Isgur-Wise functions and the Lorentz group

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The Role of Heavy Fermions in Fundamental Physics

Portoroz (Slovenia)

11-14 April 2011

On work by A. Le Yaouanc, L. Oliver and J.-C. Raynal

Motivations

At LHC, many more urgent subjects than $b \to c\ell\nu$ transitions :

- Search of the Higgs boson
- Search of New Physics (Supersymmetry ?)
- Precise study of CP violation in B mesons, as in $B_s - \overline{B}_s$
- Look for photon polarization in rare decays $b \to s\gamma$

However, there are some motivations :

- It is never too late to get new rigorous results on this subject
- $BR(\Lambda_b \to \Lambda_c \ell\nu) \simeq 5\%$ (Tevatron), $\frac{d\Gamma}{dw}$ can be studied at LHC-b
- Exclusive (HQET) $\overline{B} \to D(D^*)\ell\nu \Rightarrow |V_{cb}| = (38.7 \pm 1.1) \times 10^{-3}$
  Inclusive (OPE) $\overline{B} \to X_c\ell\nu \Rightarrow |V_{cb}| = (41.5 \pm 0.7) \times 10^{-3}$
  Consistent within errors, but the situation is not satisfactory

Isgur-Wise functions and the Lorentz group

Matrix element of a current between heavy hadrons **factorizes** (A. Falk) into a trivial **heavy quark current matrix element** and a **light cloud overlap** (that contains the long distance physics)

$$< H'(v')|J^{Q'Q}(q)|H(v) > =$$

$$< Q'(v'), \pm\frac{1}{2}|J^{Q'Q}(q)|Q(v), \pm\frac{1}{2} >< v', j', M'|v, j, M >$$

The light cloud follows the heavy quark with the same four-velocity

Isgur-Wise functions : light cloud overlaps $\xi(v.v') = < v'|v >$

Factorization valid only in absence of **hard radiative corrections**

Light cloud Hilbert space

Can demonstrate that the light cloud states form a Hilbert space

on which acts a unitary representation of the Lorentz group
$\Lambda \to U(\Lambda)$ $\qquad U(\Lambda)|v, j, \epsilon > = |\Lambda v, j, \Lambda\epsilon >$

$$\boxed{|v, j, \epsilon > = \sum_M (\Lambda^{-1}\epsilon)_M \ U(\Lambda)|v_0, j, M >}$$

$\Lambda v_0 = v$ $\qquad v_0 = (1, 0, 0, 0)$ $\qquad \Lambda^{-1}\epsilon :$ polarization vector at rest

This fundamental formula defines, in the Hilbert space $\mathcal{H}$
of a unitary representation of $SL(2, C)$ the states $|v, j, \epsilon >$
whose scalar products define the IW functions
in terms of $|v_0, j, M >$ which occur as $SU(2)$ multiplets
in the restriction to $SU(2)$ of the $SL(2, C)$ representation

Choose the simpler case of baryons with $\mathbf{j = 0}$

Baryons $\Lambda_b(v)$, $\Lambda_c(v)$ ($S_{qq} = 0, L = 0$ in quark model language)

Then, the Isgur-Wise function writes

$$\boxed{\xi(v.v') = < U(B_{v'})\phi_0 | U(B_v)\phi_0 >}$$

$|\phi_0 >$ represents the light cloud at rest and B_v, $B_{v'}$ are boosts

$$\xi(w) = < \phi_0 | U(\Lambda)\phi_0 > \qquad \Lambda v_0 = v \qquad v^0 = w$$

Λ is for instance the boost along Oz

$$\Lambda_\tau = \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix} \qquad w = ch(\tau)$$

Method completely general, for any j and any transition $j \to j'$

Decomposition into irreducible representations

The unitary representation $U(\Lambda)$ is in general reducible

Useful to decompose it into irreducible representations $U_\chi(\Lambda)$

Hilbert space $\mathcal{H}$ made of functions $\psi : \chi \in X \to \psi_\chi \in \mathcal{H}_\chi$

Scalar product in $\mathcal{H}$

$$\boxed{< \psi'|\psi > = \int_X < \psi'_\chi|\psi_\chi > d\mu(\chi)}$$

$\chi \in X$: irreducible unitary representation
$d\mu(\chi)$: a positive measure

$$(U(\Lambda)\,\psi)_\chi = U_\chi(\Lambda)\psi_\chi \qquad \psi_\chi \in \mathcal{H}_\chi$$

