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Basics of QCD for the LHC : exercises

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Abstract

Some tests, exercises and web applications are proposed to enhance the comprehension of basic aspects of perturbative QCD in collider physics. Material is organized as follows:

1. QCD fundamentals
2. QCD in the final state:e^+e^- collisions
3. QCD in the initial state: evolution and DIS
4. QCD everywhere: hadron collisions

1 Introduction

This is a collection of tests, exercises, web applications and simulations useful for a first approach to understanding QCD and its role in collider physics. The basic references where most of the exercises and examples were taken from are:

- "QCD and Collider Physics" by R.K. Ellis, J.W. Stirling, B.R. Webber (Cambridge Monographs, 1996). In addition some of the exercises reproposed here are from lectures given at CERN by B.R. Webber.
- "Introduction to QCD", by Michelangelo L. Mangano, `http://cern.ch/~mlm/talks/cern98.ps.gz`.
- "Introduction to perturbative QCD", by Paolo Nason, Lectures in European School of High-Energy Physics, 1997. `http://castore.mib.infn.it/~nason/misc/QCD-intro.ps.gz`.

Tests are collections of very easy questions on the content of the lectures, which sometimes imply a very short calculation.

The simulations can be performed with the help of the `MadGraph/MadEvent` MonteCarlo, directly from the web or by downloading the main code at `http://madgraph.phys.ucl.ac.be`. In some sense they are meant to be the most active part (and therefore enjoyable) of the tutorials. Some little practice on how to use the web interface to the code is needed.

To generate events on the clusters higher clearance has to be obtained (by sending a request by e-mail to one of the authors).

2 QCD fundamentals

2.1 Test

1. List what are the main motivations, both theoretical and experimental, for us to believe that QCD is the right theory of strong interactions.

2. Write down the quark content of the mesons beloging to the spin-0 8-multiplet of $SU(3)_f$

3. Look at the plot of R versus $\sqrt{s}$. Why is R proportional to the color? What are the red spikes in the plots? Why are they there? Why are they before each step?

4. Explain why scaling is so revealing about the nature of strong interactions at high energy. Derive the expressions for x, y, ν, W^2 in terms of the four momenta and explain their meaning in the lab frame.

5. What does it mean that a QFT is renormalizable? What is the consequence of renormazability for QCD? Can QCD be a fundamental theory? What about QED?

6. Show that the QCD Lagrangian for u, d is isospin invariant either if $m_u = m_d$ or if $m_u, m_d \to 0$.

7. Look at the plot of the cross sections at hadron colliders. What is the cross section for producing a Higgs of 120 GeV of mass? Suppose the Higgs decays into $b\bar{b}$ with branching ratio (=probability) one. What is the ratio signal over backround expected for such a channel at LHC?

8. True or False: In QCD as in QED, gauge invariance implies that it is enough to contract the an external gluon index with its four momentum, regardless of the other gluons, to get zero, i.e.,
$$k_1^\mu M^{\mu,\rho,\cdots} = 0\,. \tag{1}$$
Explain.

9. Explain why in the NLO calculation of $\sigma(e^+e^- \to \text{hadrons})$ there is no need to renormalize the strong coupling (i.e., the UV divergences cancel).

10. Λ is a physical parameter. True or False? Explain.

11. For a leading order calculation α_S at 1-loop should be used. For a NLO α_S at 2-loops, and so on. True or False? Explain.

12. Which one(s) are true? If α_S at 2-loops is used instead of the α_S at 1-loop in a leading order calculation, the result

    - is wrong.
    - does not change.
    - changes but it is not improved.

- is improved.

13. How are the unknown higher order corrections usually estimated in QCD? Why?

2.2 Exercises

2.2.1 Four-gluon vertex

By requiring that the amplitude $q\bar{q} \to gg$ is gauge invariant, we have shown that a three-gluon vertex is necessary and that it can be found by only using arguments of symmetry and dimensional analysis.

(a) Using similar arguments show that the $gg \to gg$ scattering amplitude is not gauge invariant and a four-gluon vertex is needed and build it.

