New developments for neutrino mixing in quantum field theory

Luca Smaldone

Institute of Theoretical Physics
University of Warsaw

Workshop on the Standard Model and Beyond
Corfù, 2 September 2022

Contents

1. Neutrino oscillations in quantum field theory
2. Leggett–Garg inequalities for neutrino oscillations
3. Conclusions

Neutrino oscillations in quantum field theory

The problem of flavor states

Consider the process $P_I \to P_F + l_\sigma^+ + \nu_\sigma$. Consider the S-matrix element

$$\langle \nu_\sigma l_\sigma^+ P_F | S | P_I \rangle$$

What is definition of $|\nu_\sigma\rangle$? Field mixing transformation

$$\nu_\sigma(x) \; = \; \sum_j U_{\sigma\, j} \nu_j(x)$$

between flavor fields ν_σ and mass fields ν_j. U is the mixing matrix. In the two-flavor case it is parametrized as:

$$U \; = \; \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

Mass eigenstates

Fields with definite masses can be expanded as:

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} \ + \ v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}} \,, \quad i = 1,2$$

A mass-eigenstate neutrino is defined as:

$$|\nu^r_{\mathbf{k},i}\rangle \ = \ \alpha^{r\dagger}_{\mathbf{k},i}|0\rangle_{1\,2}$$

mass vacuum is defined by:

$$\alpha^r_{\mathbf{k},i}|0\rangle_{1\,2} \ = \ \beta^r_{\mathbf{k},i}|0\rangle_{1\,2} \ = \ 0$$

Pontecorvo flavor states

Pontecorvo states[1]:

$$
\begin{aligned}
|\nu^r_{\mathbf{k},e}\rangle_P &= \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle \\
|\nu^r_{\mathbf{k},\mu}\rangle_P &= -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle
\end{aligned}
$$

Consider the amplitude of the neutrino detection process $\nu_\sigma + X_i \to e^- + X_f$:

$$
\langle e^s_{\mathbf{q},-}|\bar{e}(x)\,\gamma^\mu\,(1-\gamma^5)\,\nu_e(x)|\nu^r_{\mathbf{k},\sigma}\rangle_P\,h_\mu(x) \;\;\not\propto\;\; \delta_{\sigma e}
$$

h_μ are the matrix elements of the X part. PROBLEM: Neutrino flavor is detected by identifying the charged-lepton.

[1] S.M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978)

Mixing generator

Mixing transformation can be rewritten as

$$\begin{aligned} \nu_e(x) &= G_\theta^{-1}(t)\nu_1(x)\, G_\theta(t) \\ \nu_\mu(x) &= G_\theta^{-1}(t)\, \nu_2(x) G_\theta(t) \end{aligned}$$

Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\, \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

Flavor Vacuum

Flavor vacuum is defined by[2]:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}(0)\;|0\rangle_{1,2}$$

In the infinite volume limit:

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\;\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

where

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\, v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\,|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

[2]M.Blasone and G.Vitiello, Ann. Phys. **244**, 283 (1995)

Flavor eigenstates

Defining

$$\alpha^r_{\mathbf{k},\sigma}(t) \;\equiv\; G^{-1}_\theta(t)\, \alpha^r_{\mathbf{k},j}\, G_\theta(t) \qquad\qquad (\sigma, j) = (e,1), (\mu,2)$$

one can construct flavor eigenstates[3]

$$|\nu^r_{\mathbf{k},\sigma}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)|0\rangle_{e,\mu}$$

In fact, it are eigenstates of the neutrino-lepton charges

$$Q_{\nu_\sigma}(0)|\nu^r_{\mathbf{k},\sigma}\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle\,, \qquad\qquad Q_{\nu_\sigma}(t) \;=\; \int \mathrm{d}^3\mathbf{x}\, \nu^\dagger_\sigma(x)\nu_\sigma(x)$$

[3] M. Blasone and G. Vitiello, Phys Rev D **60**, 111302 (1999)