$\mathcal{H}_\chi$: Hilbert space of χ on which acts $U_\chi(\Lambda)$

Integral formula for the Isgur-Wise function

Notation $\boxed{\xi_\chi(w) = < \phi_{0,\chi} | U_\chi(\Lambda) \phi_{0,\chi} >}$

irreducible Isgur-Wise function corresponding to irreducible χ

General form of the IW function : $\xi(w) = \int_{X_0} \xi_\chi(w)\ d\nu(\chi)$

Isgur-Wise function as a mean value of irreducible IW functions with respect to some *positive* normalized measure ν

$\int_{X_0} d\nu(\chi) = 1$

$X_0 \subset X$ irreducible representations of $SL(2, C)$
containing a non-zero $SU(2)$ scalar subspace ($j = 0$ case)

Irreducible IW function $\xi_\chi(w)$ when ν is a δ function

Irreducible unitary representations of the Lorentz group

Naïmark (1962)

Principal series $\quad \chi = (p, n, \rho)$

$n \in Z$ and $\rho \in R \qquad (n = 0, \rho \geq 0; n > 0, \rho \in R)$

Hilbert space $\mathcal{H}_{p,n,\rho}$

$$< \phi' | \phi > = \int \overline{\phi'(z)}\ \phi(z)\ d^2 z \qquad d^2 z = d(Rez) d(Imz)$$

Unitary operator $U_{p,n,\rho}(\Lambda)$

$$(U_{p,n,\rho}(\Lambda)\phi)(z) = \left(\frac{\alpha - \gamma z}{|\alpha - \gamma z|}\right)^n |\alpha - \gamma z|^{2i\rho - 2}\ \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right)$$

$$\Lambda = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \qquad \alpha\delta - \beta\gamma = 1 \qquad (\alpha, \beta, \gamma, \delta) \in C$$

Supplementary series $\chi = (s, \rho)$

$\rho \in R \qquad (0 < \rho < 1)$

Hilbert space $\mathcal{H}_{s,\rho}$

$$< \phi' | \phi > = \int \overline{\phi'(z_1)} \; |z_1 - z_2|^{2\rho - 2} \; \phi(z_2) \; d^2 z_1 d^2 z_2$$

(non-standard scalar product)

Unitary operator $U_{s,\rho}(\Lambda)$

$$(U_{s,\rho}(\Lambda)\phi)(z) = |\alpha - \gamma z|^{-2\rho-2} \; \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right)$$

Trivial representation $\chi = t$

Hilbert space $\mathcal{H}_t = C$

$$< \phi' | \phi > = \overline{\phi'(z)} \phi(z)$$

Unitary operator $\qquad U_t(\Lambda) = 1$

Decomposition under the rotation group

Need restriction to $SU(2)$ of unitary representations χ of $SL(2,C)$

For a χ there is an orthonormal basis $\phi^{\chi}_{j,M}$ of $\mathcal{H}_\chi$ adapted to $SU(2)$

Particularizing to $j=0$: all types of representations contribute

$$\phi^{p,0,\rho}_{0,0}(z) = \frac{1}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1} \qquad (\chi=(p,0,\rho),\ \rho\geq 0)$$

$$\phi^{s,\rho}_{0,0}(z) = \frac{\sqrt{\rho}}{\pi}(1+|z|^2)^{-\rho-1} \qquad (\chi=(s,\rho),\ 0<\rho<1)$$

$$\phi^{t}_{0,0}(z) = 1 \qquad (\chi=t)$$

For $j \neq 0$ enters also the matrix element

$$D^j_{M',M}(R) = < j, M'|U_j(R)|j, M > \qquad R \in SU(2)$$

Irreducible IW functions in the case $\mathbf{j = 0}$

Need $\xi_\chi(w) = < \phi^\chi_{0,0} | U_\chi(\Lambda_\tau) \phi^\chi_{0,0} >$ (Λ_τ : boost, $w = ch(\tau)$)