(b) Show that the role of a four gluon-vertex is equivalent to the introduction of an antisymmetric, not propagating, color-octet, tensor particle $B^{\mu\nu}$, which interacts with a gluon through a vertex of the form

$$V_{Bgg} = g f^{abc}/\sqrt{2}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \ , \tag{2}$$

and has a trivial "propagator":

$$\Delta_{ab}^{\mu\nu,\rho\sigma} = -i g^{\mu\rho} g^{\nu\sigma} \delta^{ab} \ . \tag{3}$$

2.2.2 Ghost contribution

(a) Show that in $q\bar{q} \to gg$ the sum over non-physical polarization is non zero (Hint: calculate the sum over all polarization $\sum \epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu}$ and the sum over the physical states and take the difference):

$$\sum_{\text{non-physical}} |\epsilon_1^\mu \epsilon_2^\nu M_{\mu\nu}|^2 = \left| i g^2 f^{abc} t^c \frac{1}{2k_1 \cdot k_2} \bar{v}(\bar{q}) \hat{k}_1 u(q) \right|^2 \ . \tag{4}$$

(b) Show that the diagram with the ghosts exactly cancels the above contribution.

2.2.3 Color

Using the 't Hooft double line formalism, whose rules are summarized in Fig. 1 calculate the color factors of the following diagrams shown in Fig. 2.

In particular, compare the last color factor with that obtained from the diagram where the external gluon is replaced by a photon. In the former case the quark pair is in a color-octet state, while in the latter in a color-singlet state. Is the sign of the interaction between them the same?

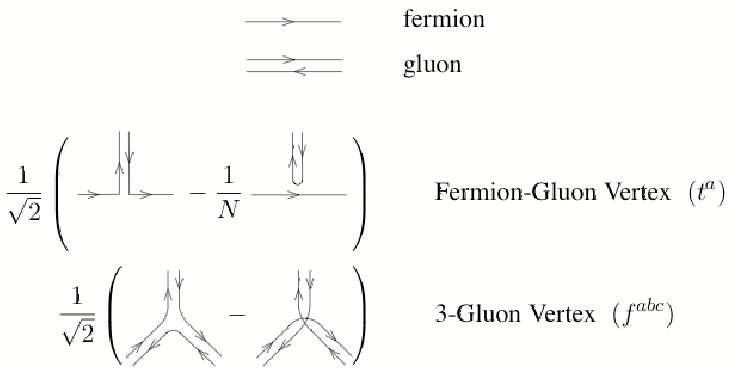

Figure 1: Double line Feynman rules useful to make fast and easy the evaluation of color factors.

2.2.4 Renormalization schemes

The QCD scale parameter Λ is defined by

$$\log \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)}\,, \tag{5}$$

where the β-function is

$$\beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} = -b\alpha_S^2 \left[1 + b'\alpha_S + b''\alpha_S^2 + \mathcal{O}(\alpha_S^3)\right]. \tag{6}$$

(a) Consider the two renormalization schemes A and B, where the couplings are related by

$$\alpha_S^B = \alpha_S^A \left[1 + c_1\alpha_S^A + c_2(\alpha_S^A)^2 + \mathcal{O}((\alpha_S^A)^3)\right]. \tag{7}$$

Show that the first two β-function coefficients b and b' are scheme-independent, whereas the third is related in the two schemes by

$$b''_B = b''_A + c_2 - b'c_1 - c_1^2\,. \tag{8}$$

(b) Show the scale parameters of the two schemes are related by

$$\Lambda_B = \Lambda_A \exp\left(\frac{c_1}{2b}\right). \tag{9}$$

Hint: Combine the two formulas for $\Lambda_{A,B}$ and calculate

$$\log \frac{\Lambda_B}{\Lambda_A} = \frac{1}{2}\int_{\alpha_A(Q)}^{\alpha_B(Q)} \tag{10}$$

and then take the limit $Q \to \infty$, $\alpha_{A,B} \to 0$.

