Xi_c^+ \to pK^-K^+)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A(\Lambda_c^+ \to \Sigma^+\pi^- K^+)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A(\Xi_c^+ \to pK^-\pi^+)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A(\Lambda_c^+ \to pK^-\pi^+)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A(\Xi_c^+ \to \Sigma^+\pi^- K^+)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A(\Lambda_c^+ \to pK^-K^+)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A(\Xi_c^+ \to \Sigma^+\pi^-\pi^+)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A(\Lambda_c^+ \to p\pi^-\pi^+)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{\sqrt{2}}{3}$	0
$A(\Xi_c^+ \to \Sigma^+ K^- K^+)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$-\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{\sqrt{2}}{3}$	0
$A(\Lambda_c^+ \to p\pi^- K^+)$	1	0	0	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0
$A(\Xi_c^+ \to \Sigma^+ K^-\pi^+)$	1	0	0	$-\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{2}}$	0	0	0	0

Decay amplitude	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
$A(\Lambda_c^+ \to \Sigma^+ K^- K^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A(\Xi_c^+ \to p\pi^-\pi^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A(\Lambda_c^+ \to \Sigma^+\pi^-\pi^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A(\Xi_c^+ \to pK^-K^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A(\Lambda_c^+ \to \Sigma^+\pi^- K^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A(\Xi_c^+ \to pK^-\pi^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A(\Lambda_c^+ \to pK^-\pi^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A(\Xi_c^+ \to \Sigma^+\pi^- K^+)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A(\Lambda_c^+ \to pK^-K^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A(\Xi_c^+ \to \Sigma^+\pi^-\pi^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A(\Lambda_c^+ \to p\pi^-\pi^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A(\Xi_c^+ \to \Sigma^+ K^- K^+)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A(\Lambda_c^+ \to p\pi^- K^+)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0
$A(\Xi_c^+ \to \Sigma^+ K^-\pi^+)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0

Note, CKM-free amplitudes

$b = 0$ (X_1–X_3), $b = 1$ (X_4–X_9), $b = 2$ (X_{10}–X_{20})

Standard approach to writing sum rules

- Going to higher orders is hard
- Each U-spin system needs to be treated individually
- Symmetries are obscure
- Doesn’t predict numbers of sum rules at different orders of breaking
- …

Systematics of U-spin Sum Rules

Disclaimer: no proofs, only results. For proofs see arxiv:2205.12975.

*some simplifications and some "−" signs are flowing around. Everything is completely generic in the paper.

Systematics of U-spin sum rules

1) Any U-spin system can be constructed from doublets
2) The movement of irreps between initial/final state and the Hamiltonian doesn't change the structure of sum rules ("crossing symmetry")

We consider a system with the following U-spin structure:

$$0 \to \left(\frac{1}{2}\right)^{\otimes n}, \qquad u = 0$$

(note, n is even)

$$0 \to \left(\frac{1}{2}\right)^{\otimes n}, \qquad u = 0$$

U-spin pairs

$$\pm\frac{1}{2} \to \pm$$

Note, only n-tuples with the same numbers of "+" and "−" are meaningful

Amplitude $A_j: \quad \underbrace{(-,-,+,-,+,\dots,+)}_{n}$ $\qquad A_j = \sum_\alpha C_{j\alpha} X_\alpha$

U-spin conjugation

U-spin conjugate $\bar{A}_j: \quad \underbrace{(+,+,-,+,-,\dots,-)}_{n}$ $\qquad \bar{A}_j = \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$

b is the order of breaking of X_α

- amplitude and its U-spin conjugate form a **U-spin pair**

$$0 \to \left(\frac{1}{2}\right)^{\otimes n}, \qquad u = 0$$

a- and s-type amplitudes

$$A_j = \sum_\alpha C_{j\alpha} X_\alpha$$
$$\bar{A}_j = \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$$

$\longrightarrow$

$$a_j \equiv A_j - \bar{A}_j, \qquad s_j \equiv A_j + \bar{A}_j$$

- all sum rules of the system can be written in terms of a- and s-type amplitudes
- $\boldsymbol{a_j}$ contain only the terms that are **odd in breaking $\boldsymbol{b}$**
- $\boldsymbol{s_j}$ contain only the terms that are **even in breaking $\boldsymbol{b}$**

Decoupling!

- **a-type** sum rules that are **valid up to odd order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- **s-type** sum rules that are **valid up to even order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- for any system there are $n/2$ **trivial a-type sum rules** at $b=0$: $a_j = 0$ [Gronau, arXiv: hep-ph/0008292]
- all sum rules at any order b have the form:

