Index theorem on T^2/Z_N orbifold with magnetic flux

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Workshop on the Standard Model and Beyond

Corfu, Greece, 7 September 2022

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Question: Why are there 3 generations of quarks and leptons?

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Accidental? ↙ ↘ Has physical meaning ?

Standard Model	Extra dimension model

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Accidental? | Has physical meaning?

Standard Model

I	II	III
$\begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} c_\mathrm{L} \\ s_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} t_\mathrm{L} \\ b_\mathrm{L} \end{pmatrix}$
u_R	c_R	t_R
d_R	s_R	b_R
$\begin{pmatrix} \nu_{e\mathrm{L}} \\ e_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu\mathrm{L}} \\ \mu_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau\mathrm{L}} \\ \tau_\mathrm{L} \end{pmatrix}$
e_R	μ_R	τ_R

Extra dimension model

6d chiral fermion

$$\begin{pmatrix} U(x,y) \\ D(x,y) \end{pmatrix}, \quad U'(x,y), \quad D'(x,y), \quad \begin{pmatrix} N(x,y) \\ E(x,y) \end{pmatrix}, \quad E'(x,y)$$

Zero modes=generation

I	II	III
$\begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} c_\mathrm{L} \\ s_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} t_\mathrm{L} \\ b_\mathrm{L} \end{pmatrix}$
u_R	c_R	t_R
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e_R	μ_R	τ_R

Degenerate

1. Introduction

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Accidental? | Has physical meaning?

Standard Model

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$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$
u_R	c_R	t_R
d_R	s_R	b_R
$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$
e_R	μ_R	τ_R

Extra dimension model

6d chiral fermion

$\begin{pmatrix} U(x,y) \\ D(x,y) \end{pmatrix}$

$U'(x,y)$

$D'(x,y)$

$\begin{pmatrix} N(x,y) \end{pmatrix}$

$\Rightarrow$

Zero modes

I	II	III
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$
u_R	c_R	t_R
d_R	s_R	b_R
$\begin{pmatrix} \nu_{eL} \\ \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$
	μ_R	τ_R

Decided by topology of extra dimension

Degenerate

1. Introduction

Atiyah-Singer index theorem

$$n_+ - n_- \propto \int_{\mathscr{M}} F^{D/2} + \int_{\mathscr{M}} R^{D/4} + (F, R \text{ mixing term})$$

$n_\pm$: chiral zero modes # D: dimension $\mathscr{M}$: a smooth manifold

1. Introduction

Atiyah-Singer index theorem

$$n_+ - n_- \propto \int_{\mathscr{M}} F^{D/2} + \int_{\mathscr{M}} R^{D/4} + (F, R \text{ mixing term})$$

Contribution of flux
D=2n

Contribution of curvature
D=4n

1. Introduction

Atiyah-Singer index theorem

$$n_+ - n_- \propto \int_{\mathscr{M}} F^{D/2} + \int_{\mathscr{M}} R^{D/4} + (F, R \text{ mixing term})$$

Contribution of flux
D=2n

Contribution of curvature
D=4n

AS index theorem on T^2

$$n_+ - n_- = \int_{T^2} \frac{F}{2\pi} = M$$

Chiral zero modes #

flux quantization#

➡ There is only contribution of flux in D=2.

2. Purpose of my talk

In previous paper, we obtain the following formula on T^2/Z_N orbifolds

$$\text{(generation \#)} = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number at fixed points

Makoto Sakamoto, **Maki Takeuchi**, Yoshiyuki Tatsuta,
Phys. Rev. D 102 (2020) 025008

2. Purpose of my talk

In previous paper, we obtain the following formula on T^2/Z_N orbifolds

Contribution of flux

$$\text{(generation \#)} = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of windi

What's physical meaning?

2. Purpose of my talk

Result of my talk

Contribution of flux

$$(\text{generation } \#) = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number at fixed points

Winding number contains localized flux and curvature.
$+1$ **comes from the sum of localized curvatures.**

2. Purpose of my talk

Result of my talk

Contribution of flux

$$(\text{generation } \#) = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number at fixed points

$+1$ **removes contribution of localized curvatures in** V_+**.**

2. Purpose of my talk

Result of my talk

$$(\text{generation \#}) = \boxed{\frac{M}{N} - \frac{V_+}{N} + 1}$$

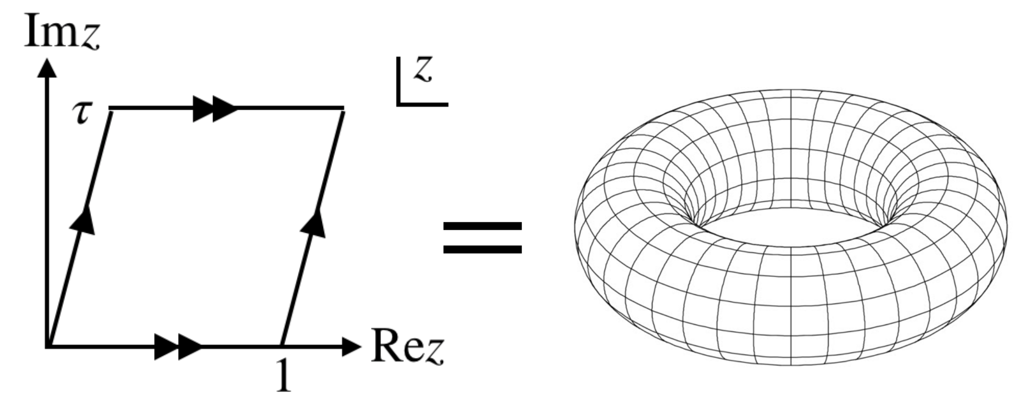
M : flux quanta、V_+ : sum of winding number at fixed points

Only contribution of flux!

3. T^2/Z_N orbifold model

- $\mathscr{M}^4 \times T^2/Z_N$ **in flux background**

$$\frac{T^2 : z \sim z+1 \sim z+\tau}{Z_N : z \sim \rho z \quad (\rho = e^{i\frac{2\pi}{N}})} =$$