Oscillation formula

Taking

$$\mathcal{Q}_{\sigma\to\rho}(t) \;=\; \langle \nu^r_{\mathbf{k},\sigma}|Q_{\nu_\rho}(t)|\nu^r_{\mathbf{k},\sigma}\rangle\,, \quad \sigma\neq\rho$$

Explicitly:

$$\mathcal{Q}_{\sigma\to\rho}(t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\omega^-_{\mathbf{k}}t\right) + |V_{\mathbf{k}}|^2\sin^2\left(\omega^+_{\mathbf{k}}t\right)\right]$$

$|U_{\mathbf{k}}|^2 = 1-|V_{\mathbf{k}}|^2$, $\omega^-_{\mathbf{k}} \equiv (\omega_{\mathbf{k},1}-\omega_{\mathbf{k},2})/2$ and $\omega^+_{\mathbf{k}} \equiv (\omega_{\mathbf{k},1}+\omega_{\mathbf{k},2})/2$.
This is the QFT oscillation formula[4]. When $m_i/|\mathbf{k}|\to 0$:

$$\mathcal{Q}_{\sigma\to\rho}(t) \;\approx\; \sin^2(2\theta)\sin^2\left(\frac{\pi L}{L_{osc}}\right)$$

with $L_{osc} = 4\pi|\mathbf{k}|/\delta m^2$, which is the standard oscillation formula.

[4] M. Blasone, P.A. Henning and G. Vitiello, Phys. Lett. B **451**, 140 (1999)

Leggett–Garg inequalities for neutrino oscillations

Macrorealism[5]:

- **Macrorealism per se: A macroscopic object is always in one of the available states, regardless of the measurement process**
- **Noninvasive measurability: It is in principle possible to measure the state of the system without affecting its dynamical evolution**

We measure a dichotomic observable $O(t)$ at three equidistant times $t_n = nt$, $n = 0, 1, 2$.

[5] A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857-860 (1985)

Macrorealism violation in neutrino oscillations

Leggett–Garg inequality (LGI):

$$C_{01} + C_{12} - C_{02} \leq 1$$

with

$$C_{ij} = \langle O(t_i) O(t_j) \rangle$$

must be fulfilled in macrorealistic systems.

Violations of LGI in neutrino oscillations have been proved by using the MINOS data[6]: neutrino oscillations are quantum!

[6] J. Formaggio, D. Kaiser, M. Murskyj, and T. Weiss, Phys. Rev. Lett. **117**, 050402 (2016)

Wigner form of LGI

We consider joint probabilities $P(m_i, m_j)$, $O(m)|m\rangle = m|m\rangle$, $m = \pm 1$. Wigner form of Leggett–Garg inequality (WLGI)[7]

$$P(m_1, m_2) - P(m_0, m_1) - P(-m_0, m_2) \ \leq \ 0$$

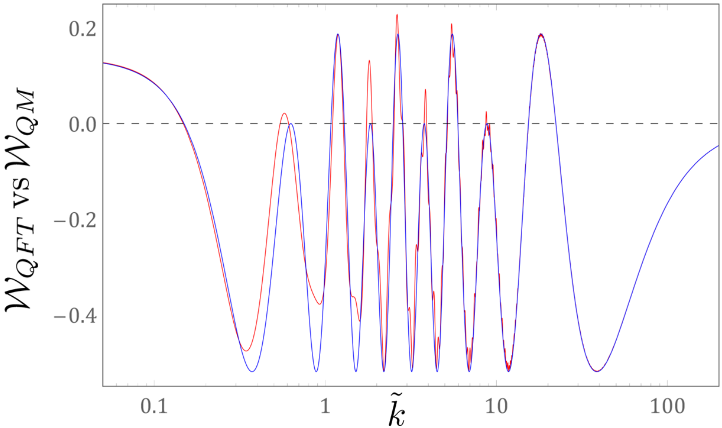
In the case of neutrino oscillations $O(t) = (Q_{\nu_e}(t) - Q_{\nu_\mu}(t))$. With $|m_0\rangle \ = \ |\nu^r_{\mathbf{k},\mu}\rangle$, $m_0 = m_1 = m_2 = 1$ one gets[8]

$$\mathcal{W}_{QFT} \equiv \mathcal{Q}_{e\to e}(t)\, \mathcal{Q}_{\mu\to e}(t) - \mathcal{Q}_{\mu\to e}(2t) \ \leq \ 0 \tag{2}$$

$\mathcal{W}_{QM}$ same expression computed from Pontecorvo states.

