Phase transitions in the early universe: a 2PI approach

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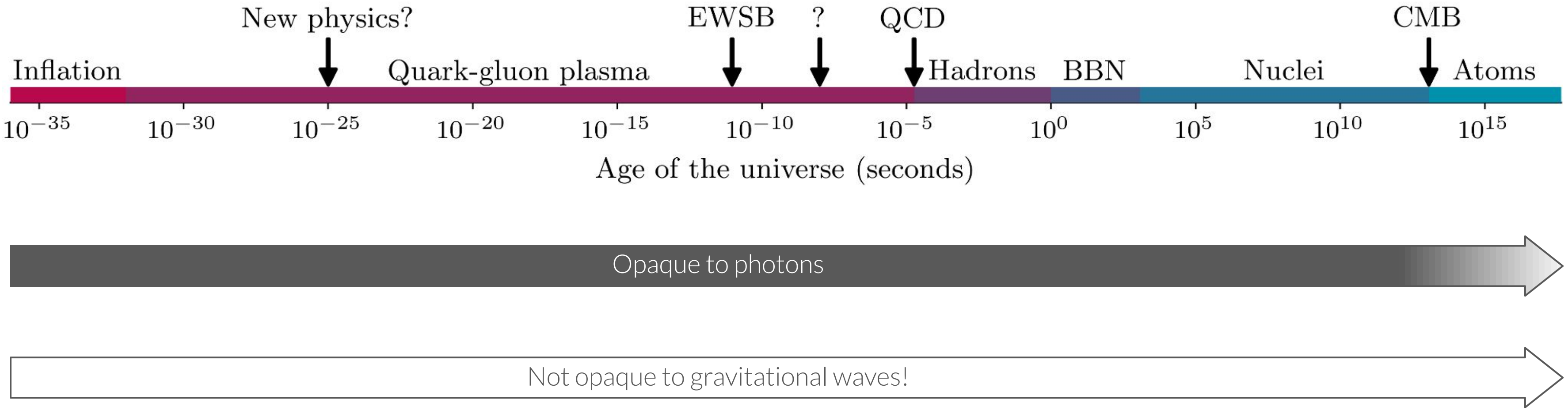
PONT, Avignon • May 3, 2023

[arxiv:2104.10687] • [arxiv:23XX.XXXXX]

Probing the early universe with gravitational waves

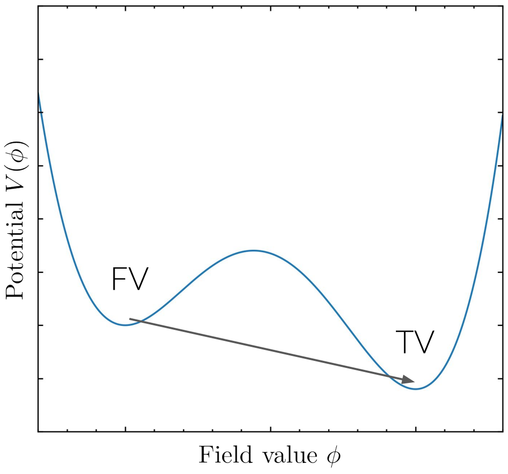


Gravitational waves are our only direct probe of the early universe – and whatever new physics may lurk in its thermal history

[Adapted from D Croon]

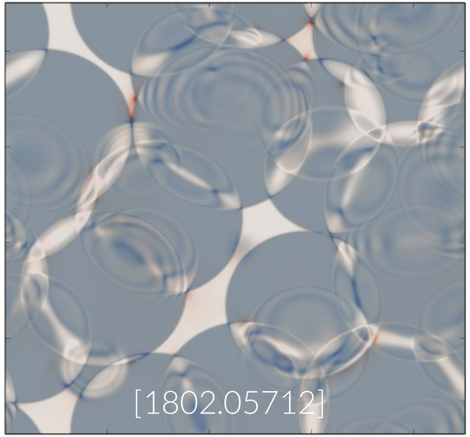
GWs from phase transitions: theory + experiment

False vacuum decay



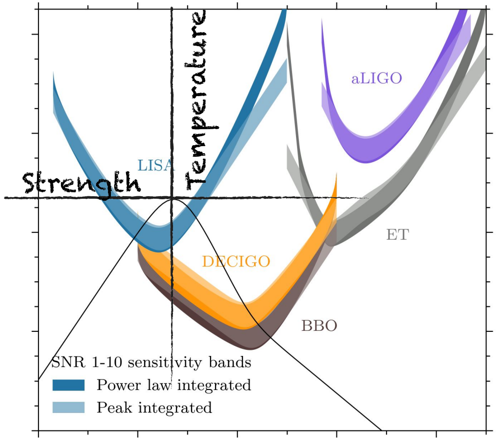
Perturbative bounce formalism: $T^*, \gamma \leftrightarrow \alpha, \beta$

Bubble collisions



Hydrodynamic simulations [arxiv:1910.13125]

GW background



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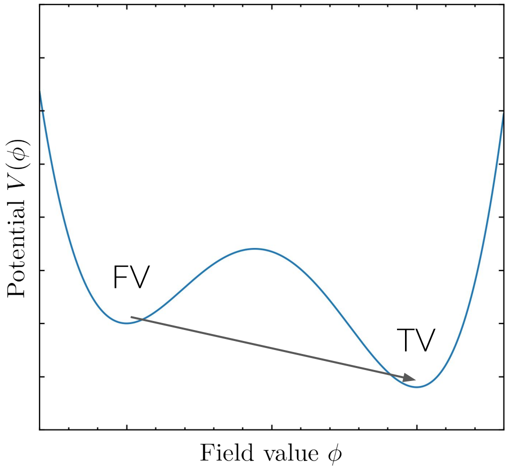