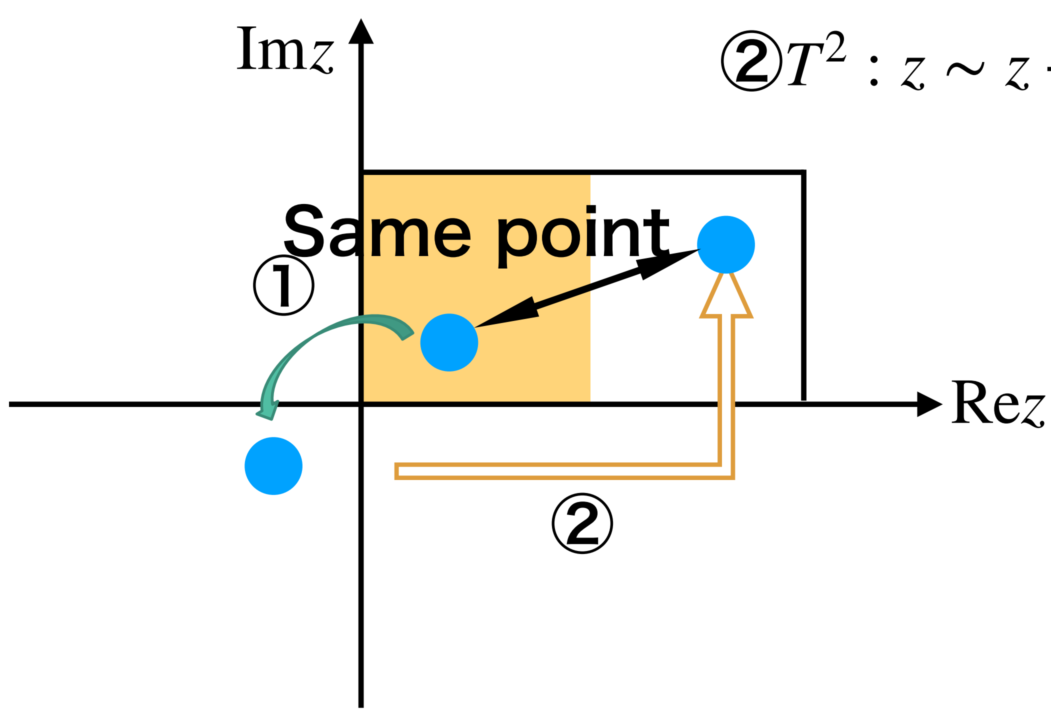


Z_N

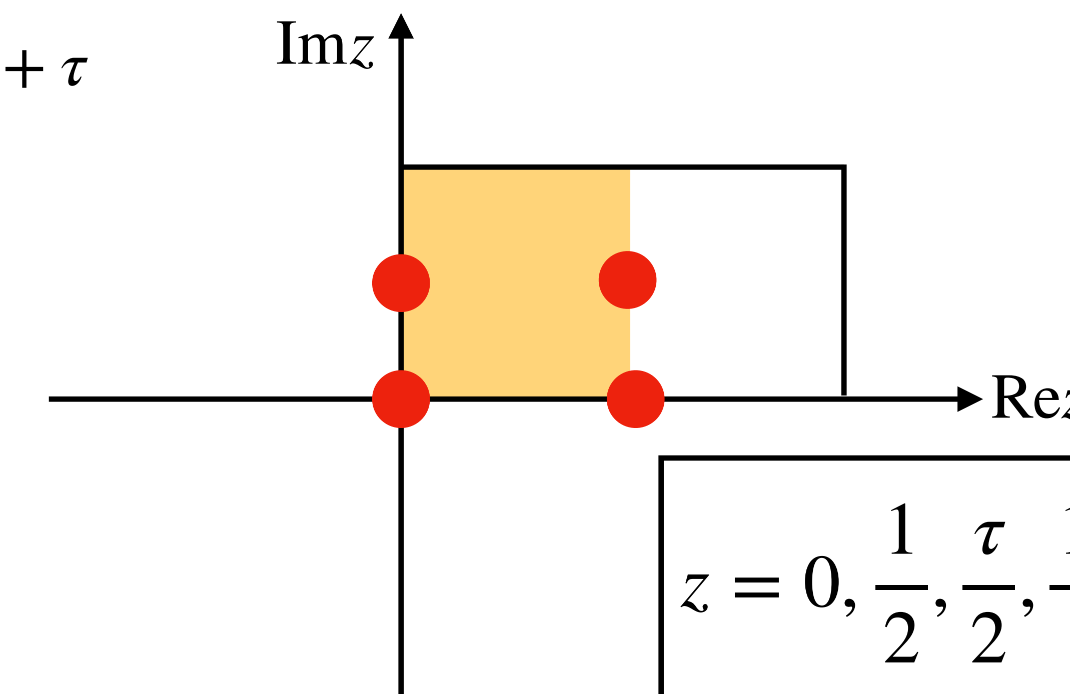
Ex) $T^2/Z_2\,(\tau = i)$

①$Z_2 : z \sim -z$

②$T^2 : z \sim z+1 \sim z+\tau$



Independent region is $1/N$ **of** T^2



$$z = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$$

Fixed point

3. T^2/Z_N orbifold model

Z_N eigen function $\psi^m_{T^2/Z_N\pm,n,j}(z)$

Boundary condition

$$\psi^m_{T^2/Z_N+,n,j}(z+1) = U_1(z)\,\psi^m_{T^2/Z_N+,n,j}(z)$$

$$\psi^m_{T^2/Z_N+,n,j}(z+\tau) = U_2(z)\,\psi^m_{T^2/Z_N+,n,j}(z)$$

$$U_1(z) = e^{iq\Lambda_1(z)},\ U_2(z) = e^{iq\Lambda_2(z)}$$

$$\psi^m_{T^2/Z_N+,n,j}(\rho z) = \rho^m \psi^m_{T^2/Z_N+,n,j}(z)$$

$$\rho = e^{i\frac{2\pi}{N}}$$

Winding number

Define winding number of Z_N eigen function $\psi^m_{T^2/Z_N+,n,j}(z)$ at fixed point z^f_j

$$\psi^m_{T^2/Z_N+,n,j}(\rho z + z^f_j) = \rho^{\chi_+}\psi^m_{T^2/Z_N+,n,j}(z + z^f_j)$$ $\Rightarrow$ **winding # :** χ_+

In case of $z^f_j = 0$, $\chi_+ = m$

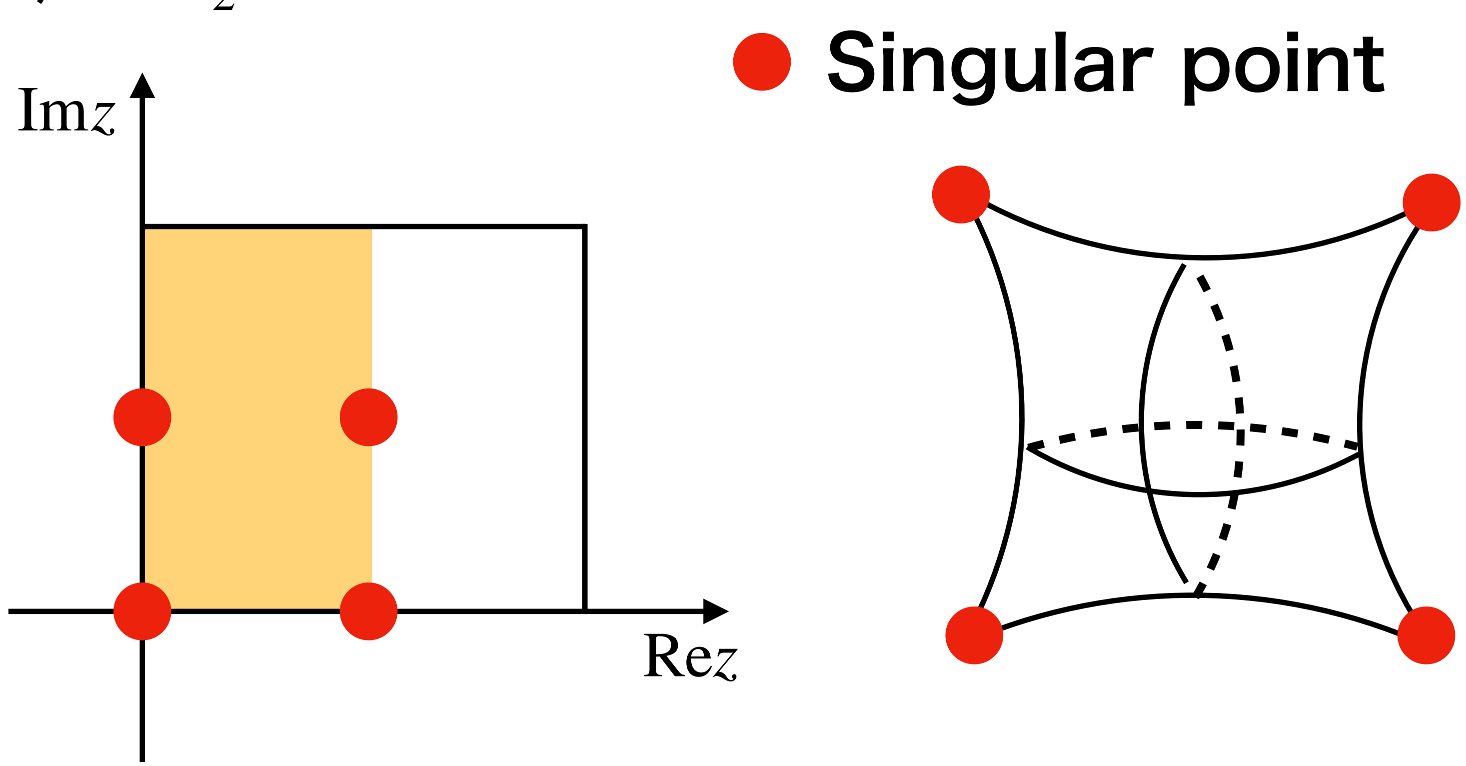
3. T^2/Z_N orbifold model

Atiyah-Singer index theorem on T^2/Z_N orbifold

It is difficult to apply AS index theorem because fixed points are singular points.

