The Dirac and Maxwell eqs. can be derived from the lagrangian

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma^m D_m - M)\Psi - \frac{1}{4}F_{mn}^2 \ . \tag{11}$$

The coupled Euler-Lagrange field eqs. are then (10), plus

$$\partial^m F_{mn} = g\bar{\Psi}\gamma_n\Psi \equiv j_n \ , \tag{12}$$

where j_n is the electromagnetic current of the charged fermion. From (12) we can derive the charge conservation

$$\partial^m j_m = 0 \rightarrow \frac{dQ}{dt} = \int d^3\mathbf{x}\ \partial^m j_m = 0\ ,$$
$$\text{where}\quad Q \ = \int d^3\mathbf{x}\ j_0(\mathbf{x})\ . \tag{13}$$

- The massless photon has two degrees of freedom.
- A photon mass $\mathcal{L}_{mass} = \frac{M_A^2}{2} A_m^2$ breaks gauge invariance and describes three degrees of freedom.
- The propagator of a massive photon is found from

$$-\frac{1}{4}F_{mn}^2 + \frac{M_A^2}{2}A_m^2 = \frac{1}{2}A^m[g_{mn}(\Box + M_A^2) - \partial_m\partial_n]A^n,$$
$$\Delta_{mn}^{-1}(x-y) = [g_{mn}(\Box + M_A^2) - \partial_m\partial_n]\delta^4(x-y) \tag{14}$$

Therefore, in momentum space (exercice)

$$\Delta^{mn}(k) = \frac{g^{mn} - \frac{k^m k^n}{M_A^2}}{k^2 - M_A^2} \ . \tag{15}$$

- Due to the current conservation $\partial^m j_m = 0$, the longitudinal polarization does not contribute to amplitudes $\rightarrow$ UV properties of the massless and massive photon theories are the same.
- Experimental limit photon mass $m_\gamma \leq 10^{-18}$ eV.

2.3. Non-abelian gauge theories.

$U(1)$ is a particular case of unitary abelian transformations. Another case of particular interest : non-abelian transformations.