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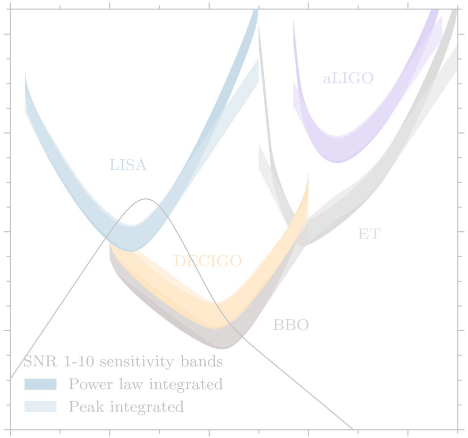
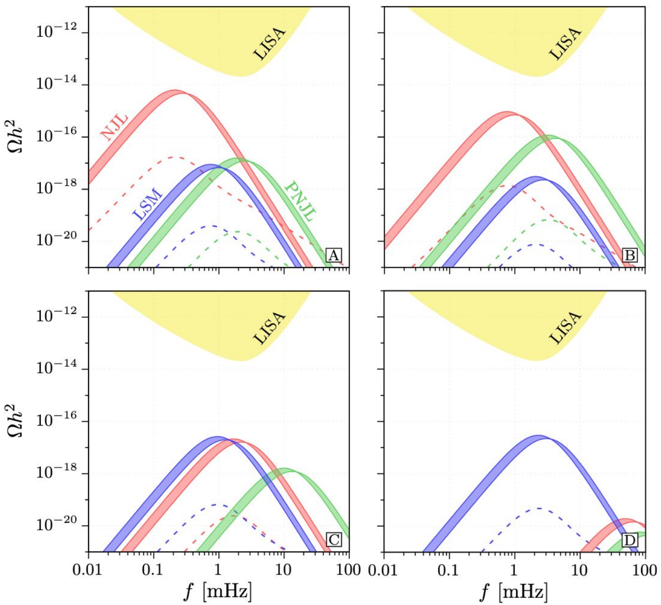
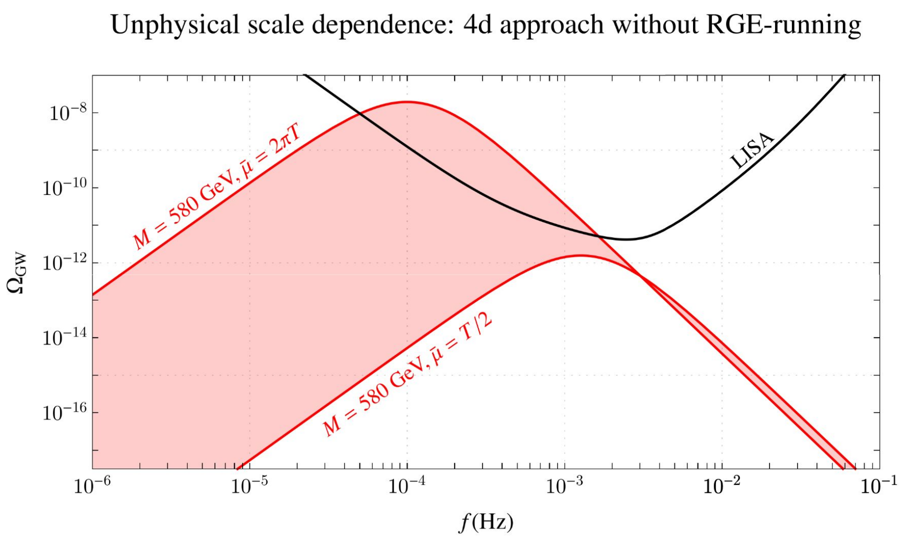


Theoretical outlook: work to do

Perturbative analysis **fails for strong couplings**

[Helmholdt, Kubo, van der Woude: 1904.07891]





Huge uncertainty even from perturbative models

[Croon, Oliver Gould, Schicho, Tenkanen, White: 2009.10080]

False vacuum decay in the direct method

Direct method: non-perturbative definition of FV decay from first principles

[Andreassen, Farhi, Frost, Schwartz: 1602.01102, 1604.06090]

$$\gamma = \left| 2\mathrm{Im}\left(\frac{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi\, e^{-S[\phi]} \delta(\tau_\Sigma[\phi])}{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi\, e^{-S[\phi]}} \right)_{\substack{T>0 \\ T=i\mathcal{T}}} \right|$$

Boundary condition

$$\phi(\pm\mathcal{T}) = \phi_F$$

Delta function

For the turning point time

Analytic continuation

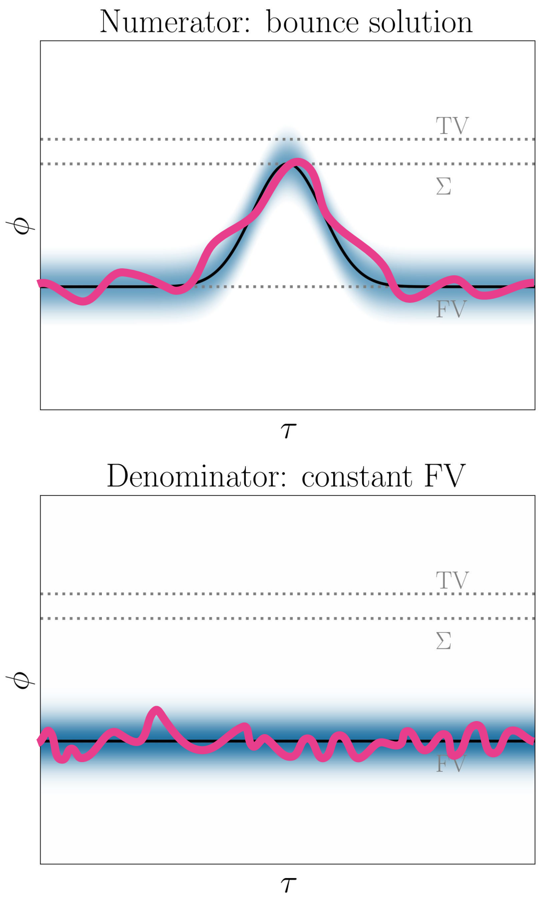
From Euclidean to Minkowski gives imaginary part

Recovering the perturbative approach

Expands locally around each classical saddle point

$$\gamma = \left| 2\mathrm{Im} \left(\frac{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi \, e^{-S[\phi]} \delta(\tau_\Sigma[\phi])}{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi \, e^{-S[\phi]}} \right)_{\substack{T>0 \\ T=i\mathcal{T}}} \right|$$

$$\simeq A \, e^{-(S[\phi_b]-S[\phi_F])}$$



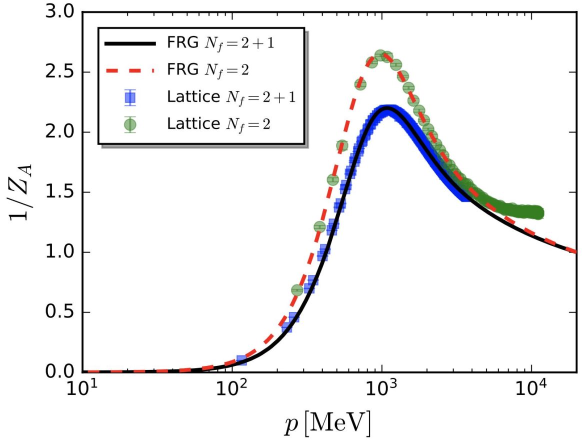
Higher orders: perturbative effective potential