➡ **We consider blow-up manifold without singular points.**

Ex) T^2/Z_2



T. Kobayashi, H. Otsuka, H. Uchida., JHEP 08, 104 (2019),046

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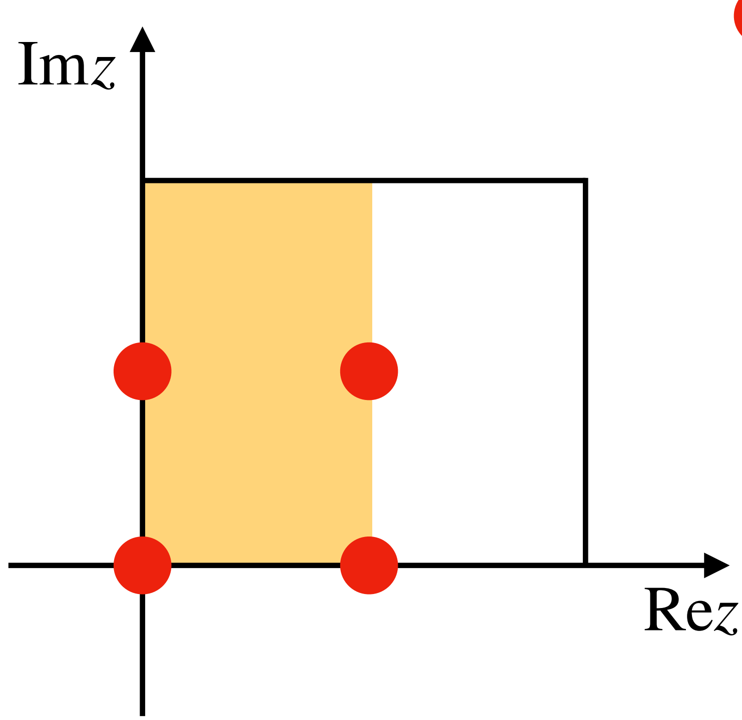
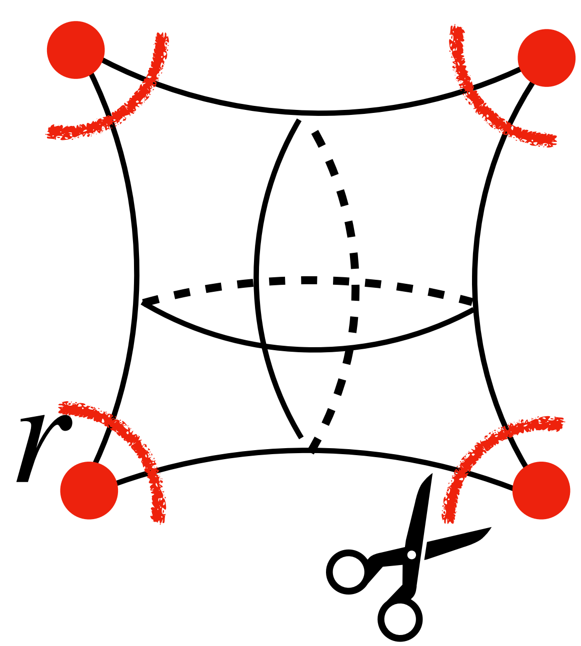


Singular point



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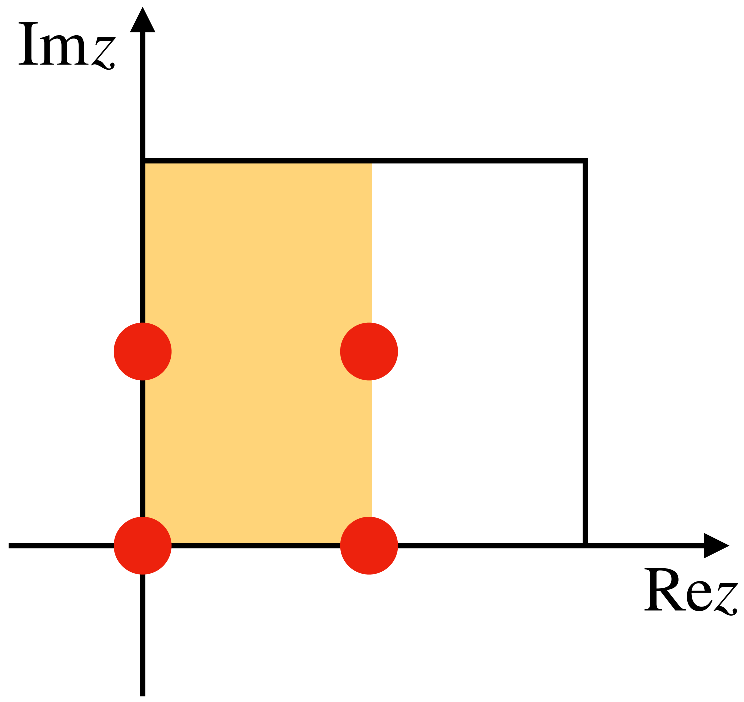
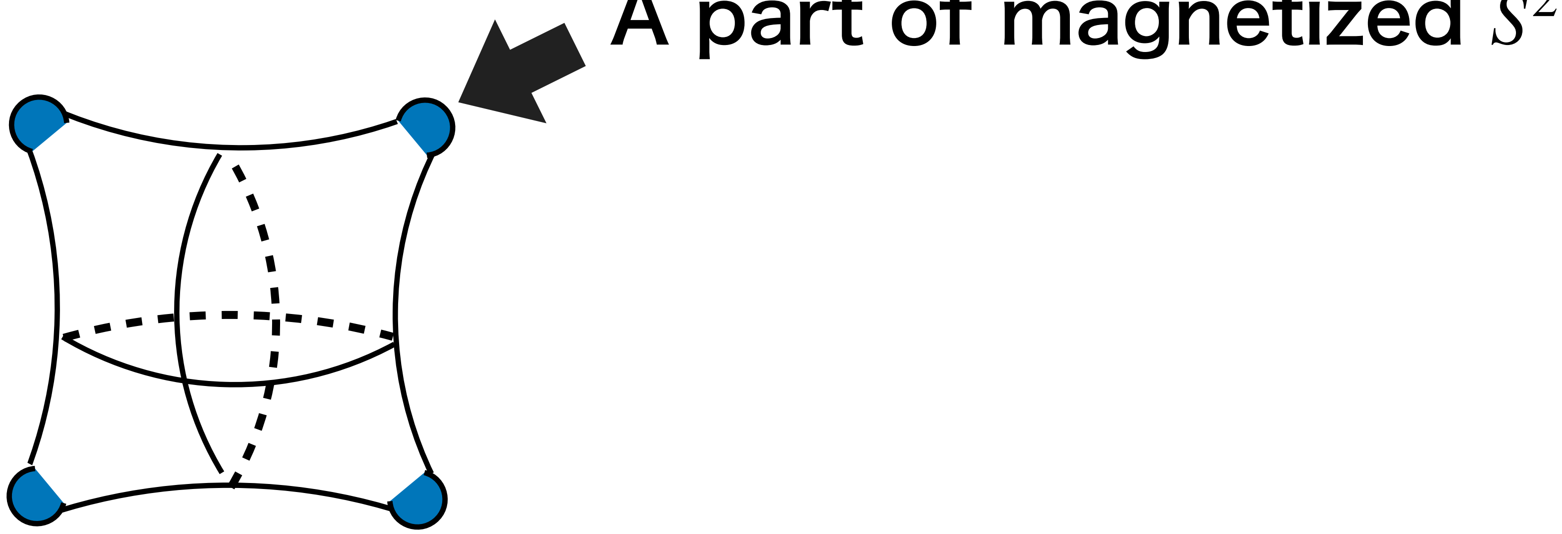