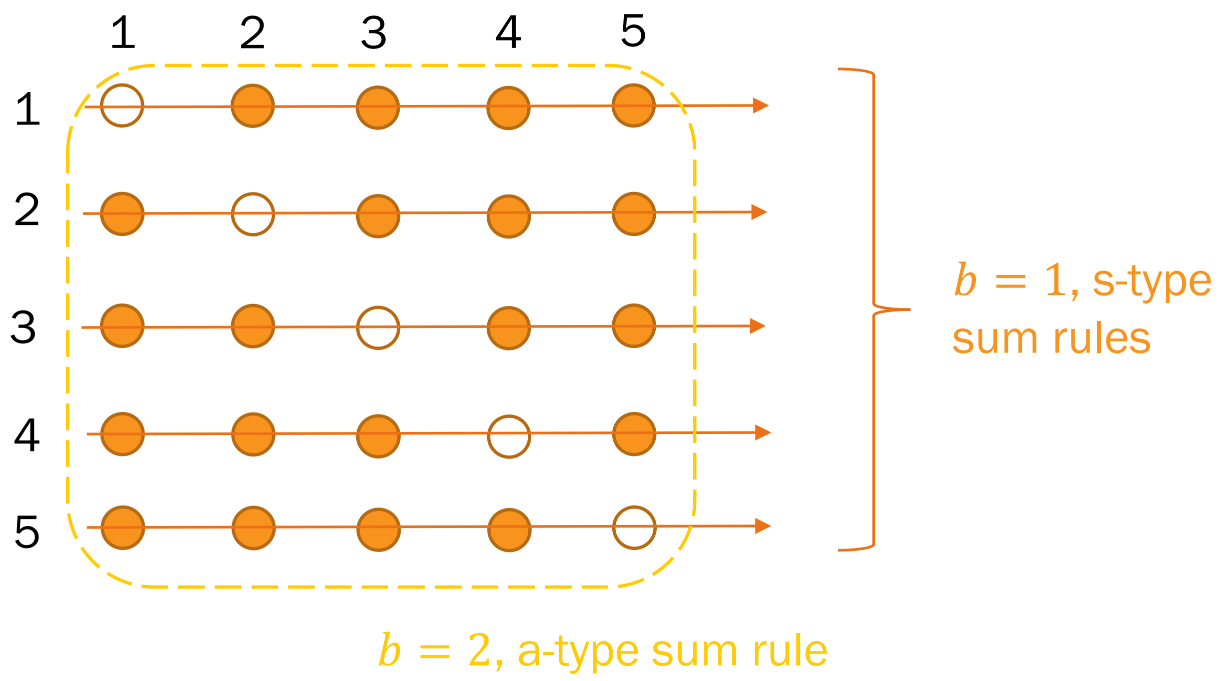
$$\sum a_j = 0 \quad \text{and} \quad \sum s_j = 0$$

Diagrammatic approach: $n = 6$ example

$$d = \frac{n}{2} - 1 = 2$$

- each node corresponds to a U-spin pair
- each node is a trivial a-type sum rule valid up to $b = 0$
- the sums of nodes in lines are s-type sum rules valid up to $b = 1$
- the sum of all nodes of the lattice is an a-type sum rule valid up to $b = 2$

$$\sum a_j = 0 \quad \text{and} \quad \sum s_j = 0$$



$$C_b = \begin{bmatrix} \Lambda_c^+ \\ \Xi_c^+ \end{bmatrix} = \begin{bmatrix} |cud\rangle \\ |cus\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad L_b = \begin{bmatrix} p \\ \Sigma^+ \end{bmatrix} = \begin{bmatrix} |uud\rangle \\ |uus\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

$$P^+ = \begin{bmatrix} K^+ \\ \pi^+ \end{bmatrix} = \begin{bmatrix} |u\bar{s}\rangle \\ -|u\bar{d}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad P^- = \begin{bmatrix} \pi^- \\ K^- \end{bmatrix} = \begin{bmatrix} |d\bar{u}\rangle \\ |s\bar{u}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_{m=-1}^{1} f_{1,m} H_m^1$$

Example: $C_b \to L_b P^+ P^-$

- Sum rules valid up to $b = 0$

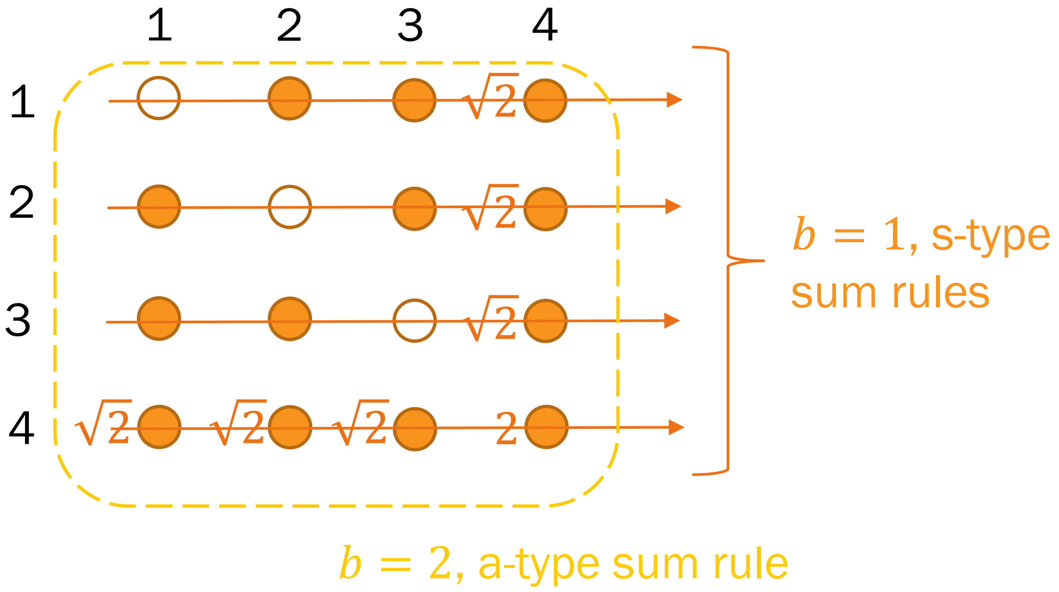
$$a_{(1,2)} = a_{(1,3)} = a_{(1,4)} = a_{(2,3)} = a_{(2,4)} = a_{(3,4)} = a_{(4,4)} = 0.$$