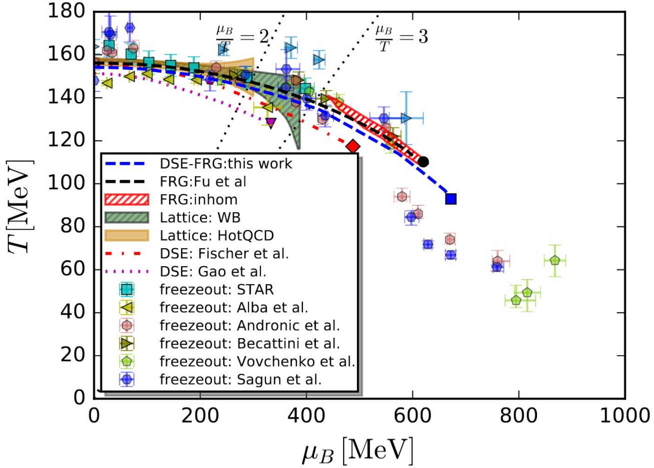
$$\gamma \simeq A e^{-(S_{\mathrm{eff}}[\phi_b]-S_{\mathrm{eff}}[\phi_F])}$$

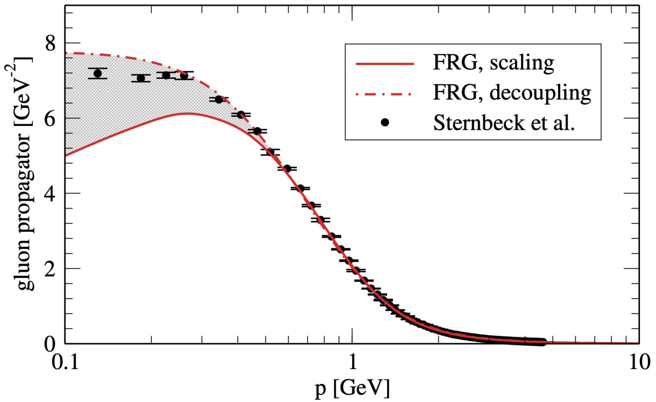
The functional renormalization group: QCD

Highly successful in QCD in last decade, e.g.

[Skokov, Friman, Redlich: 1008.4570]
[Herbst, Pawlowski, Schaefer: 1302.1426]
[Cyrol, Fister, Mitter, Pawlowski, Strodthoff: 1605.01856]
[Fu, Pawlowski, Rennecke: 1909.02991]
[Gao, Pawlowski: 2002.07500]





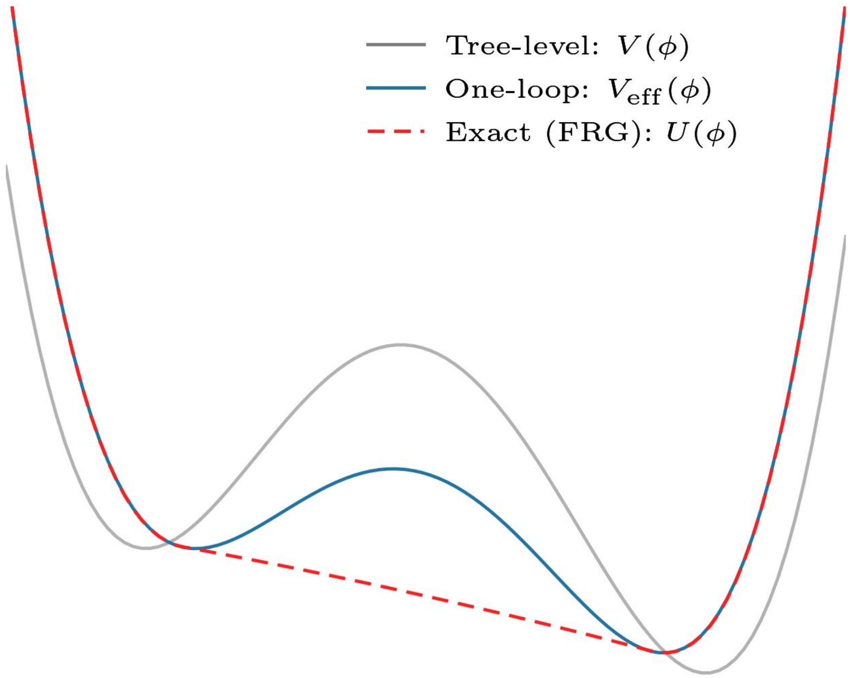


BUT: exact effective actions are convex

Maxwell construction: convexity due to non-local field configurations that interpolate between minima

Exact effective actions don't describe tunneling without modifications

[Weinberg and Wu, 1987]



Back to the drawing board: quasi-stationary patches

Generalization of saddle-point approximation: regions of quasi-stationary field configurations which dominate integral

$$\int \mathcal{D}\phi\, e^{-S[\phi]}\,\Theta_{\rm BC}[\phi] \simeq \sum_n \Theta_{\rm BC}[\bar{\phi}_n] \int \mathcal{D}\phi\, e^{-S[\phi]} w_k[\phi, \bar{\phi}_n] \ \simeq \sum_n \Theta_{\rm BC}[\bar{\phi}_n] e^{-\Gamma_k[\bar{\phi}_n]}$$

This defines a quasi-stationary effective action

$$\gamma = \left| 2{\rm Im}\left(\frac{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi\, e^{-S[\phi]}\delta(\tau_\Sigma[\phi])}{\int_{\phi_F}^{\phi_F} \mathcal{D}\phi\, e^{-S[\phi]}} \right)_{\substack{T>0\\ T=i\mathcal{T}}} \right| = \mathcal{J} e^{-(\Gamma_k[\bar{\phi}_b]-\Gamma_k[\bar{\phi}_F])}$$

“QSEA condition”

$$w_k[\phi, \bar{\phi}] = e^{-\frac{1}{2}\int R_k(\phi-\bar{\phi})^2}$$

$$\varphi^2 = (\phi-\bar{\phi})^2 \lesssim k^{-2}\mathbf{1}$$

$$R_k \text{ s.t. } \partial^2 + V'' + R_k \geq k^2$$

Non-perturbative implementation: the FRG *for fluctuations*

To formulate the QSEA in the language of the FRG, we introduce a *modified FRG* in terms of *fluctuation size* rather than momentum scale.