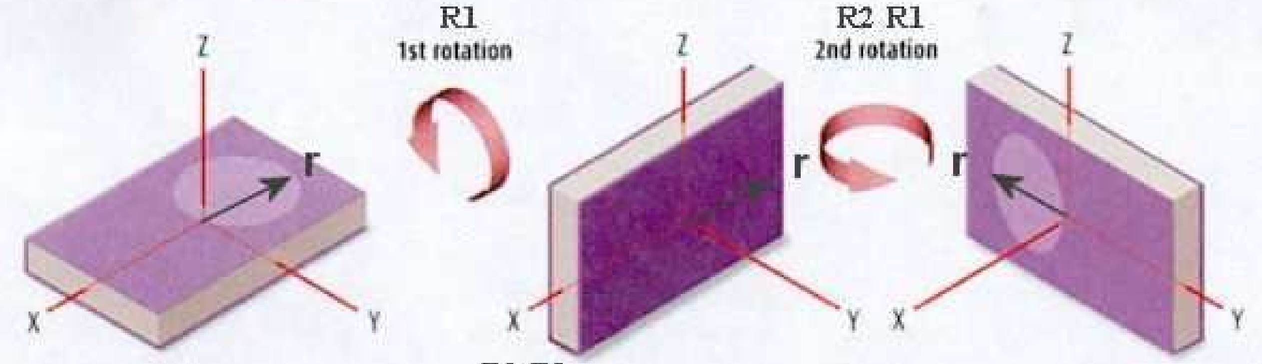
Non-commutative Operation

R1 R2 ≠ R2 R1

An object is rotated by 90° around two different axes

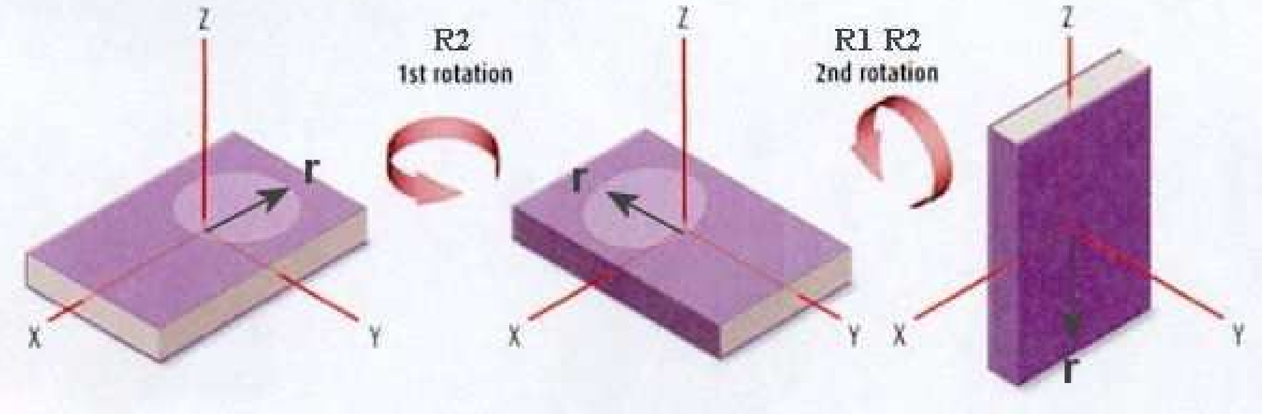
R1 : counter-clockwise rotation of 90^0 about the x axis

R2 : counter-clockwise rotation of 90^0 about the z axis



R2 R1

Applying the same rotations in reverse order leads to a different outcome



$SU(n)$ transformations are described by matrices U, satisfying

$$U^\dagger U = UU^\dagger = I \quad , \quad det\ U = 1\ . \tag{16}$$

The simplest case is $SU(2)$, proposed by Yang and Mills in 1954. Simplest representation is a doublet

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\ ,\ \Psi' = U(\theta)\Psi\ ,\ \text{where } U(\theta) = e^{\frac{i}{2}g\theta_a\tau_a}\ , \tag{17}$$

where τ_a are the Pauli matrices. The number of gauge bosons equals the number of generators (three for $SU(2)$). Simplest to introduce a matrix

$$W_m = W_m^a \frac{\tau_a}{2} = \begin{pmatrix} W_m^3 & W_m^1 - iW_m^2 \\ W_m^1 + iW_m^2 & -W_m^3 \end{pmatrix} \equiv \begin{pmatrix} W_m^3 & \sqrt{2}W_m^+ \\ \sqrt{2}W_m^- & -W_m^3 \end{pmatrix}$$

Exercice : show that

$$D_m\Psi \equiv (\partial_m - igW_m)\Psi \to (D_m\Psi)' = UD_m\Psi \ ,$$
$$\text{if } W_m \to W'_m = UW_mU^{-1} - \frac{i}{g}(\partial_m U)U^{-1} \tag{18}$$

and the infinitesimal variation in component form

$$\delta W_m^a = D_m\theta^a \equiv \partial_m\theta^a + g\epsilon_{abc}W_m^b\theta^c \tag{19}$$

The field strength is built from

$$[D_m, D_n] = -igF_{mn} \tag{20}$$

Exercice : show that

$$F_{mn} = \partial_m W_n - \partial_n W_m - ig[W_m, W_n] \ , \ F_{mn} \to F'_{mn} = UF_{mn}U^{-1}$$

For $SU(2)$ this implies (exercice :)

$$F^a_{mn} = \partial_m W^a_n - \partial_n W^a_m + g\epsilon_{abc} W^b_m W^c_n \tag{21}$$

The Yang-Mills lagrangian is

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{mn} F^{a,mn} = -\frac{1}{4}(\partial_m W^a_n - \partial_n W^a_m)^2$$
$$-\frac{g}{2}\epsilon_{abc}\partial_m W^a_n W^{b,m} W^{c,n} - \frac{g^2}{4}\epsilon_{abc}\epsilon_{ade} W^b_m W^c_n W^{d,m} W^{e,n}$$

• Non-abelian gauge bosons have self-interactions, unlike the photon ! Full Lagrangian describing interaction of Yang-Mills fields with charged fermions

$$\mathcal{L} = \bar{\Psi}(i\gamma^m D_m - M)\Psi - \frac{1}{4} F^a_{mn} F^{a,mn} \ . \tag{22}$$

Exercice : show that for an $SU(2)$ doublet

$$\bar{\Psi}(i\gamma^m D_m - M)\Psi = \bar{\Psi}^k[\delta_{kl}(i\gamma^m \partial_m - M) + \frac{g}{2}\gamma^m W_m^a (\tau_a)_{kl}]\Psi^l$$

Field eqs. are

$$(i\gamma^m D_m - M)\Psi = 0 \ ,$$
$$\partial^m F^a_{mn} + g\epsilon_{abc} A^{b,m} F^c_{mn} = -g\bar{\Psi}\gamma_n \frac{\tau_a}{2}\Psi \qquad (23)$$

on the r.h.s. is the $SU(2)$ fermionic current j^a_n $\nwarrow$

- Here $\partial^m j^a_m \neq 0$; a massive field propagator is

$$\Delta^{ab}_{mn}(k) \ = \ \delta^{ab}\frac{g_{mn} - \frac{k_m k_n}{M_A^2}}{k^2 - M_A^2} \ . \qquad (24)$$

- and the longitudinal polarization does contribute to amplitudes.