- Sum rules valid up to $b = 1$

$$s_{(1,2)} + s_{(1,3)} + \sqrt{2} s_{(1,4)} = 0$$
$$s_{(1,2)} + s_{(2,3)} + \sqrt{2} s_{(2,4)} = 0$$
$$s_{(1,3)} + s_{(2,3)} + \sqrt{2} s_{(3,4)} = 0$$
$$s_{(1,4)} + s_{(2,4)} + s_{(3,4)} + \sqrt{2} s_{(4,4)} = 0$$

- Sum rules valid up to $b = 2$

$$a_{(1,2)} + a_{(1,3)} + a_{(2,3)} + a_{(4,4)} + \sqrt{2} a_{(1,4)} + \sqrt{2} a_{(2,4)} + \sqrt{2} a_{(3,4)} = 0$$



Summary

- Systematic expansion in U-spin breaking can allow for precision theory predictions
- Our novel approach makes going to higher orders easy
- It implies that “larger systems” (multibody decays) are fundamentally different from two body decays in terms of U-spin and allow for precise predictions
- We are still at the amplitude level, going to observables is a non-trivial step that we are to take next.

Invitation to discussion: what are some interesting multibody decays that we should look at?

Backup

Number of theory parameters $<$ number of observables

Goal of Flavor Physics: overconstrain CKM

This is challenging. Due to QCD there are often more theory parameters than observables

Number of theory parameters ~~<~~ number of observables

Goal of Flavor Physics: overconstrain CKM

This is challenging. Due to QCD there are often more theory parameters than observables

Number of theory parameters ~~<~~ number of observables

Ways to approach the problem:

- Calculate the parameters (lattice)
- Measure the parameters
- Use symmetries to reduce the number of parameters ✓

SU(3) flavor

- $SU(3)$ flavor is an approximate symmetry of light quarks u, d, s
- Generators of $SU(3)$ are given by Gell-Mann matrices λ_i
- $SU(3)$ flavor contains three $SU(2)$ subgroups

Isospin (u, d):	λ_1	λ_2	λ_3
U-spin (d, s):	λ_6	λ_7	$\sqrt{3}\lambda_8 - \lambda_3$
V-spin (u, s):	λ_4	λ_5	$\sqrt{3}\lambda_8 + \lambda_3$

- Can construct rising and lowering operators for each subgroup $\hat{I}_\pm, \hat{U}_\pm, \hat{V}_\pm$
- $SU(3)$ flavor is useful, but broken by $O(30\%)$,

*ϵ is used to parametrize the breaking

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

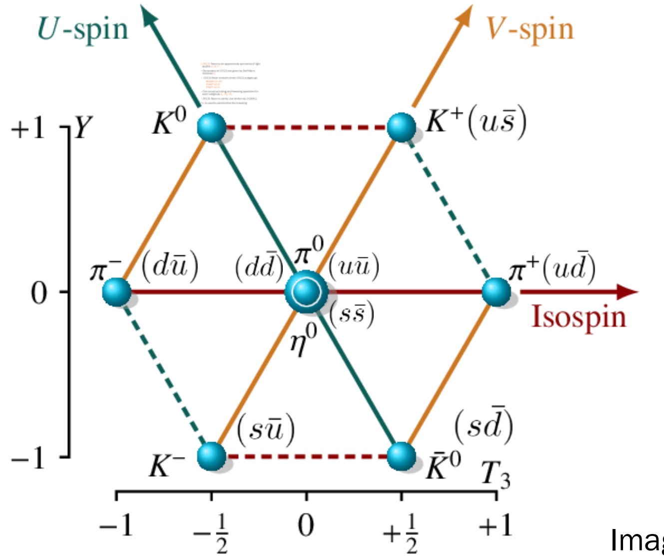


Image from arXiv:1502.07089

SU(3) breaking

- $SU(3)$ flavor is very useful, but **broken by $O(30\%)$ corrections**
- The breaking comes from **quark mass differences** and **electromagnetism**
- In this talk we focus on **U-spin**, the symmetry between d and s

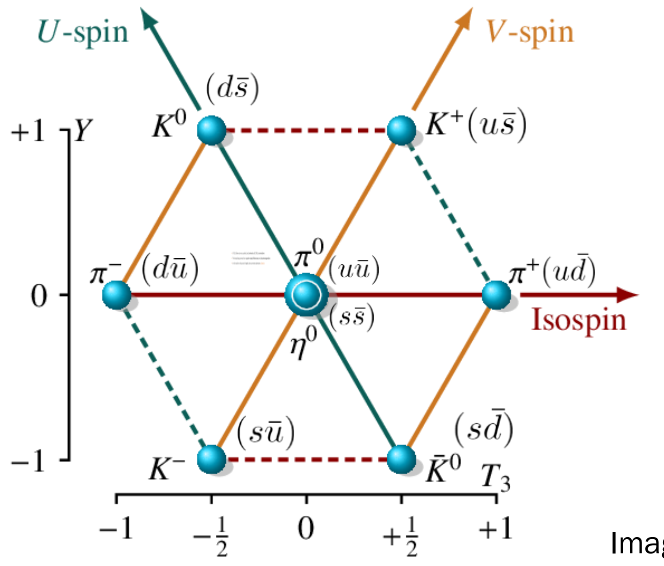


Image from arXiv:1502.07089

Example: $\overline{D}^0 \rightarrow P^+P^-$

Example: $\overline{D} \rightarrow P^+P^-$

ϵ^0: $$\frac{\mathcal{A}(\bar{D}^0 \to \pi^+ K^-)}{V_{cd}V_{us}^*} = \frac{\mathcal{A}(\bar{D}^0 \to K^+ \pi^-)}{-V_{cs}V_{ud}^*} = \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ \pi^-)}{V_{cs}V_{us}^*} = \frac{\mathcal{A}(\bar{D}^0 \to K^+ K^-)}{V_{cs}V_{us}^*}$$