Blow-up manifold



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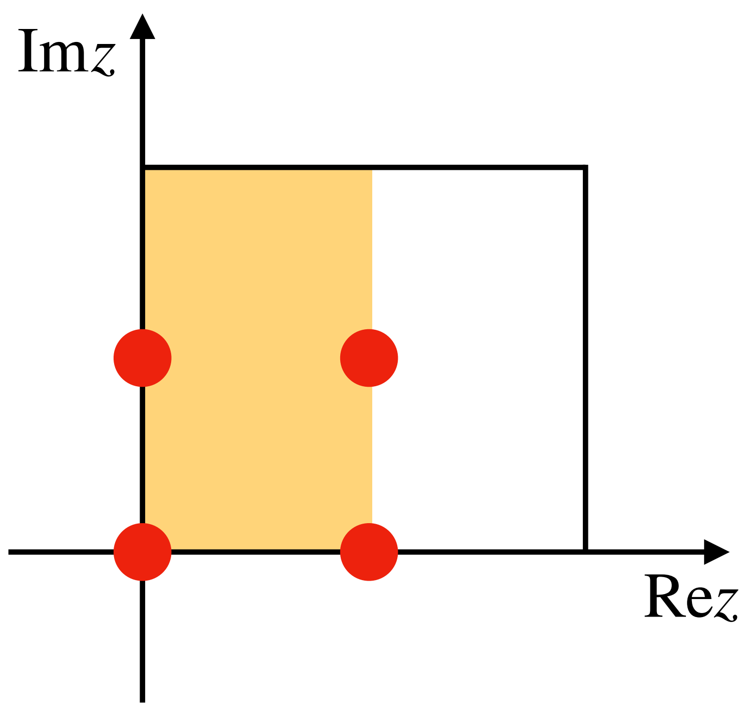
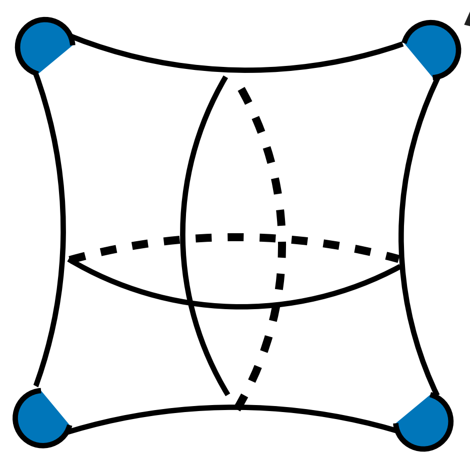


Blow-up manifold

A part of magnetized S^2



We can apply AS index theorem to blow-up manifold without singularity.

T. Kobayashi, H. Otsuka, H. Uchida., JHEP 08, 104 (2019),046

4. Blow-up of T^2/Z_N orbifold

$\psi^0_{T^2/Z_N,+}(z)$ **with winding number cannot connect to** $\psi^0_{S^2,+}(z')$ **,**

because the boundary conditions are different at $z \sim z_j^f$

4. Blow-up of T^2/Z_N orbifold

Purpose : To remove winding number

We consider in case of fixed point $z_j^f = 0$ ($m \to \chi_+$ in case of $z_j^f \neq 0$)

$$\psi^n_{T^2/Z_N,+}(\rho z) = \rho^m \psi^n_{T^2/Z_N,+}(z) \quad \Rightarrow \quad \widetilde{\psi}^n_{T^2/Z_N,+}(\rho z) = \widetilde{\psi}^n_{T^2/Z_N,+}(z)$$

“Singular” gauge transformation

$$\widetilde{\psi}_{T^2/Z_N,\pm}(z) = U_{\xi^F} U_{\xi^R} \psi_{T^2/Z_N,\pm}(z) \qquad U_{\xi^F} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^F}{2}}, \; U_{\xi^R} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^R}{4}}$$

ξ^F : localized flux at fixed point , ξ^R : localized curvature at fixed point

$$\widetilde{\psi}_{T^2/Z_N,+}(\rho z) = \rho^{\xi^F - \frac{\xi^R}{2} + m}\, \widetilde{\psi}_{T^2/Z_N,+}(z)$$

$$\boxed{\rho^{\xi^F - \frac{\xi^R}{2} + m} = 1}$$

However this gauge transformation changes flux,
this is called “singular gauge transformation”.

4. Blow-up of T^2/Z_N orbifold

Localized flux ξ^F

$$\boxed{\rho^{\xi^F - \frac{\xi^R}{2} + m} = 1} \Rightarrow \begin{aligned} \xi^F &= \frac{\xi^R}{2} - m \quad (z_j^f = 0) \\ \xi^F &= \frac{\xi^R}{2} - \chi_+ \quad (z_j^f \neq 0) \end{aligned}$$

The information of winding numbers are replaced by localized flux ξ^F and localized curvature ξ^R.

Since $\widetilde{\psi}_{T^2/Z_N,+}(z)$ have no winding numbers, these can be connected to $\psi^0_{s^2,+}(z')$.

4. Blow-up of T^2/Z_N orbifold

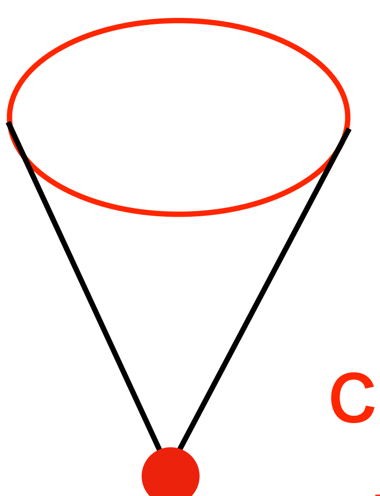
Connection condition $\tilde{\psi}^0_{T^2/Z_N,\pm}(z)$ and $\psi^0_{S^2,+}(z')$

From non-holomorphic part

$$\frac{\pi r^2}{N\,\mathrm{Im}\tau}M + \frac{\xi^F}{N} = \frac{N-1}{2N}M' \quad \overset{r\to 0}{\Rightarrow} \quad \frac{\xi^F}{N} = \frac{N-1}{2N}M'$$

ξ^F : localized flux. M' : Total flux of S^2 , $\frac{N-1}{2N}$: embedded area of S^2

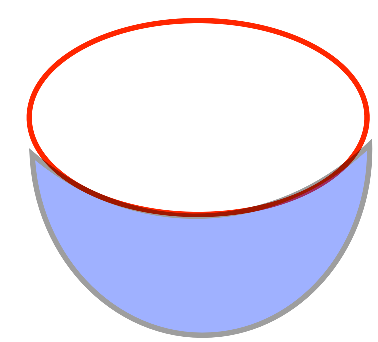
Physical meaning of connection condition



$$\boxed{\frac{\xi^F}{N} = \frac{N-1}{2N}M'}$$

Cut out flux of T^2/Z_N at fixed point

Embedded flux of S^2



5. Conclusion

Index theorem on blow-up manifold of T^2/Z_N orbifold

$$n_+ - n_- = \int_{blow-up} \frac{F}{2\pi} = \frac{M}{N} + \sum_{z_j^f} \frac{N-1}{2N} M' = \frac{M}{N} + \sum_{z_j^f} \frac{\xi^F}{N}$$

$n_\pm$: chiral zero modes number , M' : Total flux of S^2 ,

$\frac{N-1}{2N}$: embedded area of S^2 , ξ^F : localized flux at fixed point

5. Conclusion

Index theorem on blow-up manifold of T^2/Z_N orbifold

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$n_\pm$: chiral zero modes number , M' : Total flux of S^2 ,

$\frac{N-1}{2N}$: embedded area of S^2 , ξ^F : localized flux at fixed point

Only contribution of flux!