$\rightarrow$ UV properties of the massless and massive YM theories are different. The Yang-Mills boson masses should not be added by hand.

3. Spontaneous symmetry breaking.

Symmetries (Noether theorem) $\rightarrow$ conserved charges.

There are however two ways the symmetries are realized in nature :

i) Weyl-Wigner : vacuum state is invariant under the symmetry $\rightarrow$ symmetry manifest in the spectrum and interactions.

Ex: translations (momentum), rotations (angular momentum), $U(1)_{em}$ (electric charge)...

ii) Nambu-Goldstone : vacuum state not invariant under the symmetry $\rightarrow$ symmetry not manifest.

Ex : rotation (or parity) symmetry in ferromagnets, $SU(2)_{weak}$, $SU(2)_L \times SU(2)_R$ chiral symmetry of strong interactions.

Coleman : "the symmetry of the vacuum is the symmetry of the world".

Simplest example of the NG realization is the Ising model dimension d, N spins, of hamiltonian

$$H = -J \sum_{(i,j)} S_i S_j - B \sum_i S_i \ , \tag{25}$$

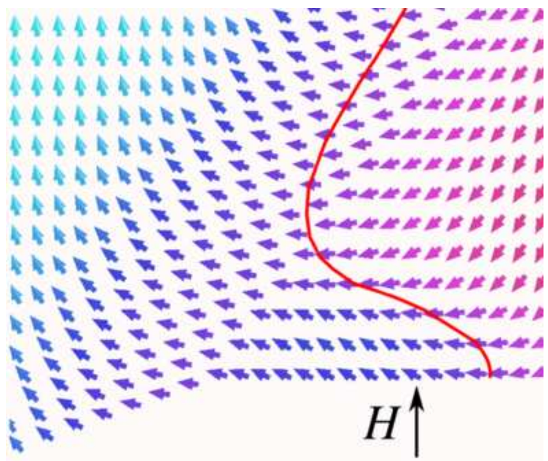
with $S_i = \pm 1$. For zero magnetic field $B = 0$ the system has a Z_2 symmetry $S_i \to -S_i$.

The magnetization

$$M = \lim_{B=0, N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \langle S_k \rangle$$

should therefore vanish. However

$$M = 0 \text{ for } T \geq T_c \ , \ M \neq 0 \text{ for } T < T_c, \text{ where } kT_c = 2dJ \tag{26}$$



3.1 The Goldstone theorem.

In a theory with continous symmetry, for every generator which does not annihilate the vacuum $\langle T^a \Phi \rangle \neq 0$ there is a massless, NG particle.

Ex: The $O(N)$ linear sigma model.

N scalar fields $\mathbf{\Phi} = (\Phi_1, \Phi_2, \cdots \Phi_N)$, with lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_m \mathbf{\Phi})^2 - V(\mathbf{\Phi}) \ , \ V(\mathbf{\Phi}) = -\frac{\mu^2}{2}\mathbf{\Phi}^2 + \frac{\lambda}{4}(\mathbf{\Phi}^2)^2 \quad (27)$$

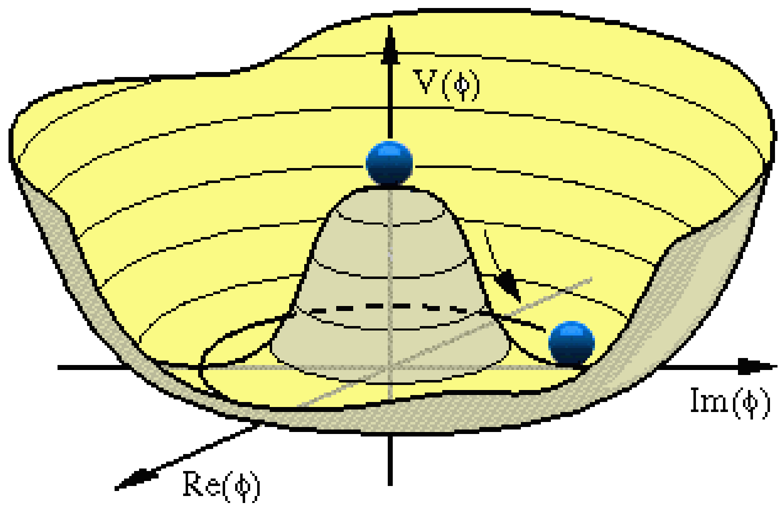
The model has a continous $O(N)$ symmetry acting as $\mathbf{\Phi} \to R\mathbf{\Phi}$, with R a rotation matrix. The potential is minimized for