Regulator *functional* added to the action freezes out *large fluctuations* $\varphi^2 \gtrsim k^{-2}\mathbf{1}$

$$R_k[\bar{\phi};p] = \left(k^2 - \frac{1}{\mathcal{V}}\int_x (p^2 + V''(\bar{\phi}(x))\right) \cdot\Theta\left(k^2 - \frac{1}{\mathcal{V}}\int_x (p^2 + V''(\bar{\phi}(x))\right)$$

Scale-dependent effective action = *QSEA* at a scale k

$$\Gamma_k[\bar{\phi}] = -W_k[J,\bar{\phi}] + \int J\bar{\phi} - \Delta S_k[\bar{\phi}]$$

$$W_k[J,\bar{\phi}] = \ln \int \mathcal{D}\phi \, \exp\left[-S[\phi] + \int J\phi - \Delta S_k[\phi,\bar{\phi}]\right]$$

Exact flow equation *receives modifications through the propagator*

$$\partial_k\Gamma_k[\bar{\phi}] = \frac{1}{2}\int_p (\partial_k R_k) G_k[\bar{\phi};-p,p]$$

Robust approximation schemes like the derivative expansion and vertex expansion that don't spoil the non-perturbativity *still can be used*

Non-perturbative implementation: the FRG *for fluctuations*

Local Potential Approximation

$$\Gamma_k[\bar{\phi}] = \int_x \left[\frac{1}{2}(\partial\bar{\phi})^2 + U_k(\bar{\phi})\right]$$

$$\partial_k U_k = \frac{1}{\mathcal{V}}\int_p (\partial_k R_k) G_k[\bar{\phi}; -p, p] \quad (\text{const. } \bar{\phi})$$

Closed form solutions

$$G_k[\bar{\phi}] = \frac{\delta(p+q)}{\tilde{k}^2(\phi) + U_k''(\phi)} \frac{2}{1 + \sqrt{1 + \frac{V''''(\phi)\tilde{k}^4(\phi)}{16\pi^2(\tilde{k}^2 + U_k'')^2}}}$$

$$\partial_k U_k = \frac{k\tilde{k}^4(\bar{\phi})\Theta(\tilde{k}^2)}{32\pi^2(\tilde{k}^2 + U_k'')} \frac{2}{1 + \sqrt{1 + \frac{V''''(\phi)\tilde{k}^4(\phi)}{16\pi^2(\tilde{k}^2 + U_k'')^2}}}$$

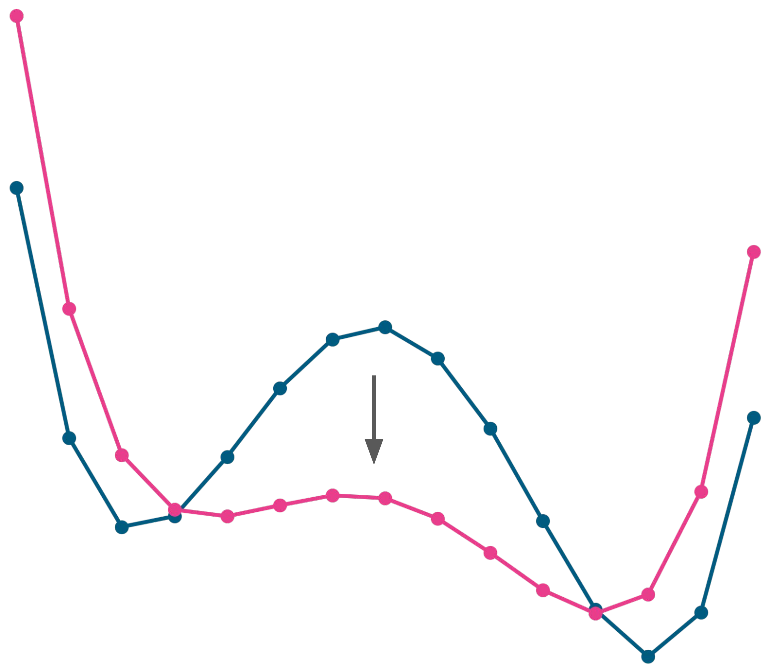
$$\tilde{k}^2 = k^2 + V''$$

Solving the flow equation

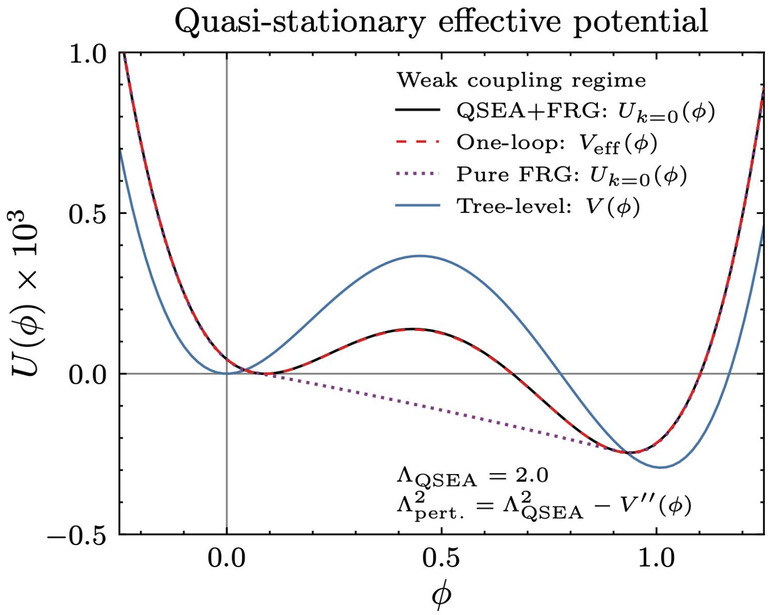
Flow equation is just a differential equation!