5. Conclusion

Reinterpretation of index formula on T^2/Z_N orbifold

$$\text{(generation \#)} = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

$$\xi^F = \frac{\xi^R}{2} - m \quad (z_j^f = 0) \quad \Rightarrow \quad \xi^F = \frac{\xi^R}{2} - \chi_+ \quad (z_j^f \neq 0)$$

$$\sum_{z_j^f} \frac{\xi^F}{N} = -\sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} = -\frac{V_+}{N} + 1$$

5. Conclusion

Reinterpretation of index formula on T^2/Z_N orbifold

$$\text{(generation \#)} = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

$+1$ removes contribution of localized curvatures in V_+.

$$\sum_{z_j^f} \frac{\xi^F}{N} = -\sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} = -\frac{V_+}{N} + 1$$

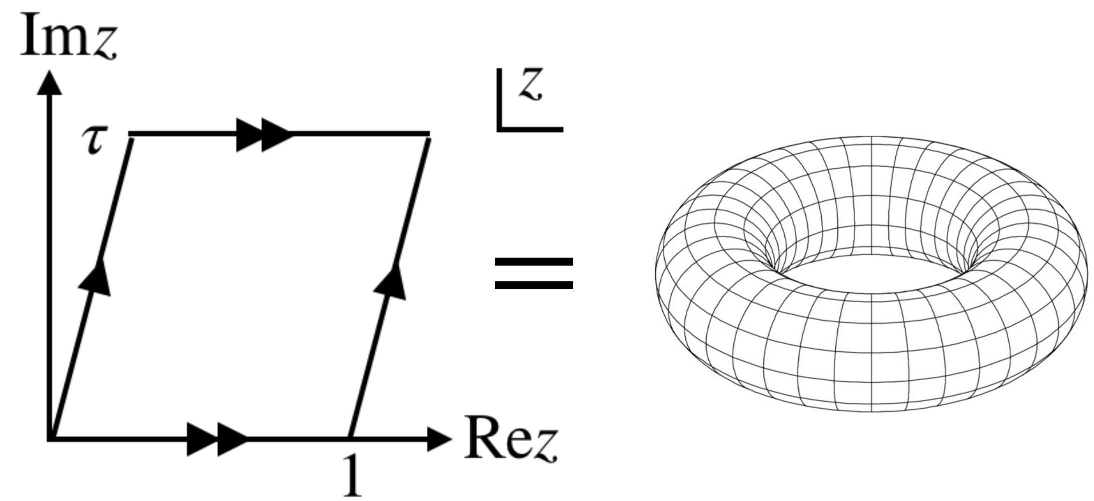
Thank you!

Back up

3. T^2 model

- $\mathscr{M}^4 \times T^2$ in flux background

$T^2 : z \sim z + 1 \sim z + \tau$



6d Dirac action

$$S = \int d^4x \int d^2z \bar{\Psi}(x,z) i\Gamma^M D_M \Psi(x,z)$$

$$D_M = \partial_M - iqA_M$$

6d chiral fermion

Mode expansion

2d chiral fermion

$$\Psi(x,z) = \sum_n \psi_{R,n}^{(4)}(x) \otimes \psi_{T^2+,n}^{(2)}(z) + \psi_{L,n}^{(4)}(x) \otimes \psi_{T^2-,n}^{(2)}(z)$$

4d chiral fermion

2d chiral fermion

$$\psi_{T^2+,n}^{(2)}(z) = \begin{pmatrix} \psi_{T^2+,n} \\ 0 \end{pmatrix}, \quad \psi_{T^2-,n}^{(2)}(z) = \begin{pmatrix} 0 \\ \psi_{T^2-,n} \end{pmatrix}$$

3. T^2 model

- $\mathscr{M}^4 \times T^2$ **in flux background**

T^2 sift corresponds to gauge transformation.

Boundary condition

$$\psi_{T^2\pm,n,j}(z+1) = U_1(z)\,\psi_{T^2\pm,n,j}(z)$$

$$\psi_{T^2\pm,n,j}(z+\tau) = U_2(z)\,\psi_{T^2\pm,n,j}(z)$$

$$U_1(z) = e^{iq\Lambda_1(z)}, U_2(z) = e^{iq\Lambda_2(z)}$$

Flux quantization $\frac{qf}{2\pi} \equiv M \in \mathbb{Z}$

M: flux quantization number f : flux

3. T^2 model

- $\mathscr{M}^4 \times T^2$ **in flux background**

Eigenvalue equation

$$\left.\begin{aligned} 2D_{\bar{z}}\psi_{T^2+,n,j} &= 2\left(\partial_{\bar{z}} - iqA_{\bar{z}}\right)\psi_{T^2+,n,j} = m_n \psi_{T^2-,n,j} \\ -2D_{z}\psi_{T^2-,n,j} &= -2\left(\partial_{z} - iqA_{z}\right)\psi_{T^2-,n,j} = m_n \psi_{T^2+,n,j} \end{aligned}\right\} \quad \begin{aligned} -4D_z D_{\bar{z}}\psi_{T^2+,n,j} &= m_n^2 \psi_{T^2+,n,j} \\ -4D_{\bar{z}} D_{z}\psi_{T^2-,n,j} &= m_n^2 \psi_{T^2-,n,j} \end{aligned}$$

$\Rightarrow$ **Zero mode has the eigenvalue** $m_n = 0$**.**

AS index theorem on T^2

$$\boxed{n_+ - n_- = \int_{T^2} \frac{F}{2\pi} = M}$$

Chiral zero modes #

flux quantization#

4. T^2/Z_N orbifold model

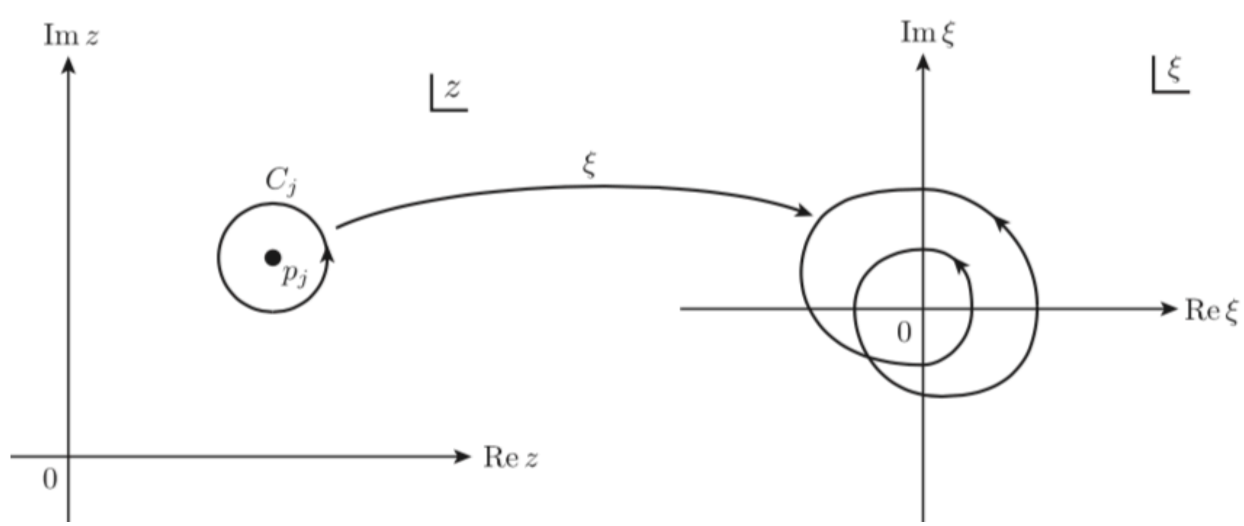
Winding number

Define winding number of Z_N eigen function $\psi^m_{T^2/Z_N+,n,j}(z)$

at fixed point z_j^f

$$\boxed{\psi^m_{T^2/Z_N+,n,j}(\rho z + z_j^f) = \rho^{\chi_+}\psi^m_{T^2/Z_N+,n,j}(z + z_j^f) \quad \Rightarrow \textbf{winding \# : } \chi_+}$$