$$\mathbf{\Phi_0}^2 = \frac{\mu^2}{\lambda} \equiv v^2 \quad (28)$$

The vacuum manifold is $O(N)$ invariant. By a rotation, the ground state can be chosen to be

$$\mathbf{\Phi_0} = (0, 0 \cdots v) \tag{29}$$

preserving an $O(N-1)$ subgroup. Goldstone's theorem: we expect $N-1$ massless particles, $O(N)/O(N-1)$.



In order to check this, we define a set of shifted fields:

$$\Phi(x) = (\pi^k(x), v + \sigma(x)) \ , \ k = 1 \cdots N-1 \ , \tag{30}$$

such that $\langle \pi^k \rangle = \langle \sigma \rangle = 0$. The lagrangian becomes

$$\begin{aligned} \mathcal{L} = \frac{1}{2}((\partial_m \pi)^2 + (\partial_m \sigma)^2) - \mu^2\sigma^2 - \sqrt{\lambda}\ \mu\sigma^3 \\ -\sqrt{\lambda}\ \mu\pi^2\sigma - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \end{aligned} \tag{31}$$

The manifest symmetry is indeed $O(N-1)$, rotating the π's. The physical masses are

$$m_\sigma^2 = 2\pi^2 \ , \ m_{\pi_k}^2 = 0 \tag{32}$$

The "pions" are massless, they are the NG bosons. $O(N-1)$ is realized a la WW, $O(N)$ is realized a la NG.

General (classical) proof of the Goldstone theorem. Consider

$$\mathcal{L} = \frac{1}{2}(\partial_m \Phi_i)^2 - V(\Phi_i) \tag{33}$$

and a global continuous symmetry

$$V(\Phi_i + \delta\Phi_i) = V(\Phi_i) \ , \ \text{with } \delta\Phi_i = i\theta^a T^a_{ij}\Phi_j \tag{34}$$

that implies

$$\frac{\partial V}{\partial \Phi_i} T^a_{ij}\Phi_j = 0 \ . \tag{35}$$

Differentiating again and taking the vev, we get

$$\langle \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_i} T^a_{ij} \Phi_j + \frac{\partial V}{\partial \Phi_i} T^a_{ik} \rangle = 0 \tag{36}$$

In the vacuum, $\mathcal{M}^2_{ki} = \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_i}$ is the scalar mass matrix, whereas $\langle \frac{\partial V}{\partial \Phi_i} \rangle = 0$. Then we get

$$\mathcal{M}^2_{ki} \, (T^a v)_i \; = \; 0 \tag{37}$$

If the vacuum is not invariant under the symmetry generator $T^a v \neq 0$, then $T^a v$ is an eigenvector of the mass matrix $\mathcal{M}^2$ corresponding to a zero eigenvalue

$\rightarrow$ the Goldstone theorem.

What happens if the symmetry is local (gauge) ?

3.2 The Higgs mechanism.

Consider an abelian gauge theory

$$\mathcal{L} = -\frac{1}{4}F_{mn}^2 + |D_m\Phi|^2 - V(\Phi) \ , \tag{38}$$

with $D_m = \partial_m + ieA_m$, $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$, and scalar potential

$$V = -\mu^2|\Phi|^2 + \lambda(|\Phi|^2)^2 = -\frac{\mu^2}{2}(\Phi_1^2+\Phi_2^2) + \frac{\lambda}{4}(\Phi_1^2+\Phi_2^2)^2 \ , \tag{39}$$

invariant under the local $U(1)$ transformations

$$\Phi \to e^{i\alpha(x)}\Phi \ , \ A_m \to A_m - \frac{1}{e}\partial_m\alpha \tag{40}$$

We expand around the vacuum state

$$\Phi_0 = \sqrt{\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}}\ ,\ \Phi(x) = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2) \qquad (41)$$