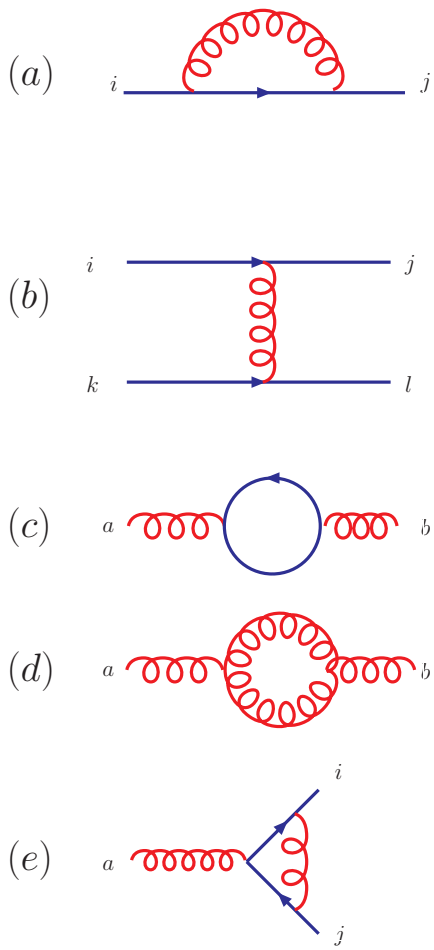

Figure 2: Sample of QCD diagrams.

(c) The QCD effective charge is

$$\alpha_S = \alpha_0 - g_0^4 b \mathcal{I}_d + \dots \tag{11}$$

where g_0 is the bare charge and $\mathcal{I}_d$ is the dimensionless integral

$$\mathcal{I}_d = \frac{\mu^{4-d}}{(2\pi)^d} \int \frac{d^d k}{(k^2+m^2)^2} \tag{12}$$

$$= \frac{\mu^{4-d}}{(2\pi)^d} \Gamma_d \int_0^\infty k^{d-1} \frac{dk}{(k^2+m^2)^2} \tag{13}$$

where $\Gamma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the the $d-1$ dimensional surface of a d-dimensional unit hypersphere. Show that for $d = 4 - 2\epsilon$

$$\mathcal{I}_d = \frac{1}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log(4\pi) - \gamma_E + \log\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon) \right], \tag{14}$$

and find the relation between the $\Lambda_{MS}$and $\Lambda_{\overline{MS}}$ where in the MS-scheme we absorb only the $1/\epsilon$ into the renormalized charge, whereas in the $\overline{MS}$-scheme we also absorb the $\log(4\pi) - \gamma_E$. The following formulas are useful:

$$\int_0^1 u^{a-1}(1-u)^{b-1} = \beta(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)\,, \tag{15}$$

$$\Gamma(a+1) = a\Gamma(a)\,, \quad \Gamma(1) = 1\,, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}\,, \quad \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,. \tag{16}$$

2.3 Web

2.3.1 α_S extractions

Browse the Particle Data Group (PDG) site and find the plot that summarizes the determination of $\alpha_S(M_Z)$ from different experiments. Which measurement is the best one? At which scale has it been performed? Which extraction has the central value mostly off? How is that measuremement performed? Divide (when possible) the measurements at low energy from those at high energy. Is there a systematic behaviour? Find the plot which shows the evolution of α_S with the scale and understand which data go in it.

2.3.2 Try MadGraph out

Logon to the MadGraph web site and register. Familiarize with the code by generating a few processes in QED and QCD trying to guess which diagrams appear. What is the minimum number of jets have to be asked for in e^+e^- collisions so that the triple gauge vertex appear? Calculate the cross section for $u\bar{u} \to \gamma\gamma, u\bar{u} \to gg$, fixing the c.m.s energy at 100 GeV and leaving the acceptance cuts as in the default. Which one of the two processes gives as a larger cross section, taking in to account the difference in the couplings,i.e., setting $\alpha_S = \alpha_{EM}$?