ex)winding # = 2

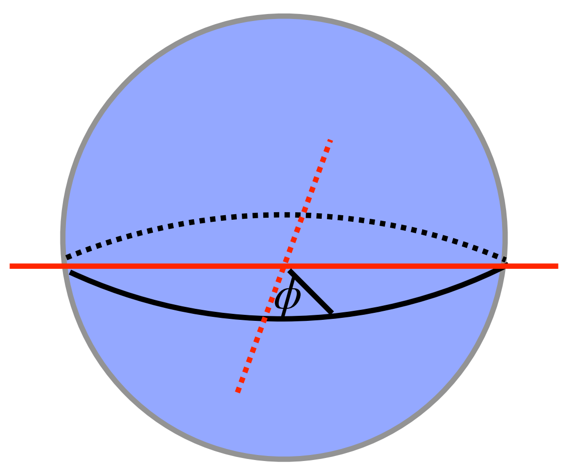


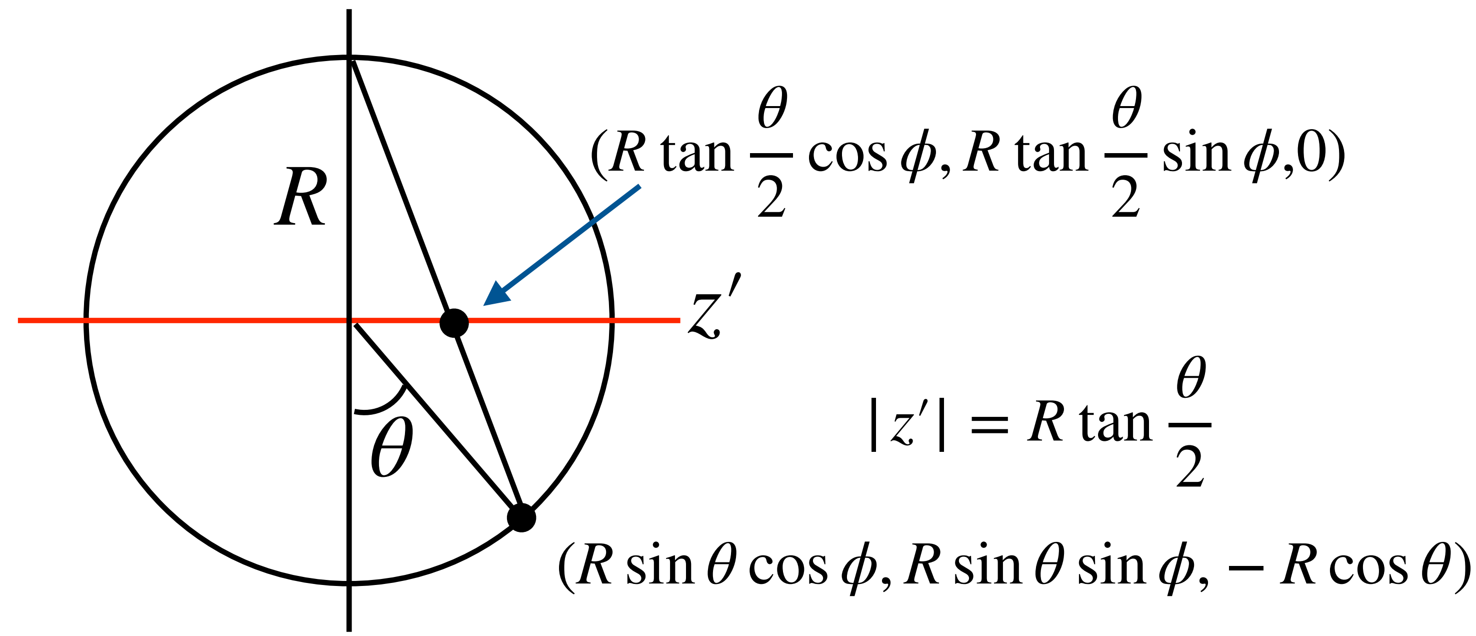
Winding number depends on eigen function & fixed point

5. Magnetized S^2

Magnetized S^2

How to take the coordinate z' of S^2





AS index theorem on S^2

$$n_+ - n_- = \int_{S^2} \frac{F}{2\pi} = M'$$

Chiral zero modes # flux quantization#

5. Magnetized S^2

Magnetized S^2

Dirac equation

$$\frac{R^2+|z'|^2}{R} i\left(\partial_{\bar{z}'} + i\frac{1}{2}\omega_{\bar{z}'} - iA_{\bar{z}'}\right)\psi^n_{s^2,+}(z') = m_n \psi^n_{s^2,-}(z')$$

$$\frac{R^2+|z'|^2}{R} i\left(\partial_{z'} - i\frac{1}{2}\omega_{z'} - iA_{z'}\right)\psi^n_{s^2,-}(z') = m_n \psi^n_{s^2,+}(z')$$

$$\omega_{\bar{z}'} = \frac{i}{2}\frac{2}{R^2+|z'|^2}z',\ \omega_{z'} = -\frac{i}{2}\frac{2}{R^2+|z'|^2}\bar{z}',\ A_{\bar{z}'} = \frac{i}{2}\frac{M'}{R^2+|z'|^2}z',\ A_{z'} = -\frac{i}{2}\frac{M'}{R^2+|z'|^2}\bar{z}'$$