ϵ^1: $$\frac{\mathcal{A}(\bar{D}^0 \to \pi^+ K^-)}{V_{cd}V_{us}^*} + \frac{\mathcal{A}(\bar{D}^0 \to K^+ \pi^-)}{-V_{cs}V_{ud}^*} = \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ \pi^-)}{V_{cs}V_{us}^*} + \frac{\mathcal{A}(\bar{D}^0 \to K^+ K^-)}{V_{cs}V_{us}^*}$$

Summary

- One of the main goals of FP is to overconstrain CKM matrix
- This is challenging due to non-perturbative QCD
- One way to approach the challenge is to use approximate symmetries of QCD to reduce the number of unknown theory parameters
- However, the symmetries at our disposal are approximate and are broken by $O(30\%)$ corrections, so symmetry limit relations are not good enough anymore

Higher order Sum Rules are the way to go!

Status before our work:

- It is mostly understood how to construct the higher order relations, but **no well-established PT** ✓
- Going to **higher orders is hard** ✓
- Not clear **how to go from amplitudes to observables**

Outline

- Definitions and assumptions
- Standard approach to U-spin sum rules
- Systematics of U-spin sum rules
- Concluding remarks

DISCLAIMER: the discussion to follow is about mathematics of U-spin amplitude sum rules

Definitions and assumptions

U-spin set

- Fundamental doublets under U-spin are:

$$\begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad \begin{bmatrix} \bar{s} \\ -\bar{d} \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

- **U-spin set** is a set of amplitudes (processes) that are related by U-spin
- U-spin set is defined via listing the U-spin properties of:
  - initial/final state
  - and the Hamiltonian
- U-spin limit Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_m f_{u,m} H_m^u$$

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- **U-spin set** is a set of amplitudes (processes) that are related by U-spin
- U-spin set is defined via listing the U-spin properties of:
  - initial/final state
  - and the Hamiltonian

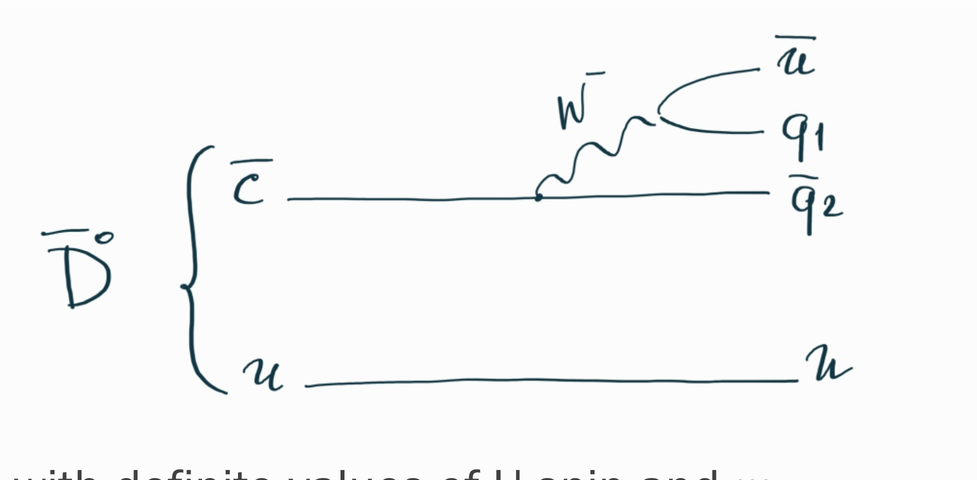
assumption: initial/final state particles are in pure multiplets

- U-spin limit Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_m f_{u,m} H_m^u$$

assumption: only one u is present!

Effective Hamiltonian for $\bar{D}^0 \to P^+P^-$



- Integrating out W, we obtain the following operators: $(\bar{u}q_1)(\bar{q}_2 c)$, where $q_{1,2} = d, s$
- since $q_{1,2} = d, s$ are components of U-spin doublets, the possible values of U-spin are 0 and 1

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

- Operators with definite values of U-spin and m_u

$$H_1^1 = (\bar{u}s)(\bar{d}c), \qquad H_{-1}^1 = -(\bar{u}d)(\bar{s}c), \qquad H_0^1 = \frac{(\bar{u}s)(\bar{s}c) - (\bar{u}d)(\bar{d}c)}{\sqrt{2}}$$

$$H_0^0 = \frac{(\bar{u}s)(\bar{s}c) + (\bar{u}d)(\bar{d}c)}{\sqrt{2}}$$

$$f_{1,1} = V_{cd}^* V_{us}, \qquad f_{1,-1} = -V_{cs}^* V_{ud}, \qquad f_{1,0} = \frac{V_{cs}^* V_{us} - V_{cd}^* V_{ud}}{\sqrt{2}} \approx \sqrt{2}\,(V_{cs}^* V_{us})$$

$$f_{0,0} = \frac{V_{cs}^* V_{us} + V_{cd}^* V_{ud}}{2} \approx 0$$

Approximations hold up to $O(\lambda^4)$, $\lambda \approx 0.22$

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_{u=0,1} \sum_{m=-u}^{u} f_{u,m} H_m^u$$