3 e^+e^- collisions

3.1 Test

1. How should we intepret the leading order calculation for $\sigma(e^+e^- \to q\bar{q})$?
2. List the properties that a function of the four momenta has to enjoy to be an "infrared-safe" quantity.
3. Explain what is the physics leading to the idea of factorization.
4. Derive the soft Feynman rules for a qqg and ggg.
5. Show by explicit calculation that the form of the virtual corrections

$$\frac{d^2\sigma^{\text{VIRT}}}{dEd\cos\theta} = -\sigma^{LO} C_F \frac{\alpha_S}{\pi} \int_0^{\sqrt{s/2}} \frac{dE'}{E'} \int_{-1}^{1} \frac{d\cos\theta'}{1-\cos^2\theta'} 2\delta(E')[\delta(1-\cos\theta')+\delta(1+\cos\theta')]+\ldots \tag{17}$$

   cancels the soft and collinear divergences present in $\sigma(e^+e^- \to q\bar{q}g)$.
6. Present the physical explanation of the angular ordering.
7. Explain the Chudakov effect by angular ordering.
8. Explain preconfiment.
9. Derive the amplitude for soft gluon emission from a $q\bar{q}g$ final state.
10. Show by explicit calculation in the example above that interference from different color flows is suppressed $1/N_c^2$. Generalize to any color flow.
11. Write down the definition of a two-jet cross section (Sterman-Weinberg) and see in which configurations $e^+e^- \to q\bar{q}gg$ would contribute to it.

3.2 Exercises

3.2.1 $e^+e^- \to q\bar{q}$

(a) Derive the expression for the amplitude squared $e^+e^- \to q\bar{q}$,in terms of the invariants, s, t, u, for massless quarks. Include only photon exchange. Express it in terms of the c.m.s. variables $\cos\theta, \phi$ and write the differential cross section:

$$\frac{d\sigma}{d\cos\theta} = N_c(\sum_f Q_f^2)\frac{\pi\alpha^2}{2s}(1+\cos\theta^2)\,. \tag{18}$$

Which quarks should be included in the sum over flavors f?

(b) Include the diagram where a Z is exchanged and recall that the interaction vertex $q\bar{q}Z$ is given by:

$$\frac{-ig_w}{2\sqrt{2}}\gamma_\mu(V_f - A_f\gamma_5)\,, \tag{19}$$

and the axial and vector couplings of the fermions to the Z are

$$V_f = T_f^3 - 2Q_f \sin^2\theta_W\,, \qquad A_f = T_f^3\,, \tag{20}$$

with $T_f^3 = 1/2$for $f = \nu, u, \ldots$ and $T_f^3 = -1/2$ for $f = e, d, \ldots$.
What happens to the $\cos\theta$ distribution?

3.2.2 $e^+e^- \to q\bar{q}g$

(a) Show that the phase space for the unpolarized decay into three massless objects can be written as:

$$d\Phi_3 = \frac{1}{(2\pi)^5}\frac{s}{32}dx_1dx_2d\alpha d(\cos\beta)d\gamma \tag{21}$$

where s is the c.m.s. energy and $x_i = 2E_i/\sqrt{s}$ are the fractional energies for the quark and anti-quark.

(b) Calculate the matrix element squared for $e^+e^- \to q\bar{q}g$. Use the fact that we interested only in azimuthal averaged quantities and therefore we neglect angular correlations betweeen the initial state plane and the final state one, so we can write

$$|M|^2 = \frac{1}{s^2}L^{\mu\nu}H_{\mu\nu} \to \frac{1}{s^2}(L^{\mu\nu}g_{\mu\nu})(H^{\rho\sigma}g_{\rho\sigma}) \tag{22}$$

where $L^{\mu\nu}$ and $H^{\mu\nu}$ are the leptonic and hadronic tensors that come from the squaring of the the corresponding currents. The result to be be found is:

$$\sigma^{q\bar{q}g} = \sigma^{LO}C_F\frac{\alpha_S}{2\pi}\int dx_1dx_2\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} \tag{23}$$

where $\sigma^{LO} = N_c(\sum_f Q_f^2)4\pi\alpha^2/(3s)$.

(c) Perform the same calculation for a scalar gluon and verify that

$$\sigma^{q\bar{q}s} = \sigma^{LO}\int dx_1dx_2\frac{x_3^2}{2(1-x_1)(1-x_2)}\,. \tag{24}$$

3.2.3 Thrust distribution

The thrust is defined as:

$$T = \max_{\mathbf{n}}\frac{\sum_i|\mathbf{p}_i\cdot\mathbf{n}|}{\sum_i|\mathbf{p}_i|} \tag{25}$$

and in the case of $e^+e^- \to q\bar{q}g$ process it corresponds to the $\max\{x_i\}$, where the $x_i = 2E_i/\sqrt{s}$ are the energy fractions of each parton.