⇒ **Zero mode has the eigenvalue $m_n = 0$.**

Zero mode

$$\psi^0_{s^2,+}(z') = \frac{f_+(z')}{(R^2+|z'|^2)^{\frac{M'-1}{2}}}$$

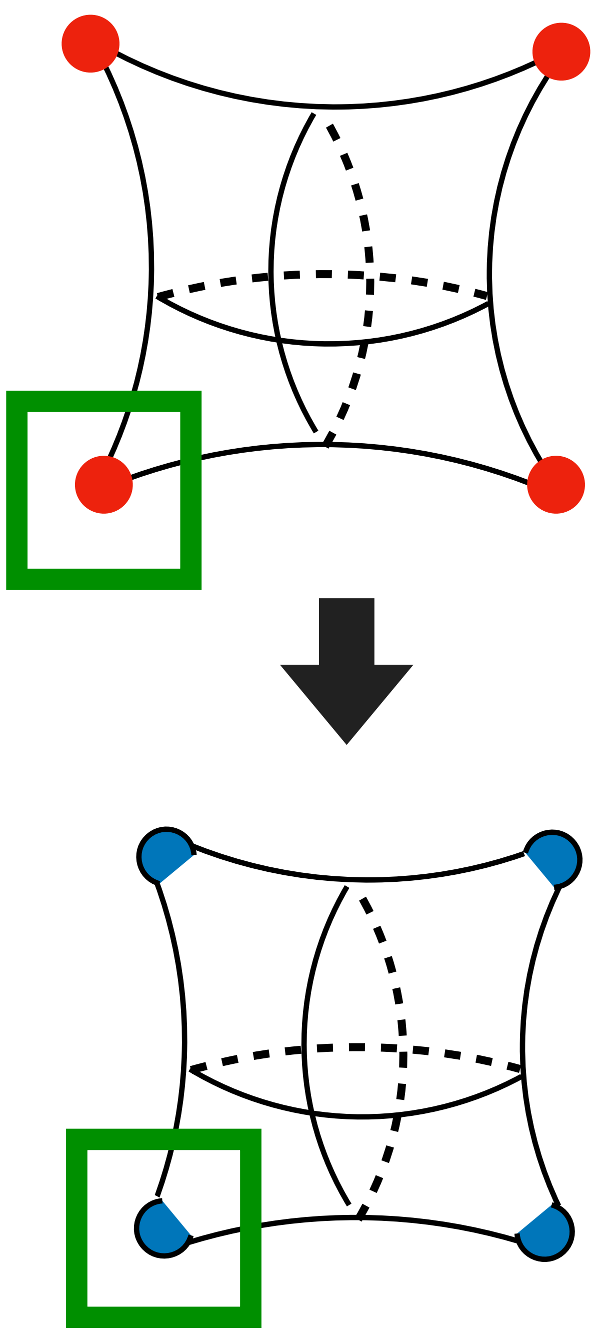
Lowest mode with - chirality

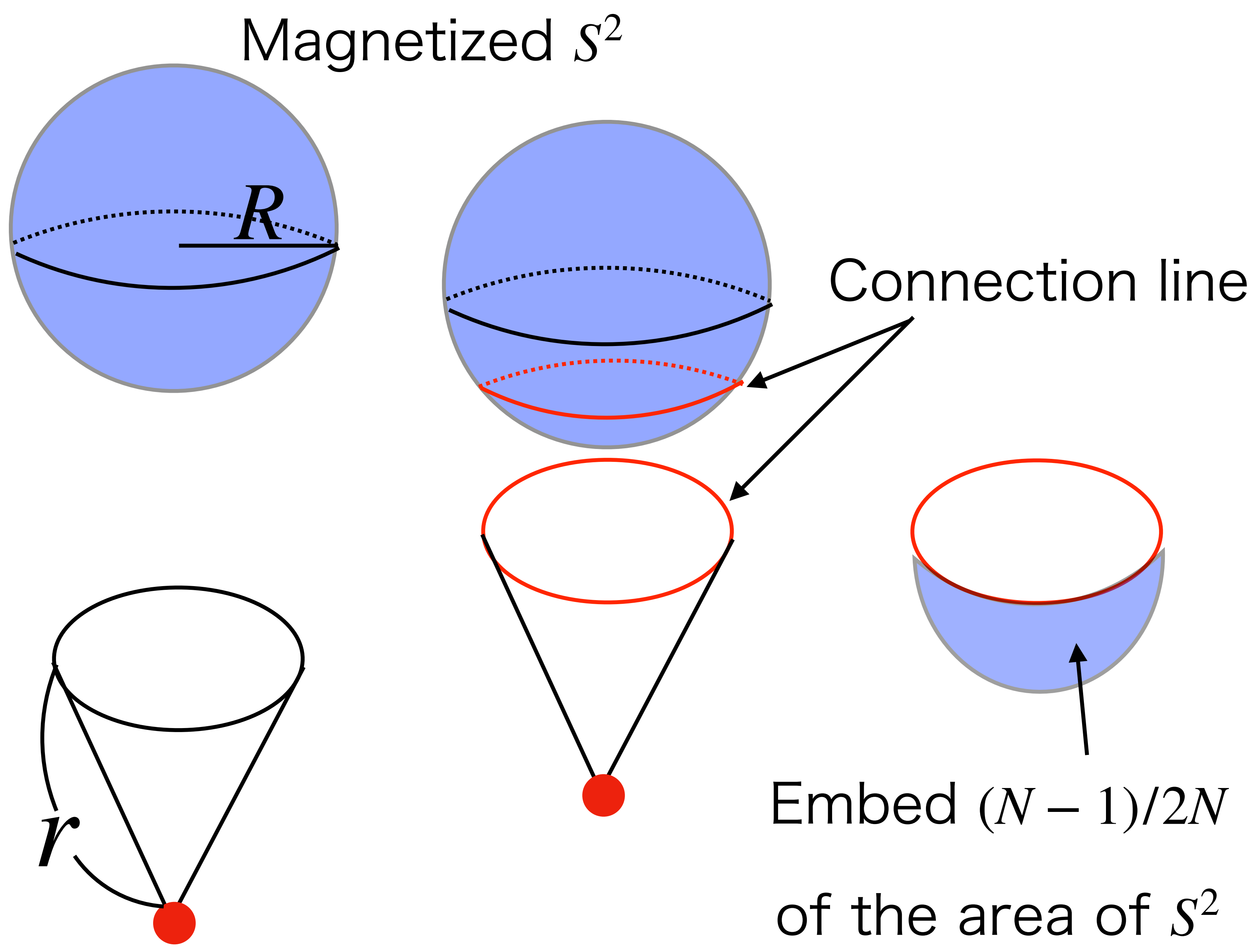
$$\psi^1_{s^2,-}(z') = \frac{f_-(z')}{(R^2+|z'|^2)^{\frac{M'+1}{2}}}$$

$f_+(z'), f_-(z')$: holomorphic function

4. Blow-up of T^2/Z_N orbifold

To apply AS index theorem, we introduce blow-up manifold





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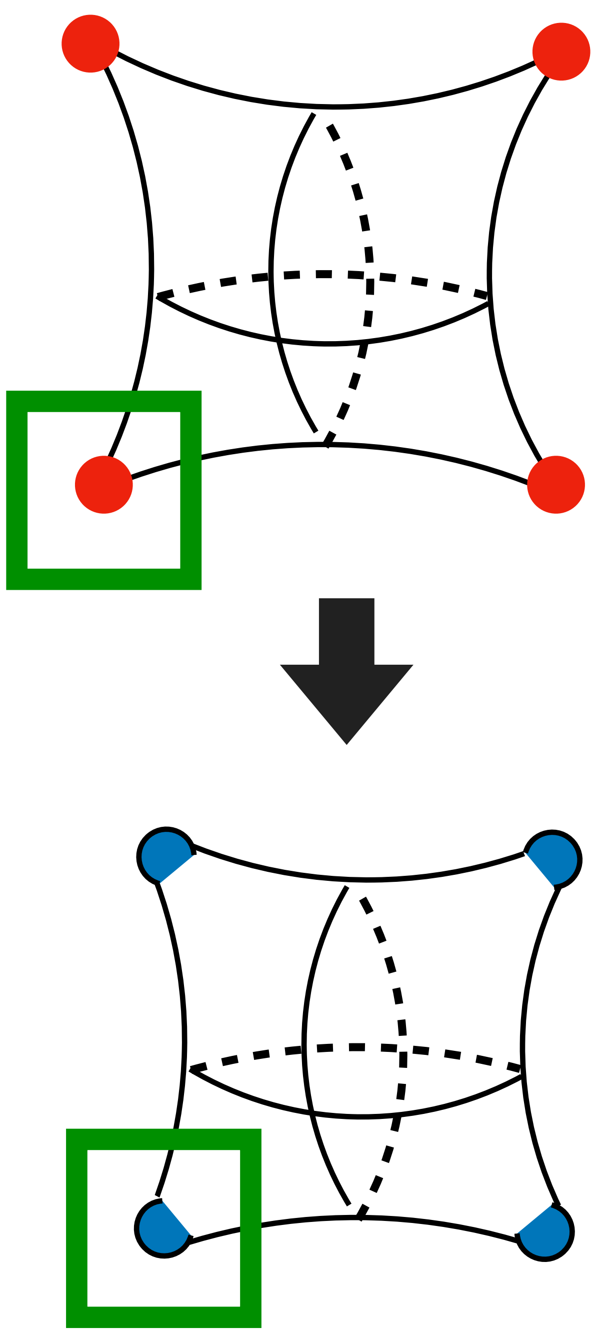
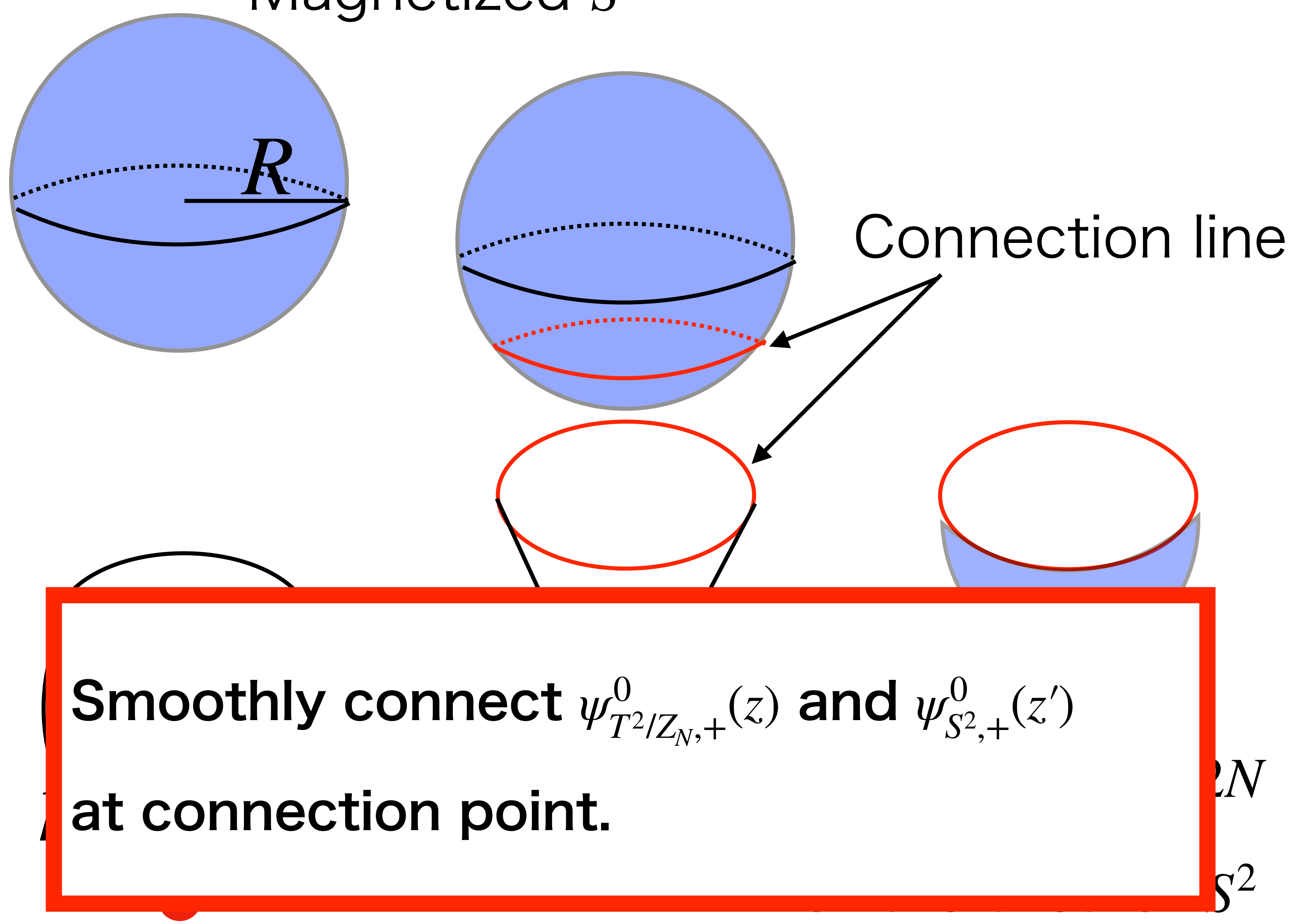
To apply AS index theorem, we introduce blow-up manifold