Example: $\bar{D}^0 \to P^+P^-$

Initial and final state multiplets:

$$\bar{D}^0 = |u\bar{c}\rangle = |0,0\rangle\,, \qquad P^+ = \begin{bmatrix} K^+ \\ \pi^+ \end{bmatrix} = \begin{bmatrix} |u\bar{s}\rangle \\ -|u\bar{d}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2},+\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \end{bmatrix}, \qquad P^- = \begin{bmatrix} \pi^- \\ K^- \end{bmatrix} = \begin{bmatrix} |d\bar{u}\rangle \\ |s\bar{u}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2},+\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \end{bmatrix}$$

U-spin set of processes:

$$\bar{D}^0 \to \pi^+K^-, \qquad \bar{D}^0 \to K^+\pi^-, \qquad \bar{D}^0 \to \pi^+\pi^-, \qquad \bar{D}^0 \to K^+K^-$$

Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_{m=-1}^{1} f_{1,m} H_m^1$$

$u = 1$

$$f_{0,0} = \frac{V_{cs}^* V_{us} + V_{cd}^* V_{ud}}{2} \approx 0$$

$$H_1^1 = (\bar{u}s)(\bar{d}c), \qquad H_{-1}^1 = -(\bar{u}d)(\bar{s}c), \qquad H_0^1 = \frac{(\bar{u}s)(\bar{s}c) - (\bar{u}d)(\bar{d}c)}{\sqrt{2}}$$

Approximations hold up to $O(\lambda^4)$, $\lambda \approx 0.22$

$$f_{1,1} = V_{cd}^* V_{us}, \qquad f_{1,-1} = -V_{cs}^* V_{ud}, \qquad f_{1,0} = \frac{V_{cs}^* V_{us} - V_{cd}^* V_{ud}}{\sqrt{2}} \approx \sqrt{2}\,(V_{cs}^* V_{us})$$

Expansion in the U-spin breaking

- On the fundamental level the U-spin breaking comes from the **mass difference between strange and down quarks**
- The small parameter is $\epsilon = \frac{m_s - m_d}{\Lambda_{QCD}} \sim 0.3$
- The breaking is realized via spurion $H_\epsilon \propto \Delta m\,(s\bar{s} - d\bar{d})$ with $u = 1,\ m = 0$

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_m f_{u,m} H_m^u \longrightarrow \mathcal{H}_{\text{eff}} = \sum_{m,b} f_{u,m} \left(H_m^u \otimes H_\epsilon^{\otimes b} \right)$$

b is the order of U-spin breaking

$$H_\varepsilon^{\otimes b} \equiv \underbrace{H_\varepsilon \otimes \cdots \otimes H_\varepsilon}_{b}$$

Standard approach to U-spin sum rules

Standard approach to writing sum rules

1) Basis rotation: from physical to U-spin basis
2) Wigner-Eckart theorem

Physical and U-spin bases

- **Physical basis** is the basis in which each particle in the initial/final state is represented by a component of a U-spin multiplet with definite value of U-spin, and the Hamiltonian operators are written as tensor products of operators from U-spin limit Hamiltonian and possibly several insertions of U-spin breaking spurion.

$|in\rangle$, $|out\rangle$ and H_{eff} are written as tensor products of U-spin reps

$$\mathcal{A}_j = \langle \text{out}|\mathcal{H}_{\text{eff}}|\text{in}\rangle_j$$

j is a multiindex that contains information about m-QN of initial, final state and the Hamiltonian

- **U-spin basis** is a basis in which initial state, final state and all terms in the Hamiltonian have definite values of total U-spin.

$$\mathcal{A}_j = f_{u,m}\, A_j^{(\text{phys})} = f_{u,m} \sum c_{ji} A_i^{(U\text{-spin})}$$

CKM

CKM-free amplitudes in physical basis, often denote as A_j

Rotational matrix between physical and U-spin bases

Amplitudes in U-spin basis

Wigner-Eckart theorem

Theorem. Matrix elements of spherical tensor operators in the basis of angular momentum eigenstates can be always written as a product of two constants: one that is independent on the orientation of angular momentum and one that is dependent.

Mathematically this statement can be written as follows:

An amplitude in U-spin basis, $A_i^{(U-spin)}$ →

$$\langle u_2; m_2 | O(u,m) | u_1; m_1 \rangle = C^{u_2,m_2}_{\substack{u_1,m_1\\u,m}} \langle u_2 | O(u) | u_1 \rangle$$

Clebsch-Gordan coefficient, **depends on m-QN**

Reduced matrix element, **doesn't depend on m-QN**

number of amplitudes in phys basis > number of RME → sum rules

Standard approach to writing sum rules

1) Basis rotation: from physical to U-spin basis

2) Wigner-Eckart theorem

Amplitude in the **physical basis** (states and the Hamiltonian are given by tensor products):

$$\mathcal{A}_j = \langle \text{out} | \mathcal{H}_{\text{eff}} | \text{in} \rangle_j$$