[7] D. Saha, S. Mal, P. K. Panigrahi, D. Home, Phys. Rev. A, **91**, 032117 (2015)
[8] M. Blasone, F. Illuminati, L. Petruzziello and L. S., [arXiv:2111.09979 [quant-ph]]

$\mathcal{W}_{QFT}$ vs $\mathcal{W}_{QM}$



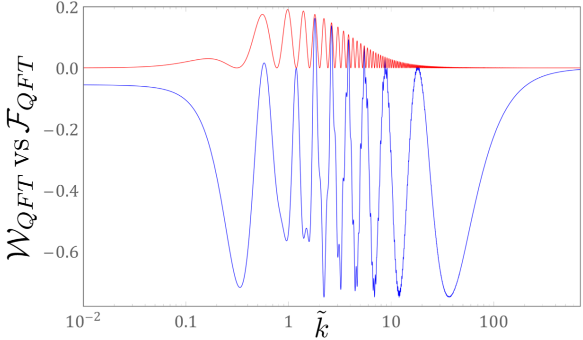
Violation of WLGI, as a function of $\tilde{k} \equiv |\mathbf{k}|/\sqrt{m_1\, m_2}$, in QFT (red) vs QM (blue), $m_1 = 2$, $m_2 = 30$, $t = 1$, $\theta = \pi/6$.

$\mathcal{W}_{QFT}$ vs $\mathcal{F}_{QFT}$

The function

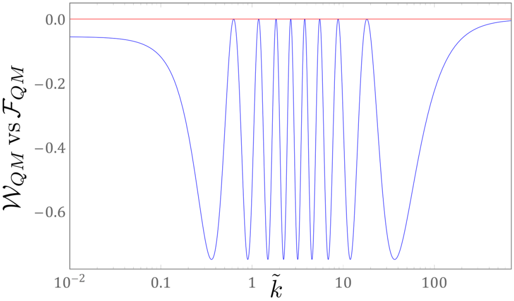
$$\mathcal{F}_{QFT} \;\equiv\; \mathcal{Q}_{\sigma\to\sigma}(t)\,(1-\mathcal{Q}_{\sigma\to\sigma}(t)) - \frac{1}{4m_{e\mu}^2}\,|\langle[Q_{\nu_\sigma}(t)\,,\,Q_M]\rangle_\sigma|^2$$

with $Q_M \equiv \int d^3\mathbf{x}\,\boldsymbol{\nu}^\dagger M_\nu \boldsymbol{\nu}$ bounds $\mathcal{W}_{\mathcal{QFT}}$ from above.



$\mathcal{W}_{QFT}$ (blue) vs $\mathcal{F}_{QFT}$ (red), $m_1 = 2$, $m_2 = 30$, $t = 1$, $\theta = \pi/4$.

$\mathcal{W}_{QM}$ vs $\mathcal{F}_{QM}$



$\mathcal{W}_{QM}$ (blue) vs $\mathcal{F}_{QM}$ (red), $m_1 = 2$, $m_2 = 30$, $t = 1$, $\theta = \pi/4$.

Violation of WLGI in QFT is more generic than in QM:

QFT is less classical than QM.

This agrees with general results on Bell's inequalities[9]

[9] S. J. Summers and R. Werner, Commun. Math. Phys. **110**, 247 (1987)

Conclusions

Conclusions

- Problem: to define flavor states (in presence of mixing), in QFT language
- The study of mixing transformation reveals that flavor and mass representations are unitarily inequivalent
- In order to preserve conservation of the flavor charge in the vertex, at tree level, we have to work in the flavor basis
- Flavor Fock space approach permits to construct flavor eigenstates: corrections to the oscillation formula!
- Test of macrorealism through WLGI: QFT less-classical than QM!

Thank you for the attention!