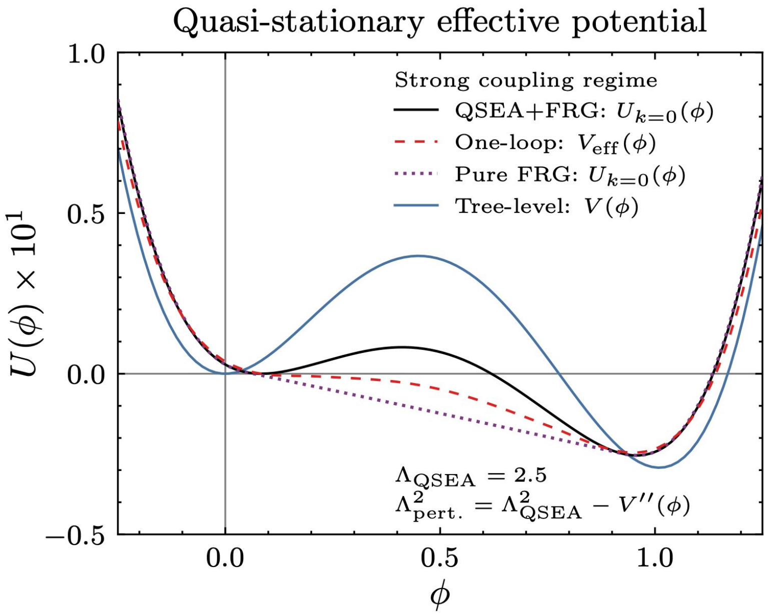
Can be straightforwardly solved in a few lines Mathematica or with SciPy's built-in differential equation solvers

Evaluation time ~ seconds



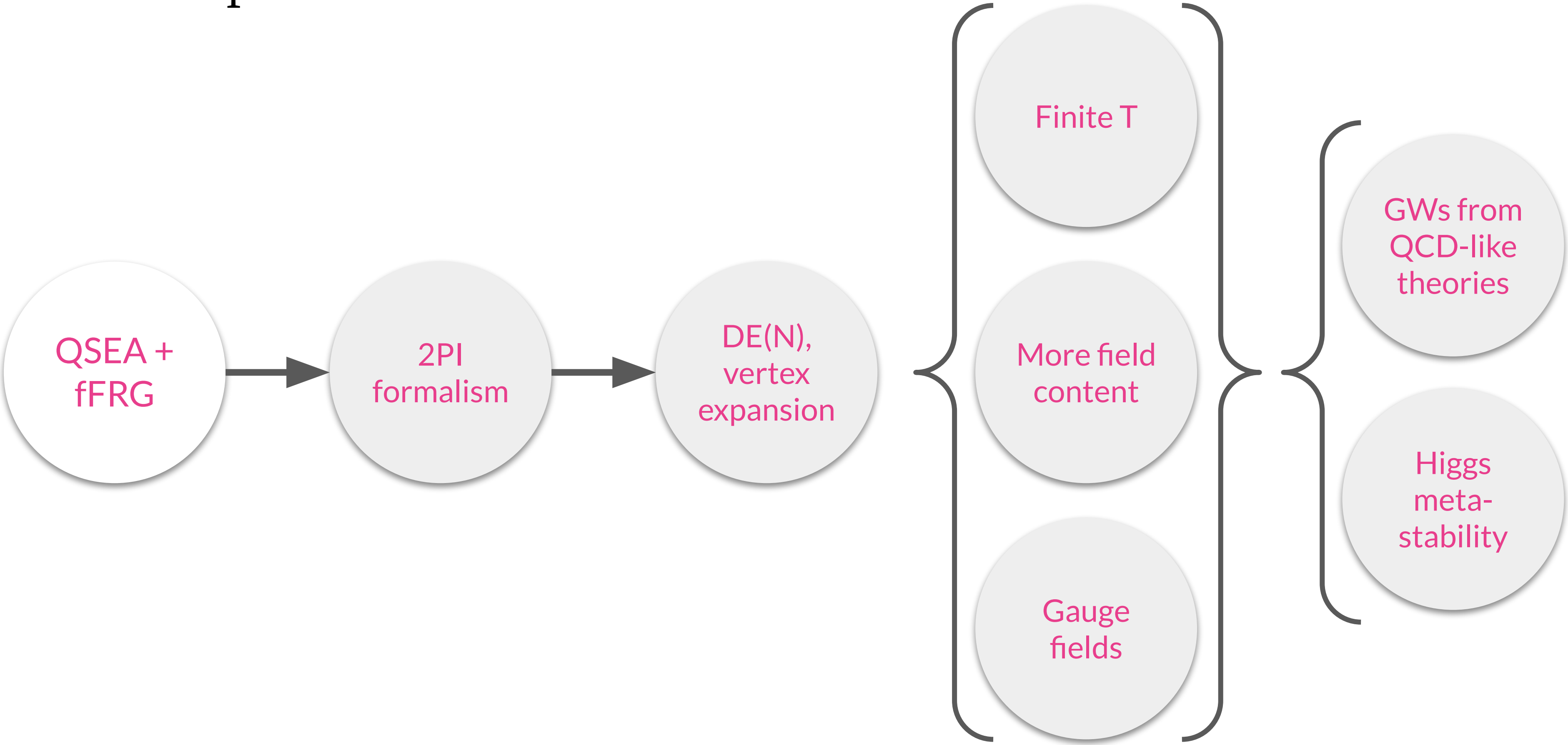
Results and comparison with perturbation theory





Disagrees qualitatively as you approach larger couplings

Next steps


QSEA + fFRG
2PI formalism
DE(N), vertex expansion
Finite T
More field content
Gauge fields
GWs from QCD-like theories
Higgs meta-stability

Introducing: the 2PI formalism

1PI action

$$\Gamma_{\mathrm{1PI}}[\bar\phi] = -W[J] + \int_x J_x \bar\phi_x$$

$$W[J] = \ln \int \mathcal{D}\phi \exp\left[-S[\phi] + \int_x J_x \phi_x\right]$$