Transformed elements $U_\chi(\Lambda_\tau)\phi^\chi_{0,0}$

$$\left(U_{p,0,\rho}(\Lambda_\tau)\phi^{p,0,\rho}_{0,0} \right)(z) = \frac{1}{\sqrt{\pi}}(e^\tau + e^{-\tau}|z|^2)^{i\rho - 1}$$

$$\left(U_{s,\rho}(\Lambda_\tau)\phi^{s,\rho}_{0,0} \right)(z) = \frac{\sqrt{\rho}}{\sqrt{\pi}}(e^\tau + e^{-\tau}|z|^2)^{-\rho - 1}$$

$$U_t(\Lambda_\tau)\phi^t_{0,0} = 1$$

Using the scalar products for each class of representations

$$\xi_{p,0,\rho}(w) = \frac{sin(\rho\tau)}{\rho \; sh(\tau)}$$

$$\xi_{s,\rho}(w) = \frac{sh(\rho\tau)}{\rho \; sh(\tau)} \qquad (0 < \rho < 1)$$

$$\xi_t(w) = 1$$

Integral formula for the IW function in the case $j = 0$

$$\xi(w) = \int_{[0,\infty[} \frac{sin(\rho\tau)}{\rho\ sh(\tau)}\ d\nu_p(\rho) + \int_{]0,1[} \frac{sh(\rho\tau)}{\rho\ sh(\tau)}\ d\nu_s(\rho) + \nu_t$$

ν_p and ν_s are positive measures and ν_t a real number ≥ 0

$$\int_{[0,\infty[} d\nu_p(\rho) + \int_{]0,1[} d\nu_s(\rho) + \nu_t = 1$$

One-parameter family $\qquad \xi_x(w) = \frac{sh(\tau\sqrt{1-x})}{sh(\tau)\sqrt{1-x}} = \frac{sin(\tau\sqrt{x-1})}{sh(\tau)\sqrt{x-1}}$

covers all irreducible representations $\rightarrow$ simplifies integral formula

$\xi(w) = \int_{[0,\infty[} \xi_x(w)\ d\nu(x) \quad$ (ν positive measure $\int_{[0,\infty[} d\nu(x) = 1$)

$\xi_{p,0,\rho}(w) = \xi_x(w)$	$x = 1 + \rho^2, \rho \in [0,\infty[$	$\Leftrightarrow$	$x \in [1,\infty[$
$\xi_{s,\rho}(w) = \xi_x(w)$	$x = 1 - \rho^2, \rho \in]0,1[$	$\Leftrightarrow$	$x \in]0,1[$
$\xi_t(w) = \xi_x(w)$	$x = 0$		$x \in \{0\}$

$\rightarrow$ a transparent deduction of constraints on the derivatives $\xi^{(n)}(1)$

Constraints on the derivatives of the Isgur-Wise function

Derivative $\xi^{(k)}(1)$: *expectation value* of a polynomial of degree k

$$\xi^{(k)}(1) = (-1)^k \ 2^k \frac{k!}{(2k+1)!} < \prod_{i=1}^{k} (x + i^2 - 1) >$$

In terms of moments $\qquad \mu_n = < x^n >$

$$\xi(1) = \mu_0 = 1$$

$$\xi'(1) = -\frac{1}{3} \ \mu_1$$

$$\xi''(1) = \frac{1}{15} \ (3\mu_1 + \mu_2)$$

...

Moments μ_k in terms of derivatives $\xi(1)$, $\xi'(1)$, ... $\xi^{(k)}(1)$

$$\mu_0 = \xi(1) = 1$$

$$\mu_1 = -3 \ \xi'(1)$$

$$\mu_2 = 3 \left[3 \ \xi'(1) + 5 \ \xi''(1)\right]$$

...

Constraints on moments of a variable with positive values

$det\left[(\mu_{i+j})_{0\leq i,j\leq n}\right] \geq 0$

$det\left[(\mu_{i+j+1})_{0\leq i,j\leq n}\right] \geq 0$

Lower moments

$\mu_1 \geq 0$

$\mu_2 \geq \mu_1^2$

...

That imply for the derivatives of the Isgur-Wise function

$\rho_\Lambda^2 \geq 0$

$\xi''(1) \geq \frac{3}{5}\rho_\Lambda^2(1+\rho_\Lambda^2)$

...