Wigner-Eckart theorem:

$$\langle u_2; m_2 | O(u, m) | u_1; m_1 \rangle = C^{u_2, m_2}_{\substack{u_1, m_1 \\ u, m}} \langle u_2 | O(u) | u_1 \rangle$$

$$\mathcal{A}_j = f_{u,m} \sum_\alpha C_{j\alpha} X_\alpha$$

Reduced matrix element, α is a multiindex that contains information about u, u_1, u_2 (and b)

number of amplitudes in phys basis > number of RME → sum rules

$$\mathcal{A}_j = f_{u,m} \sum_\alpha C_{j\alpha} X_\alpha$$

X_α is a short notation for reduced matrix elements

Example: $C_b \to L_b P^+ P^-$

- Below is the matrix $C_{j\alpha}$ up to $b = 2$
- To find the **sum rules** one needs to find the **null space of the matrix $C^T_{j\alpha}$**

Decay amplitude	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to pK^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to pK^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to pK^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to pK^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{\sqrt{2}}{3}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$-\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{\sqrt{2}}{3}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	1	0	0	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	1	0	0	$-\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{2}}$	0	0	0	0

Decay amplitude	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to pK^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to pK^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to pK^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to pK^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0

Note, CKM-free amplitudes

$b = 0$ (X_1–X_3) $b = 1$ (X_4–X_9) $b = 2$ (X_{10}–X_{20})

Systematics of U-spin Sum Rules

Disclaimer: no proofs, only results. For proofs see arxiv:2205.12975.

*some simplifications and some "−" signs are flowing around. Everything is completely generic in the paper.

$$0 \to \left(\frac{1}{2}\right)^{\otimes n}, \qquad u = 0$$

U-spin pairs

$$\pm\frac{1}{2} \to \pm$$

Note, only n-tuples with the same numbers of "+" and "−" are meaningful

Amplitude $\quad A_j : \quad \underbrace{(-,-,+,-,+,\ldots,+)}_{n} \qquad A_j = \sum_\alpha C_{j\alpha} X_\alpha$

U-spin conjugation

U-spin conjugate $\quad \bar{A}_j : \quad \underbrace{(+,+,-,+,-,\ldots,-)}_{n} \qquad \bar{A}_j = \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$

$(-1)^b$ depends on the order of breaking

- amplitude and its U-spin conjugate form a **U-spin pair**

a- and s-type amplitudes

$$A_j = \sum_\alpha C_{j\alpha} X_\alpha$$
$$\bar{A}_j = \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$$

→

$$a_j \equiv A_j - \bar{A}_j, \qquad s_j \equiv A_j + \bar{A}_j$$

- all sum rules of the system can be written in terms of a- and s-type amplitudes
- $\boldsymbol{a_j}$ contain only the terms that are **odd in breaking $\boldsymbol{b}$**
- $\boldsymbol{s_j}$ contain only the terms that are **even in breaking $\boldsymbol{b}$**

Decoupling!

- **a-type** sum rules that are **valid up to odd order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- **s-type** sum rules that are **valid up to even order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- for any system there are $n/2$ **trivial a-type sum rules** at $b = 0$: $a_j = 0$ [Gronau, arXiv: hep-ph/0008292]
- all sum rules at any order b have the form:

$$\sum a_j = 0 \quad \text{and} \quad \sum s_j = 0$$

Coordinate notation

Amplitude $A_j: \quad \underbrace{(-,-,+,-,+,\ldots,+)}_{n}$

U-spin conjugate $\bar{A}_j: \quad \underbrace{(+,+,-,+,-,\ldots,-)}_{n}$

We choose to denote U-spin pairs via n-tuples that start with "$-$" sign

To write an amplitude pair in coordinate notation:

- enumerate all positions of n-tuple starting with "0"
- the first position is always fixed to be "$-$" according to our convention
- the coordinate notation is a string of $n/2-1$ numbers, the positions of "$-$" signs excluding the first one
- amplitudes in coordinate notation can be viewed as nodes of $d = n/2-1$ dimensional lattice

Example, $n = 8$: $\quad (\underset{0}{-},\underset{1}{-},\underset{2}{-},\underset{3}{+},\underset{4}{-},\underset{5}{+},\underset{6}{+},\underset{7}{+}) = (1,2,4)$

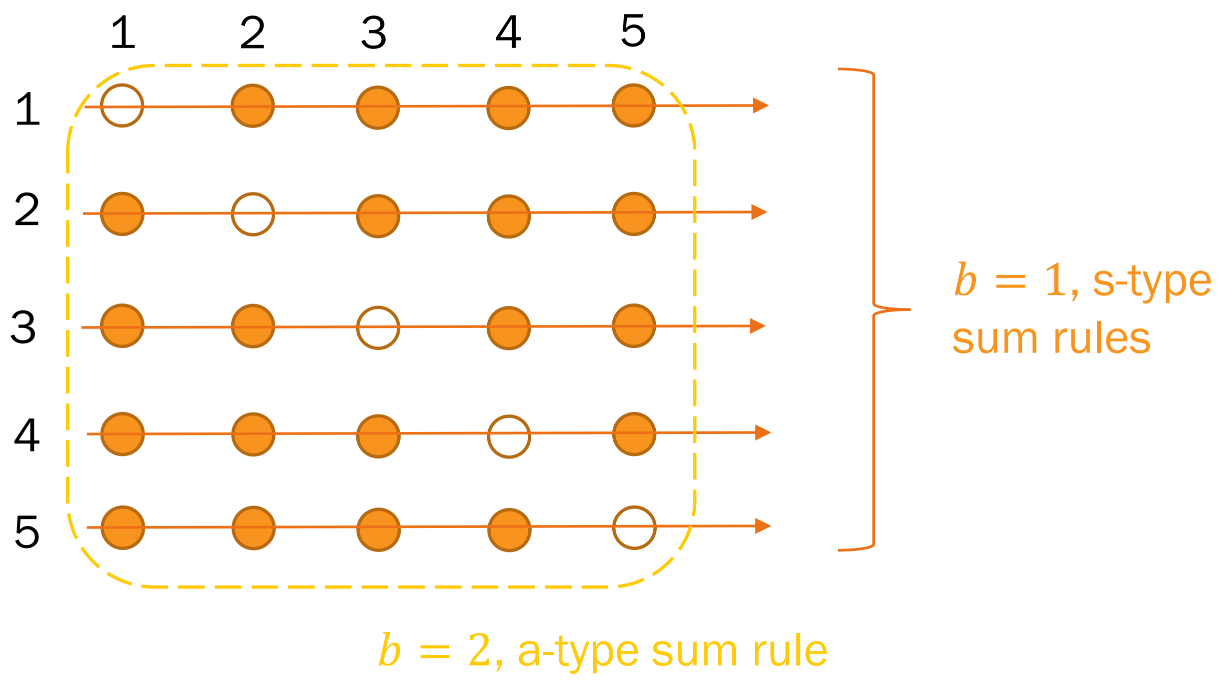
$$(1,2,4) = (1,4,2) = (2,1,4) = (2,4,1) = (4,1,2) = (4,2,1)$$