2PI action

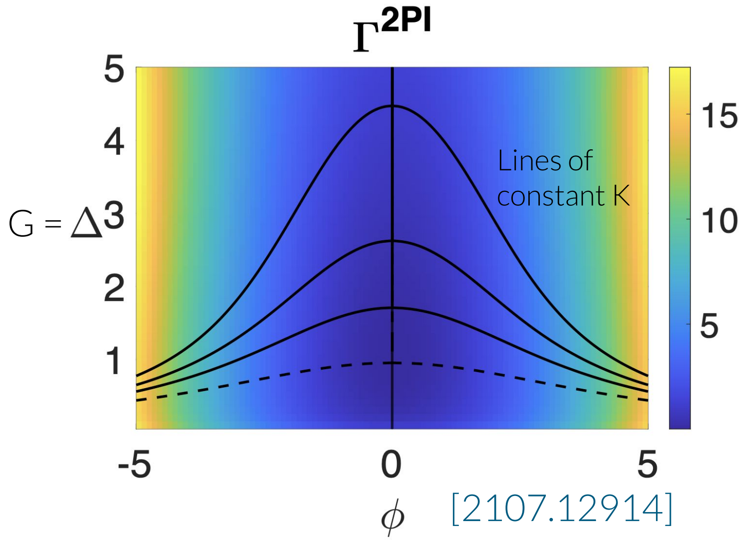
$$\Gamma_{\mathrm{2PI}}[\bar\phi, G] = -W[J,K] + \int_x J_x \bar\phi_x + \frac{1}{2}\int_{x,y} K_{xy}(G_{xy} + \bar\phi_x \bar\phi_y)$$

$$W[J,K] = \ln \int \mathcal{D}\phi \exp\left[-S[\phi] + \int_x J_x \phi_x + \frac{1}{2}\int_{x,y} K_{xy}\phi_x\phi_y\right]$$

In addition to usual one-point source, introduce two-point source

The resulting action has external dependence on propagator

At a given value, -K is directly analogous to the regulator; the difference is that it is now selected by the Legendre transform for the propagator [1908.02214]



[2107.12914]

The QSEA condition can be implemented in 2PI

1PI QSEA

QSEA condition

- Choose R_k s.t. $\varphi^2 \lesssim k^{-2}\mathbf{1}$
- In practice: $\partial^2 + V'' + R_k \geq k^2$
- Regulator is positive semidefinite

Quasi-stationary effective action

- = modified FRG action

General flow equation

$$\partial_k \Gamma_k[\bar\phi] = \frac{1}{2}\int_p (\partial_k R_k) G_k[\bar\phi; -p, p]$$

2PI QSEA

QSEA condition

- Choose G_k s.t. $\langle\varphi^2\rangle = G \leq k^{-2}\mathbf{1}$
- In practice: $G_k = \min\{G_{1\mathrm{PI}}, k^{-2}\mathbb{1}\}$
- K is negative semidefinite

Quasi-stationary effective action

- $\Gamma_{1\mathrm{PI}}[\bar\phi, k] = \Gamma_{2\mathrm{PI}}[\bar\phi, G_k] - \frac{1}{2}\int K G_k$

General flow equation

$$\partial_k \Gamma_{2\mathrm{PI}}[\bar\phi, G_k] = \frac{1}{2}\int K \partial_k G_k$$

$$\partial_k \Gamma_{1\mathrm{PI}}[\bar\phi] = -\frac{1}{2}\mathrm{Tr}\left[(\partial_k K_k)\Theta(-K_k) G_k\right]$$

First steps: 2PI + LPA

The flow equations can be remarkably simplified (including a change of variables) in the LPA

Looks just like usual LPA, but with mass dressing

$$E_k^2 \simeq k^2 + U_k'' + \frac{k^2 (U_k''')^2}{32\pi^2 (k^2 + U_k'')}$$

Next steps: DE2, VE

More involved, but b.c. of general flow equation is tractable (unlike 1PI)

2PI LPA flow equations

$$\partial_k U_k = \frac{k^5}{32\pi^2 E_k^2} \Theta(E_k^2)$$

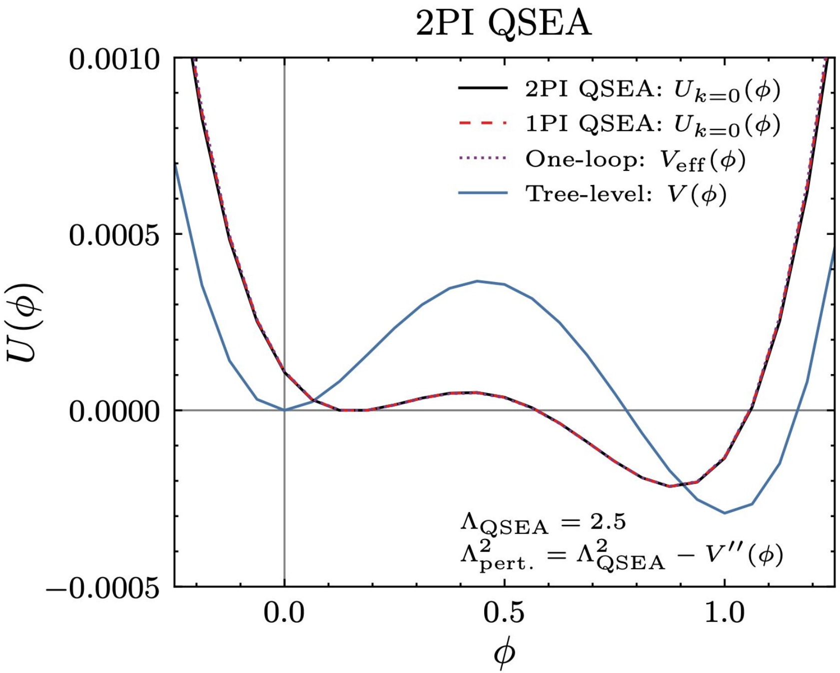
$$\partial_k E_k^2 = k^2 \sqrt{\frac{(E_k^{2\prime})^2}{8\pi^2 E_k^2 (E_k^2 - k^2 - U_k'')}}$$