Same results as with the Sum Rule approach

Consistency test for any Ansatz of the Isgur-Wise function

• We have the integral representation

$$\xi(w) = \int_{[0,\infty[} \frac{sin(\rho\tau)}{\rho \ sh(\tau)} \ d\nu_p(\rho) + \int_{]0,1[} \frac{sh(\rho\tau)}{\rho \ sh(\tau)} \ d\nu_s(\rho) + \nu_t$$

• ν_p and ν_s are positive measures and ν_t real ≥ 0 satisfying

$$\int_{[0,\infty[} d\nu_p(\rho) + \int_{]0,1[} d\nu_s(\rho) + \nu_t = 1$$

• One can invert the integral formula by Fourier transforming

• One can check if a given Ansatz for the Isgur-Wise function $\xi(w)$ satisfies the integral representation with *positive measures*

Phenomenological examples

Example 1 exponential (principal series contributes)

$$\xi(w) = exp[-c(w-1)] = \frac{2}{\pi}\frac{e^c}{c}\int_0^\infty \rho^2\, K_{i\rho}(c)\, \frac{sin(\rho\tau)}{\rho\ sh(\tau)}\, d\rho\ [w = ch(\tau)]$$

Inconsistent : $K_{i\rho}(c)$ can be negative $\rightarrow d\nu_p(\rho)$ is not positive

Example 2 (principal and supplementary series contribute)

$$\xi(w) = \left(\frac{2}{1+w}\right)^{2c} = \frac{4^{2c}}{\pi}\int_0^\infty \rho^2\ \frac{|\Gamma(2c+i\rho-1)|^2}{\Gamma(4c-1)}\ \frac{sin(\rho\tau)}{\rho\ sh(\tau)}\ d\rho$$

$$+\ \theta(1-2c)\ (1-2c)\ 2^{4c}\ \frac{sh((1-2c)\tau)}{(1-2c)\ sh(\tau)}$$ valid for slope $c \geqslant \frac{1}{4}$

Example 3 (principal series contributes)

$$\xi(w) = \frac{1}{\left[1+\frac{c}{2}(w-1)\right]^2} = \frac{8}{c^2}\int_0^\infty \frac{\rho^2}{sh(\pi\rho)}\ \frac{sh(\gamma\rho)}{sh(\gamma)}\ \frac{sin(\rho\tau)}{\rho\ sh(\tau)}\ d\rho$$

$(cos\gamma = \frac{2}{c} - 1)$ valid for slope $c \geqslant 1$

Other theoretical results

The Isgur-Wise function is a function of positive type

From Bjorken-like Sum Rules one can demonstrate

$$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}\,'}{v\,'^0}\ \psi(v')^*\ \xi(v.v')\ \psi(v) \geq 0 \qquad \text{for any } \psi(v)$$

$\Rightarrow$ strong constraints on the possible forms of the IW function

The Sum Rule and Lorentz group approaches are equivalent

- The Lorentz group approach implies that $\xi(w)$ is of positive type
- The Sum Rule approach implies the Lorentz group approach

Examples of one-parameter functions satisfying the theoretical constraints

Isgur-Wise function for baryons $j^P = 0^+$ $\qquad \Lambda_b \to \Lambda_c \ell \nu$

$$\boxed{\xi_\Lambda(w) = \left(\frac{2}{w+1}\right)^{2\rho_\Lambda^2} \qquad \text{with} \qquad \rho_\Lambda^2 \geq \frac{1}{4}}$$

Rigorous lower bound (Isgur et al. SR) : $\qquad \rho_\Lambda^2 \geq 0$

Isgur-Wise function for mesons $j^P = \frac{1}{2}^-$ $\qquad \overline{B} \to D(D^*)\ell\nu$

One can apply the method to mesons (spin complications)

$$\boxed{\xi(w) = \left(\frac{2}{w+1}\right)^{2\rho^2} \qquad \text{with} \qquad \rho^2 \geq \frac{3}{4}}$$

Rigorous lower bound (Bjorken + Uraltsev SR) : $\qquad \rho^2 \geq \frac{3}{4}$

New rigorous results for non-perturbative physics in HQET

• Decomposing into irreducible representations a unitary representation of the Lorentz group → integral formula for the Isgur-Wise function with positive measures

• Explicitly given for $j = 0$ ($\Lambda_b \to \Lambda_c \ell \nu$)

• Derivatives of the IW function given in terms of moments of a positive variable → inequalities between the derivatives

• Sum Rules → IW function is a function of positive type

• Application : exponential form of the IW function is inconsistent

• Equivalence between Sum Rule and Lorentz group approaches

• Consistency test for any Ansatz of the IW function

• Application to phenomenological examples

• Can be generalized for any j ($j = \frac{1}{2}$ for mesons $\overline{B}_d \to D^{(*)} \ell \nu$)