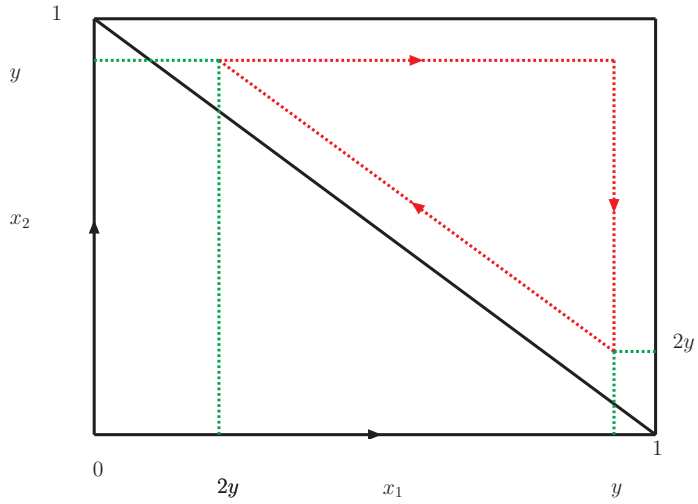

Figure 3: Countour in the (x_1, x_2) phase space plane corresponding to the JADE measure for jet definition. The differential cross section has to be integrated on the countour to obtain the thrust T. $y = T$.

(a) Calculate the thrust distribution for a vector gluon:

$$\frac{d\sigma}{dT} = \int dx_1 dx_2 \frac{d\sigma}{dx_1 dx_2} \delta(T - \max\{x_i\}) \,. \tag{26}$$

Convince yourself that the above result is obtained by integrating the differential cross section over the (JADE) countour in the (x_1, x_2) plane shown by the dashed line in Fig. 3. Compare your result with:

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \log\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{1-T} \right] . \tag{27}$$

(b) Calculate the thrust distribution for a scalar gluon and compare your result with:

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi} \left[\frac{9(2-T)T - 8}{2(1-T)} + \log\left(\frac{2T-1}{1-T}\right) \right] . \tag{28}$$

(c) Plot the two distributions for $2/3 < T < 1$ in a log scale and compare with the data of Fig. 4. Why the QCD prediction at order $\mathcal{O}(\alpha_S)$ start to differ from the data when T approaches 1? What about at 2/3?

3.2.4 $e^+e^- \to Q\bar{Q}g$

(a) Compute the differential cross section for the case of massive final state using a program for symbolic calculations (such as `FORM` or `Mathematica+FeynCalc`) and compare your

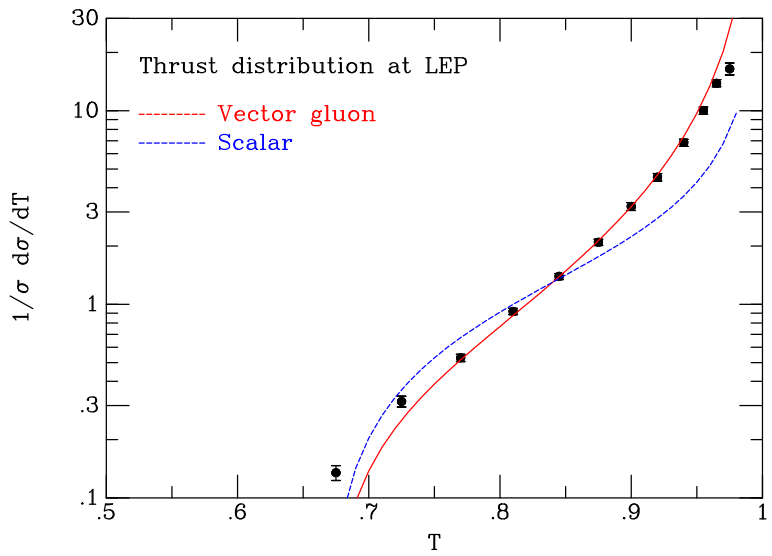

Figure 4: The thrust distribution measured at LEP, showing data from the DELPHI collaboration.