Usual (Non-QSEA) LPA

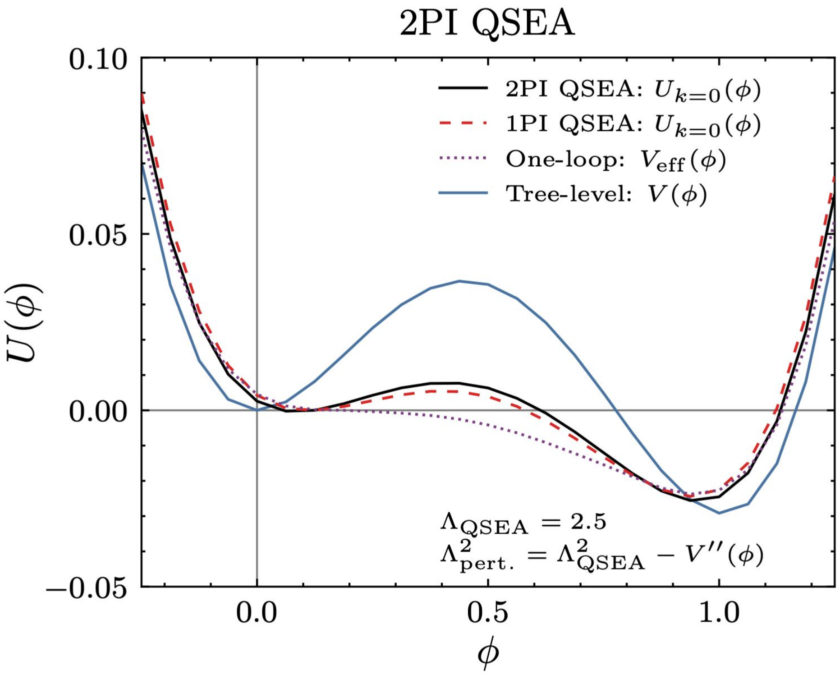
$$\partial_k U_k = \frac{k^5}{32\pi^2 E_k^2}$$

$$E_k^2 = k^2 + U_k''$$

Results + comparison ***PRELIMINARY***



Weak coupling: all agree as expected



Strong coupling: Small but non-zero disagreement with 1PI QSEA

Likely due to higher order contributions to regulator

Outlook: a new program of research

New quasi-stationary effective action for false vacuum decay implemented in a modified FRG for fluctuations that is robust to strong couplings and is versatile + easy to use

Today's work with Djuna Croon, Pete Millington:

- ➔ Extending the QSEA to the 2PI formalism

Works in progress with Djuna Croon

- ➔ Significant update to [arxiv:2104.10687] coming soon
- ➔ In-depth follow-up on new insights, additional tests
- ➔ Extending to finite T, more general field content

Works in progress with Djuna Croon, Rachel Houtz, Ansh Bhatnagar:

- ➔ Improving warm DM constraints on axions with the FRG

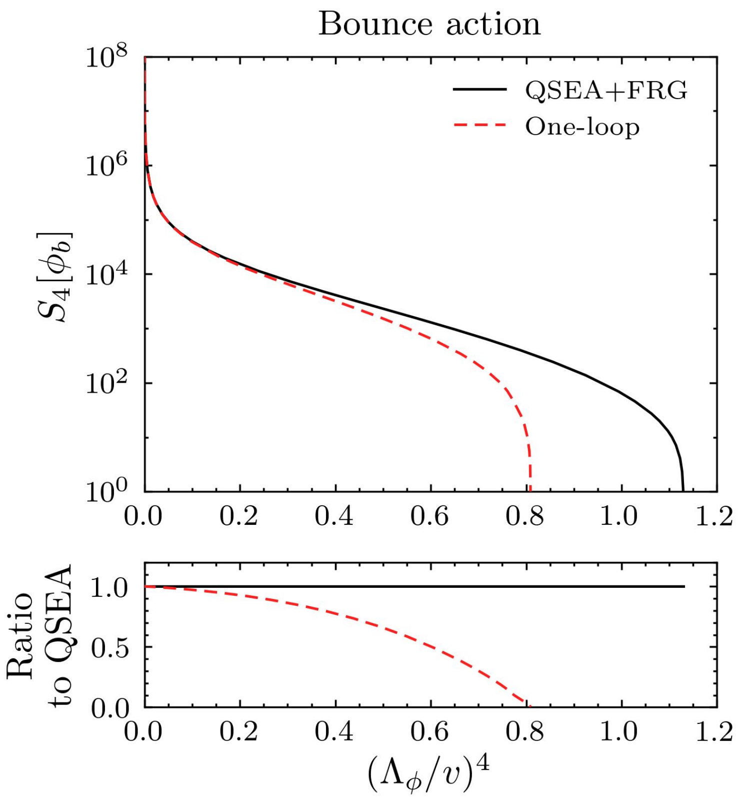
Future directions

- ➔ Using the QSEA to make new sphaleron rate calculations
- ➔ GW signals from chiral phase transitions in QCD-like dark sectors

The big point: decay rates for strong interactions

Significant differences with perturbation theory at large coupling

Unlike perturbation theory, FRG + QSEA is **robust to strong couplings**



Decay rates: the Callan-Coleman formalism

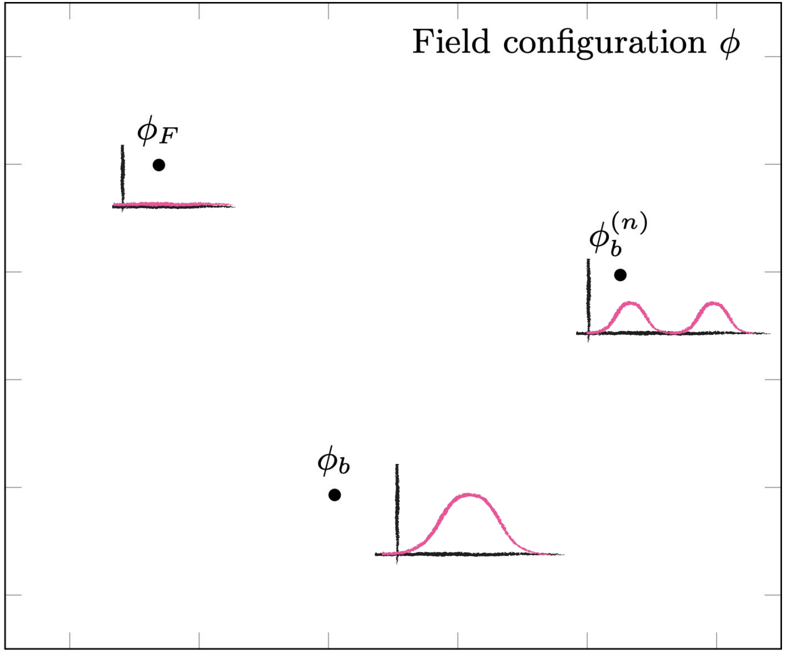
[Coleman, 1977]
[Callan and Coleman, 1977]

Decay rate = imaginary part of FV energy

$$\gamma = -2\,\mathrm{Im}\,\mathcal{E} \simeq \frac{2}{\mathcal{V}}\mathrm{Im}\ln\int_{\phi_F}^{\phi_F}\mathcal{D}\phi\, e^{-S[\phi]}$$

To evaluate, use direct method or potential deformation along with saddle-point approximation

$$\gamma = \left|2\,\mathrm{Im}\,\frac{\int_{\mathrm{hits}\ \Sigma}\mathcal{D}\phi\, e^{-S[\phi]}}{i\mathcal{V}\int\mathcal{D}\phi\, e^{-S[\phi]}}\right| \simeq Ae^{-(S[\phi_b]-S[\phi_F])}$$



Problems:

- ➔ Saddle point method: fundamentally perturbative
- ➔ When taken to all orders, must be zero!
- ➔ Off by a factor of two

Beyond perturbation theory: exact effective actions

What about powerful existing tools for non-perturbative physics like the functional renormalization group?