From the quadratic mass terms we find $m_1^2 = 2\mu^2$, $m_2 = 0$, so ϕ_2 is the Goldstone boson. New features appear from the kinetic term

$$|D_m\Phi|^2 = \frac{1}{2}(\partial_m\phi_i)^2 + evA_m\partial^m\phi_2 + \frac{e^2v^2}{2}A_m^2 + \cdots \quad (42)$$

$\rightarrow$ the gauge boson acquired a mass $M_A^2 = e^2v^2$. But this can only happen if

$$A_m(M_A = 0) + \phi_2 \rightarrow A_m(M_A \neq 0) \qquad (43)$$

This is indeed true and can be seen in various ways:

i) The quadratic term can be diagonalized

$$-\frac{1}{4}F_{mn}^2 + \frac{1}{2}(\partial_m\phi_2)^2 + \sqrt{2}evA_m\partial^m\phi_2 + \frac{e^2v^2}{2}A_m^2$$
$$= -\frac{1}{4}(\partial_m B_n - \partial_n B_m)^2 + \frac{e^2v^2}{2}B_m^2 \;, \tag{44}$$

where $B_m = A_m + \frac{1}{ev}\partial_m\phi_2$. ϕ_2 disappeared from the quadratic part, and is absorbed into the longitudinal component of the gauge field.

ii) The Goldstone can be eliminated altogether in the unitary gauge

$$\Phi(x) = \frac{1}{\sqrt{2}}\, e^{\frac{i\theta(x)}{v}}\, (v + \rho(x)) \tag{45}$$

by the trans. $\Phi \to \Phi' = e^{-\frac{i\theta}{v}}\Phi$, $A_m \to A'_m = A_m + \frac{1}{ev}\partial_m\theta$.

In the unitary gauge, the lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F'_{mn})^2 + (\partial_m - ieA'_m)\Phi'(\partial^m + ieA'^m)\Phi' - \mu^2\Phi'^2 - \lambda\Phi'^4$$

Higgs mechanism, non-abelian case

Consider a gauge group G of rank r and scalar fields in some irreducible n-dim. representation

$$\mathcal{L} = -\frac{1}{4}F^a_{mn}F^{a,mn} + |[(\partial_m - igT^a A^a_m)\Phi]|^2 - V(\Phi) \quad (46)$$

with V the scalar potential minimized for $\langle\Phi\rangle = v$, and $H \in G$ the subgroup of rank s leaving v invariant

$$T^a v = 0 \quad , \quad a = 1 \cdots s$$
$$T^a v \neq 0 \quad , \quad a = s+1 \cdots r \quad (47)$$

Unitary gauge parametrization ($\swarrow$ Goldstone's)

$$\Phi(x) = e^{i \sum_{a=s+1}^{r} T_a \frac{\xi_a(x)}{v}} \; \frac{\rho(x) + v}{\sqrt{2}} \; , \tag{48}$$

where $\langle \xi_a \rangle = \langle \rho \rangle = 0$. The gauge trans.

$$\Phi(x) \to \Phi'(x) = U\Phi \; , \text{ with } U = e^{-i \sum_{a=s+1}^{r} T_a \frac{\xi_a(x)}{v}}$$

$$A_m \to A'_m = U \; (A_m + \frac{i}{g}\partial_m) \; U^{-1} \tag{49}$$

eliminates the Goldstone's from the lagrangian. The resulting mass matrix of the vector fields is then

$$M^2_{ab} = g^2 (T_a v)^\dagger (T_b v) \; ; \tag{50}$$

$r - s$ gauge bosons become massive.

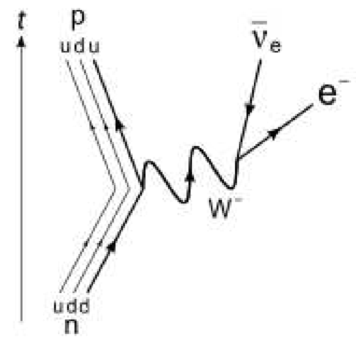
$$A_m^a + \xi_a \;\rightarrow\; A'^a_m = A_m^a - \frac{1}{v} D_m \xi^a + \cdots \tag{51}$$

massless $\nwarrow$ $\qquad\qquad$ $\nwarrow$ massive

4. The electroweak sector of the Standard Model.

4.1. Gauge group and matter content.

Standard model = "unified" description of weak and electromagnetic interactions. From the Fermi theory of weak interactions



with $G_F/\sqrt{2} = g^2/8M_w^2$, we know that we need at least a charged gauge boson $W_m^{\pm}$ and the photon A_m.