Diagrammatic approach: $n = 6$ example

$$d = \frac{n}{2} - 1 = 2$$

- each node corresponds to a U-spin pair
- each node is a trivial a-type sum rule valid up to $b = 0$
- the sums of nodes in lines are s-type sum rules valid up to $b = 1$
- the sum of all nodes of the lattice is an a-type sum rule valid up to $b = 2$

$$\sum a_j = 0 \quad \text{and} \quad \sum s_j = 0$$



Example: $C_b \to L_b P^+ P^-$

Initial and final state multiplets:

$$C_b = \begin{bmatrix} \Lambda_c^+ \\ \Xi_c^+ \end{bmatrix} = \begin{bmatrix} |cud\rangle \\ |cus\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad L_b = \begin{bmatrix} p \\ \Sigma^+ \end{bmatrix} = \begin{bmatrix} |uud\rangle \\ |uus\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

$$P^+ = \begin{bmatrix} K^+ \\ \pi^+ \end{bmatrix} = \begin{bmatrix} |u\bar{s}\rangle \\ -\left|u\bar{d}\right\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad P^- = \begin{bmatrix} \pi^- \\ K^- \end{bmatrix} = \begin{bmatrix} |d\bar{u}\rangle \\ |s\bar{u}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_{m=-1}^{1} f_{1,m} H_m^1$$

What is next?

- generalize coordinate notation
- generalize the lattice algorithm to account for a triplet representation

Generalization

- Recall that all higher U-spin irreps can be build from doublets. For example, for $C_b \to L_b P^+ P^-$ we need Hamiltonian with $u = 1$:

$$H_1^1 = |{+}{+}\rangle\,, \qquad H_{-1}^1 = |{-}{-}\rangle\,, \qquad H_0^1 = \frac{|{+}{-}\rangle + |{-}{+}\rangle}{\sqrt{2}}$$

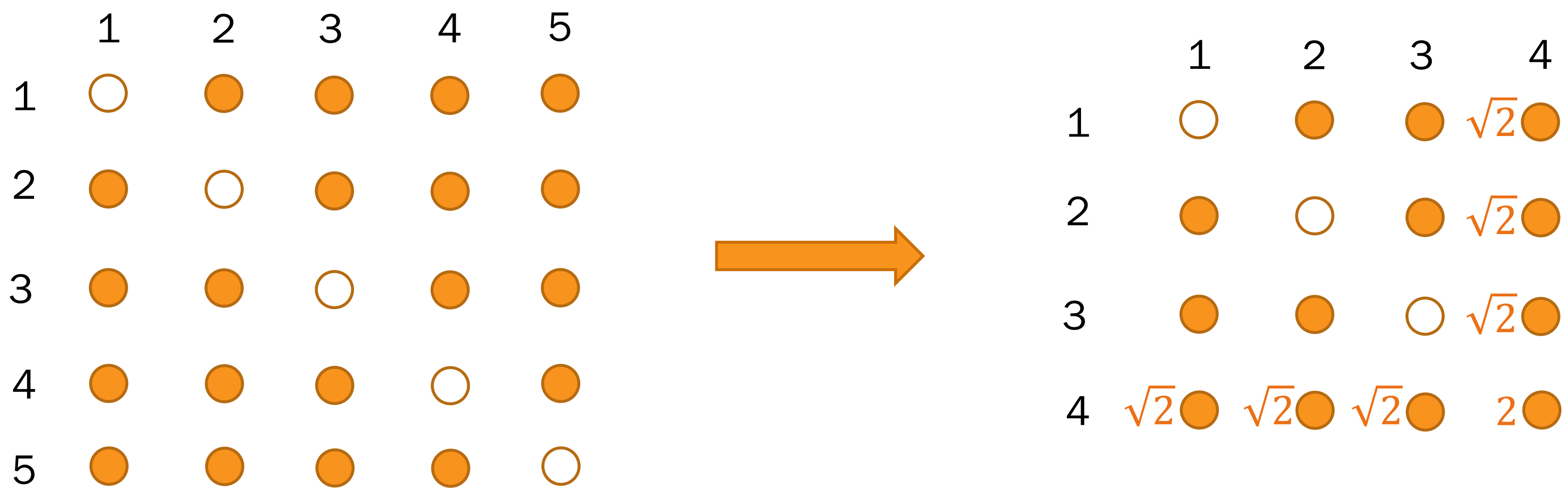
- Introduce n, the number of "would be doublets", the minimum number of doublets needed to build all the irreps of the system. For example, for $C_b \to L_b P^+ P^-$ $n = 6$.