The Isgur-Wise function is a function of positive type

For any N and any complex numbers a_i and velocities v_i

$\sum_{i,j=1}^{N} a_i^* a_j \ \xi(v_i.v_j) \geq 0$ or, in a covariant form

$\int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}\,'}{v\,'^0} \ \psi(v')^* \ \xi(v.v') \ \psi(v) \geq 0$ for any $\psi(v)$

From the Sum Rule $(w_i = v_i.v', w_j = v_j.v', w_{ij} = v_i.v_j)$

$$\xi(w_{ij}) = \sum_n \sum_L \tau_L^{(n)}(w_i)^* \tau_L^{(n)}(w_j)$$
$$\sum_{0\leq k\leq L/2} C_{L,k} \ (w_i^2-1)^k (w_j^2-1)^k (w_i w_j - w_{ij})^{L-2k}$$

Legendre polynomial. Use rest frame $v' = (1,0,0,0)$

$$\sum_{i,j=1}^{N} a_i^* a_j \ \xi(v_i.v_j) = 4\pi \sum_{i,j=1}^{N} \sum_n \sum_L \frac{2^L (L!)^2}{(2L+1)!} \sum_{m=-L}^{m=+L}$$
$$\left[a_i \ \tau_L^{(n)}\left(\sqrt{1+\vec{v}_i^{\,2}}\right) \mathcal{Y}_L^m(\vec{v}_i)\right]^* \left[a_j \ \tau_L^{(n)}\left(\sqrt{1+\vec{v}_j^{\,2}}\right) \mathcal{Y}_L^m(\vec{v}_j)\right] \geq 0$$

One example : application to the exponential form

$$\xi(w) = exp\left[-c(w-1)\right]$$

$$I = \int \frac{d^3\vec{v}}{v^0}\frac{d^3\vec{v}\,'}{v\,'^0}\ \phi(|\vec{v}\,'|)^*\ exp\left[-c((v.v')-1)\right]\ \phi(|\vec{v}|)$$

$$= 16\pi^3\ \frac{e^c}{c}\int_{-\infty}^{\infty} K_{i\rho}(c)\ |\tilde{f}(\rho)|^2\ d\rho$$

$$f(\eta) = sh(\eta)\ \phi(sh(\eta))$$

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} exp[-z\ ch(t)]\ e^{\nu t}\ dt$$ Macdonald function

Whatever the slope $c > 0$, $K_{i\rho}(c)$ *takes negative values*

Asymptotic formula

$$K_{i\rho}(c) \sim \sqrt{\frac{2\pi}{\rho}}\ e^{-\rho\pi/2}\ cos\left[\rho\left(log\left(\frac{2\rho}{c}\right)-1\right)-\frac{\pi}{4}\right] \qquad (\rho >> c)$$

Therefore there a function $\psi(v)$ for which the integral $I < 0$

The exponential form is inconsistent with the Sum Rules

Sum Rule and Lorentz group approaches are equivalent

- The Lorentz group approach implies that $\xi(w)$ is of positive type

$\xi(w) = < U(B_{v'})\psi_0 | U(B_v)\psi_0 >$ $\qquad (B_v :$ boost $v_0 \to v)$

$\sum_{i,j=1}^{N} a_i^* a_j \ \xi(v_i.v_j) = \| \sum_{j=1}^{N} a_j U(B_{v_j})\psi_0 \|^2 \geq 0$

- The Sum Rule approach implies the Lorentz group approach

A function $f(\Lambda)$ on the group $SL(2, C)$ is of positive type when

$\sum_{i,j=1}^{N} a_i^* a_j \ f(\Lambda_i^{-1}\Lambda_j) \geq 0$ $\qquad (N \geq 1$, complex a_i, $\Lambda_i \in SL(2, C))$

Theorem (Dixmier) : for any function $f(\Lambda)$ of positive type exists a unitary representation $U(\Lambda)$ of $SL(2, C)$ in a Hilbert space $\mathcal{H}$ and an element $\phi_0 \in \mathcal{H} \to f(\Lambda) = < \phi_0 | U(\Lambda)\phi_0 >$

Definition of $f(\Lambda_i^{-1}\Lambda_j) = \xi(v_i.v_j) = \xi(v_0.\Lambda_i^{-1}\Lambda_j v_0)$