result with

$$\frac{1}{\sigma^{LO}}\frac{d^2\sigma}{dx_1dx_2} = \frac{1}{\beta}C_F\frac{\alpha_S}{2\pi}\left[\frac{2(x_1+x_2-1-\rho/2)}{(1-x_1)(1-x_2)} - \frac{\rho}{2}\left(\frac{1}{(1-x_1)^2}+\frac{1}{(1-x_2)^2}\right) + \frac{1}{1+\rho/2}\frac{(1-x_1)^2+(1-x_2)^2}{(1-x_1)(1-x_2)}\right], \tag{29}$$

where

$$\rho = \frac{4m^2}{s} \leq 1\,, \qquad \beta = \sqrt{1-\rho} \tag{30}$$

and σ^{LO} is defined as in Eq. (23).

(b) Verify that the massless limit corresponds to Eq. (23). Study the soft and collinear limits. Is the collinear divergence still there? Write the soft and collinear approximation of the amplitude in the case the gluon is close to the quark:

$$\frac{1}{\sigma^{LO}}\frac{d^2\sigma}{dzd\theta^2} = C_F\frac{\alpha_S}{\pi}\frac{1}{z}\frac{\theta^2}{(\theta^2+\rho)^2} \tag{31}$$

where $z = 2E_g/\sqrt{s}$ is the energy fraction of the gluon and θ the angle between the gluon and the quark. Plot the behaviour of the matrix element in the massless and massive cases and compare with Fig. 5. Explain this behaviour in terms of angular momentum conservation.

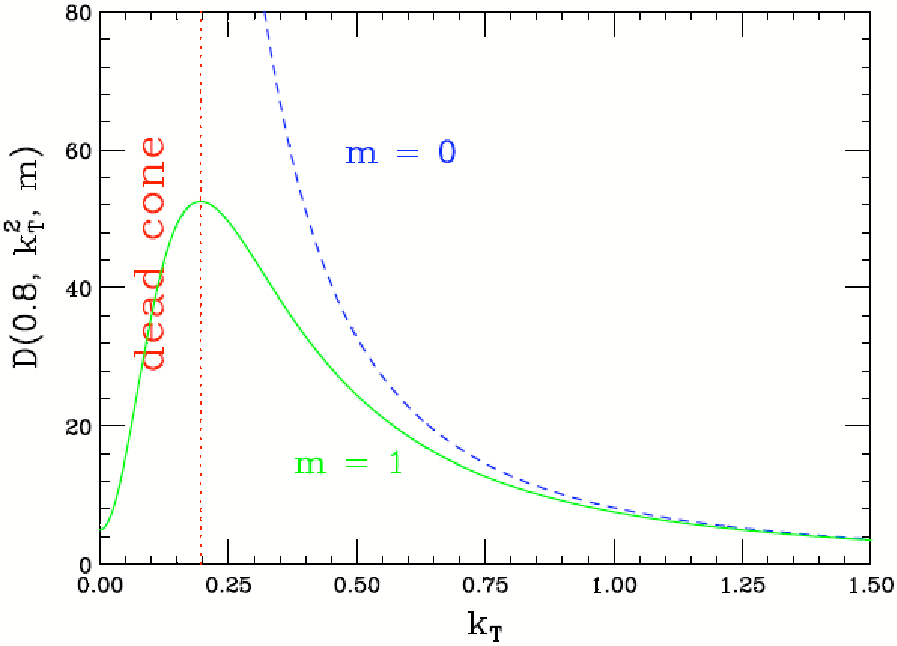

Figure 5: Dead cone: emission of collinear (soft) gluons from a massive quark is suppressed by angular momentum conservation.