Generalize the coordinate notation as follows:

- order irreps of the system in some arbitrary, but fixed order. For example, for $C_b \to L_b P^+ P^-$ we can choose $u_0 = u_1 = u_2 = u_3 = 1/2$, $u_4 = 1$.
- we label entries of n-tuple by indices of the corresponding irreps.
- the coordinate notation is then given by $n/2 - 1$ numbers, the positions of "$-$" signs. For example, for $C_b \to L_b P^+ P^-$ we can have:

$$(\underset{0}{-}, \underset{1}{+}, \underset{2}{+}, \underset{3}{+}, \underset{4}{--}) = (4, 4) \qquad (\underset{0}{-}, \underset{1}{+}, \underset{2}{-}, \underset{3}{+}, \underset{4}{-+}) = (2, 4)$$

Generalized lattice



Reminder: $C_b \to L_b P^+ P^-$

Decay amplitude	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to pK^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to pK^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to pK^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to pK^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{\sqrt{2}}{3}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$-\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{\sqrt{2}}{3}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	1	0	0	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	1	0	0	$-\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{2}}$	0	0	0	0

Decay amplitude	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to pK^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to pK^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to pK^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to pK^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0

Note, CKM-free amplitudes

$b=0$ (X_1–X_3) $b=1$ (X_4–X_9) $b=2$ (X_{10}–X_{20})

Example: $C_b \to L_b P^+ P^-$

- Sum rules valid up to $b = 0$

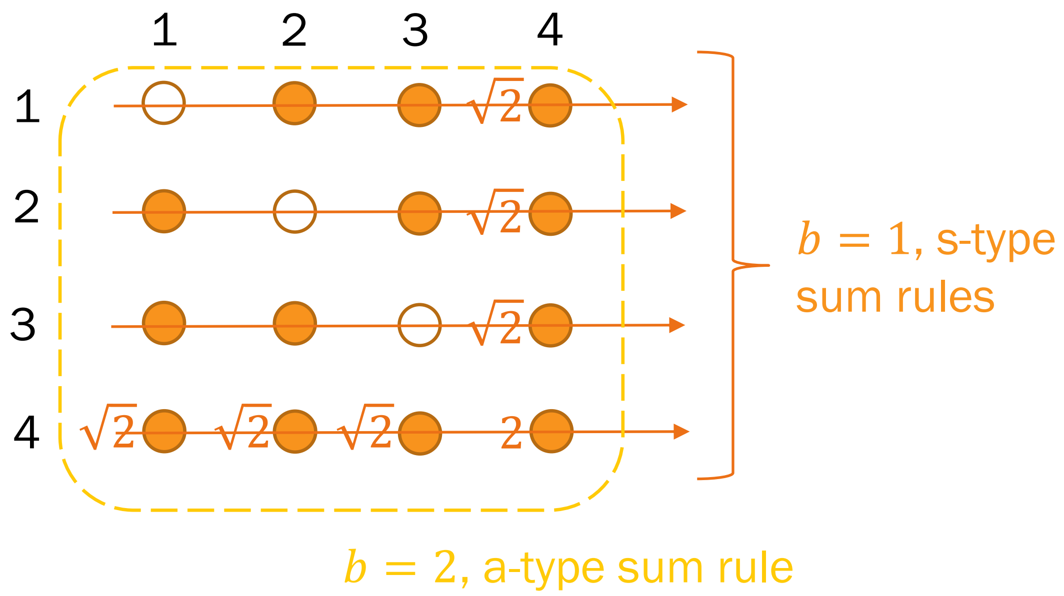
$$a_{(1,2)} = a_{(1,3)} = a_{(1,4)} = a_{(2,3)} = a_{(2,4)} = a_{(3,4)} = a_{(4,4)} = 0.$$

- Sum rules valid up to $b = 1$

$$s_{(1,2)} + s_{(1,3)} + \sqrt{2}s_{(1,4)} = 0$$
$$s_{(1,2)} + s_{(2,3)} + \sqrt{2}s_{(2,4)} = 0$$
$$s_{(1,3)} + s_{(2,3)} + \sqrt{2}s_{(3,4)} = 0$$
$$s_{(1,4)} + s_{(2,4)} + s_{(3,4)} + \sqrt{2}s_{(4,4)} = 0$$

- Sum rules valid up to $b = 2$

$$a_{(1,2)} + a_{(1,3)} + a_{(2,3)} + a_{(4,4)} + \sqrt{2}a_{(1,4)} + \sqrt{2}a_{(2,4)} + \sqrt{2}a_{(3,4)} = 0$$



Conclusions

What did we do?

- Performed systematic study of U-spin amplitude sum rules & found reach mathematical structure
- We have a way to write all the sum rules to any order of the symmetry breaking & know the number of sum rules without any calculation
- New method makes going to higher orders easy
- Uniform form of sum rules for any U-spin system
- Results can be also applied to semileptonic decays (both U-spin and Isospin)

We now fully understand how to derive the higher order amplitude sum rules!

What is next?

- Going from amplitude level to observables – not trivial!
- Systematics of $SU(3)$ flavor sum rules
- Breaking in phase space
- Treatment of resonances