Smoothly connect $\psi^0_{T^2/Z_N,+}(z)$ **and** $\psi^0_{S^2,+}(z')$
at connection point.

4. Blow-up of T^2/Z_N orbifold

"Singular" gauge transformation

$$\widetilde{\psi}_{T^2/Z_N,\pm}(z) = U_{\xi^F} U_{\xi^R} \psi_{T^2/Z_N,\pm}(z) \qquad U_{\xi^F} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^F}{2}}, \ U_{\xi^R} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^R}{4}}$$

$\psi_{T^2/Z_N,+}(z)$

Field strength F

$$\int_{T^2/Z_N} \frac{F}{2\pi} = \frac{M}{N}$$

Gauge field A

+

Winding number χ_+

$\widetilde{\psi}_{T^2/Z_N,+}(z)$

Field strength $F + \delta F$

$$\int_{T^2/Z_N} \frac{F+\delta F}{2\pi} = \frac{M}{N} + \frac{\xi^F}{N}$$

ξ^F : localized flux at fixed point

Gauge field $A + \delta A$ $\qquad \delta A = iU_{\xi^F} dU_{\xi^F}^{-1}$

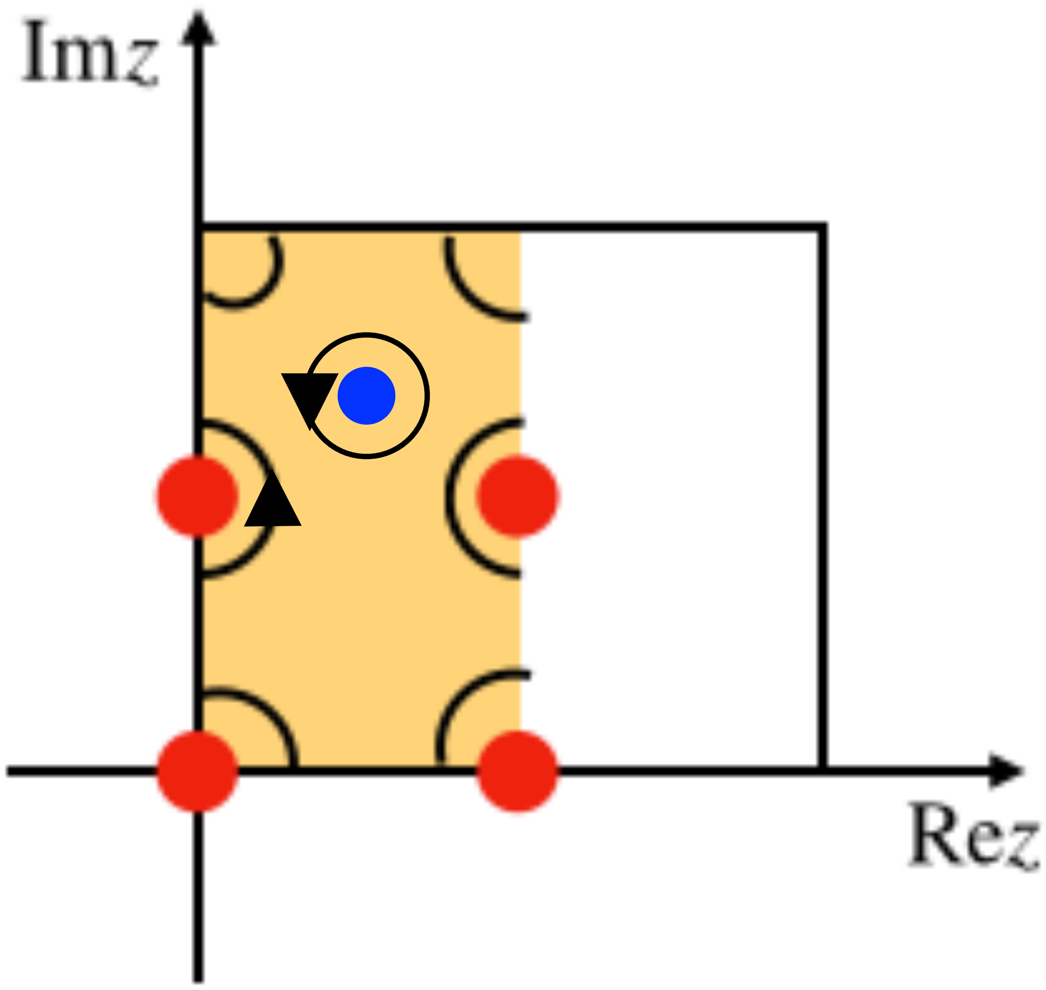
Localized curvature ξ^R

$$\int_{T^2/Z_N \ fixed\, point} \frac{\delta R}{2\pi} = \frac{\xi^R}{N}$$

+ No winding number

6. Blow-up of T^2/Z_N orbifold

Localized curvature at fixed point ξ^R



At fixed points
→Winding is closed at 180°

At the other points
→Winding is closed at 360°

At fixed points, there is deflection angle $2\pi - \dfrac{2\pi}{N} = 2\pi \dfrac{N-1}{N}$**.**

6. Blow-up of T^2/Z_N orbifold

Localized curvature at fixed point ξ^R

Gauss-Bonnet theorem

$$\int_{\mathcal{M}} \frac{R}{2\pi} = \chi(\mathcal{M})$$

R : Curvature χ : Euler characteristic

$$+$$

$$(deflection\ angle) = 2\pi\chi$$

→ Localized curvature at fixed point $\dfrac{\xi^R}{N} = \dfrac{N-1}{N}$

5. Conclusion

Reinterpretation of index formula on T^2/Z_N orbifold

$$(\text{generation }\#) = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

Index theorem implies the existence of l new zero modes = localized modes

$$\sum_{z_j^f} \frac{\xi^F}{N} = -\sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} + l = -\frac{V_+}{N} + 1 + l$$