Flows from classical action → exact 1pI effective action

Regulator function $R_k(p)$ added to the action freezes out IR modes with $p \lesssim k$

$$\Delta S_k = \frac{1}{2}\int_p R_k \phi^2$$

Scale-dependent effective action for the theory at a scale k

$$\Gamma_k[\bar{\phi}] = -W_k[J] + \int J\bar{\phi} - \Delta S_k[\bar{\phi}]$$
$$W_k[J] = \ln \int \mathcal{D}\phi \, \exp\left[-S[\phi] + \int J\phi - \Delta S_k[\phi]\right]$$

Exact flow equation flows the effective action across different scales

$$\partial_k \Gamma_k[\bar{\phi}] = \frac{1}{2}\int_p (\partial_k R_k) G_k[\bar{\phi}; -p, p] = \frac{1}{2}\int_p \frac{\partial_k R_k}{\frac{1}{\mathcal{V}}\Gamma_k^{(2)} + R_k}$$

Robust approximation schemes like the derivative expansion and vertex expansion that don't spoil the non-perturbativity

[Good review: Dupuis et al., arxiv:2006.04853]

The local potential approximation

Zero'th order of derivative expansion; flow equation for effective *potential*

LPA

$$\Gamma_k[\bar\phi] = \int_x \left[\frac{1}{2}(\partial\bar\phi)^2 + U_k(\bar\phi)\right]$$

$$\partial_k U_k = \frac{1}{\mathcal{V}}\int_p (\partial_k R_k) G_k[\bar\phi; -p, p] \quad (\text{const. } \bar\phi)$$

Closed form solutions

$$G_k[\bar\phi] = \frac{\delta(p+q)}{\tilde k^2(\phi) + U_k''(\phi)} \frac{2}{1+\sqrt{1+\frac{V''''(\phi)\tilde k^4(\phi)}{16\pi^2(\tilde k^2 + U_k'')^2}}}$$

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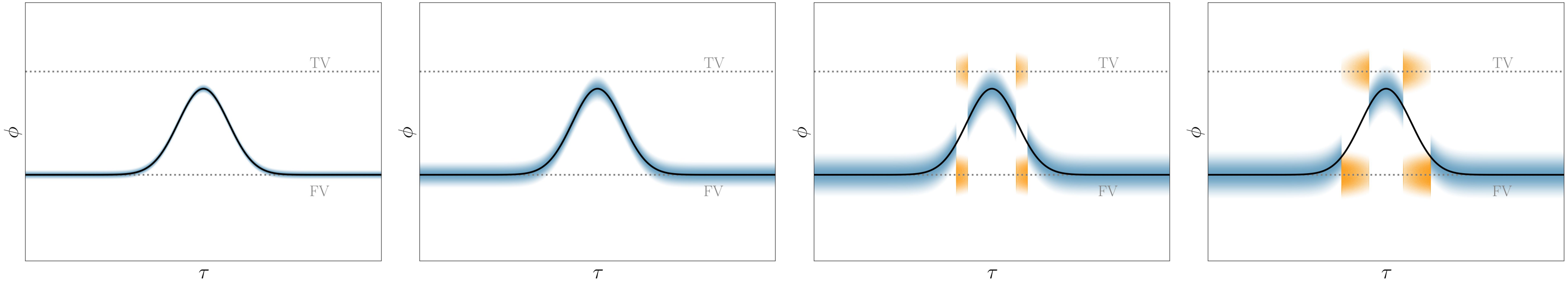
$$\tilde k^2 = k^2 + V''$$

BUT: exact effective actions are convex

Fluctuations very constrained
Potential basically classical

Fluctuations constrained at some field values,
becoming non-local at others

Regulator zero (physical)
Fluctuations non-local
Convex potential



High k → k = 0

BUT: exact effective actions are convex

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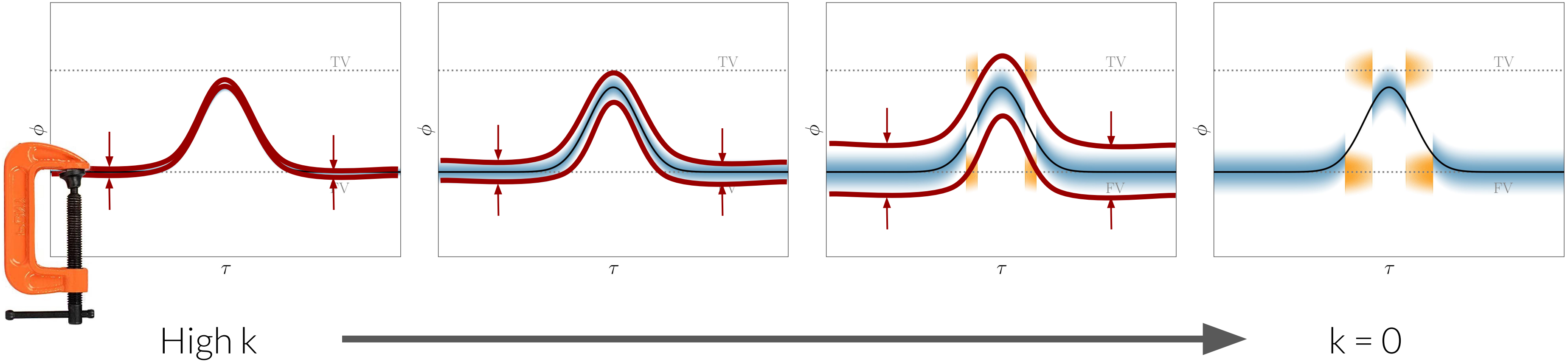

Understanding the fFRG flow

Only “clamped” in unstable regions – and there only minimally (= massless theory)

No more tension between constraints and locality