3.2.5 Jet rates in the soft limit$^{(*)}$

(a) Derive the expression of the differential cross section for $e^+e^- \to q\bar{q}g$ in the soft limit, in the terms of the gluon energy E and the cosine of angle between the gluon and the quark (or anti-quark) $\cos\theta$:

$$\frac{d^2\sigma^{\text{REAL}}}{dE d\cos\theta} = \sigma^{LO} C_F \frac{2\alpha_S}{\pi} \frac{1}{E} \frac{1}{1-\cos^2\theta} \,. \tag{32}$$

(b) Without calculating the virtual contributions, guess their final form in order to cancel the soft and collinear divergences:

$$\frac{d^2\sigma^{\text{VIRT}}}{dE d\cos\theta} = -\sigma^{LO} C_F \frac{2\alpha_S}{\pi} \int_0^{\sqrt{s/2}} \frac{dE'}{E'} \int_{-1}^{1} \frac{d\cos\theta'}{1-\cos^2\theta'} \frac{1}{2} \delta(E')[\delta(1-\cos\theta')+\delta(1+\cos\theta')]+\ldots \tag{33}$$

(c) Define the two and three jet rates using the JADE measure, $y = M^2/s$ and calculate the two- and three-jet rates up to order α_S. First, identify the regions of the phase space contributing to the two jet rates

$$\begin{aligned} &\text{Region I}: E < y\sqrt{s} \qquad \text{and} \qquad 0 < \cos\theta < 1\,, \\ &\text{Region II}: E > y\sqrt{s} \qquad \text{and} \qquad 1 - \frac{y\sqrt{s}}{E} < \cos\theta < 1\,, \end{aligned} \tag{34}$$

and then perfom the integration:

$$\frac{\sigma_{2-\text{jet}}}{\sigma^{LO}} = \frac{1}{\sigma^{LO}} \left[2\int_{R_1} d\sigma^{\text{REAL}} + 2\int_{R_2} d\sigma^{\text{REAL}} + \int d\sigma^{\text{VIRTUAL}} \right] \tag{35}$$

Compare your result with:

$$\sigma_{2-\text{jet}} = \sigma^{LO}\left[1 - C_F\frac{\alpha_S}{\pi}\log^2 y + \ldots\right] \tag{36}$$

$$\sigma_{3-\text{jet}} = \sigma^{LO} C_F\frac{\alpha_S}{\pi}\log^2 y + \ldots \tag{37}$$

and, ignoring self-gluon interaction, exponentiate the above result to find the $\sigma_{(\text{n}+2)-\text{jet}}$ rate.

(d) Estimate the average number of the jets, $\langle n_{\text{jet}}\rangle$ and how the average number of particles in the final states (identify each particle with a jet at small y) scales with the c.m.s energy.

(e) Estimate the average invariant mass of the jets as a function of the c.m.s energy.

(f) Estimate the average thrust.

3.3 Web and MC Simulations

3.3.1 e^+e^- cross section

Surf onto the Particle Data Group web-site and find the plot of R, the ratio of the hadronic cross section over that of $\mu^+\mu^-$ in e^+e^- collisions.
(Hint: `http://pdg.lbl.gov/2006/hadronic-xsections/hadron.html`). Compare your leading order calculation for $e^+e^- \to q\bar{q}$ to the data. Is this a strong evidence that the number of colors is three? Evaluate the steps in R due to the opening of $c\bar{c}$ and $b\bar{b}$ channels and compare with the experimental data.

3.3.2 Thrust distributions

Use `MadGraph/MadEvent` to obtain the thrust distributions in $e^+e^- \to 3j$ for a vector and a scalar gluon and compare your results with the analytic ones of Fig. 4. For details on the thrust definitions see Ex. 3.2.3.

3.3.3 $e^+e^- \to Q\bar{Q}g$

Use `MadGraph/MadEvent` and verify that there are no collinear divergences to be regulated and the cross section is finite with just a minimum cut on the energy of the gluon. Plot the behaviour of the cross sections as a function of the quark mass and verify that it has a logarithmic behaviour.

3.3.4 The BZ angle in $e^+e^- \to 4$jets: abelian vs non-abelian

Use `MadGraph/MadEvent` to produce two event samples, one for standard QCD and one with an abelian QCD model for

$$e^+e^- \to Z \to 4j\,. \tag{38}$$

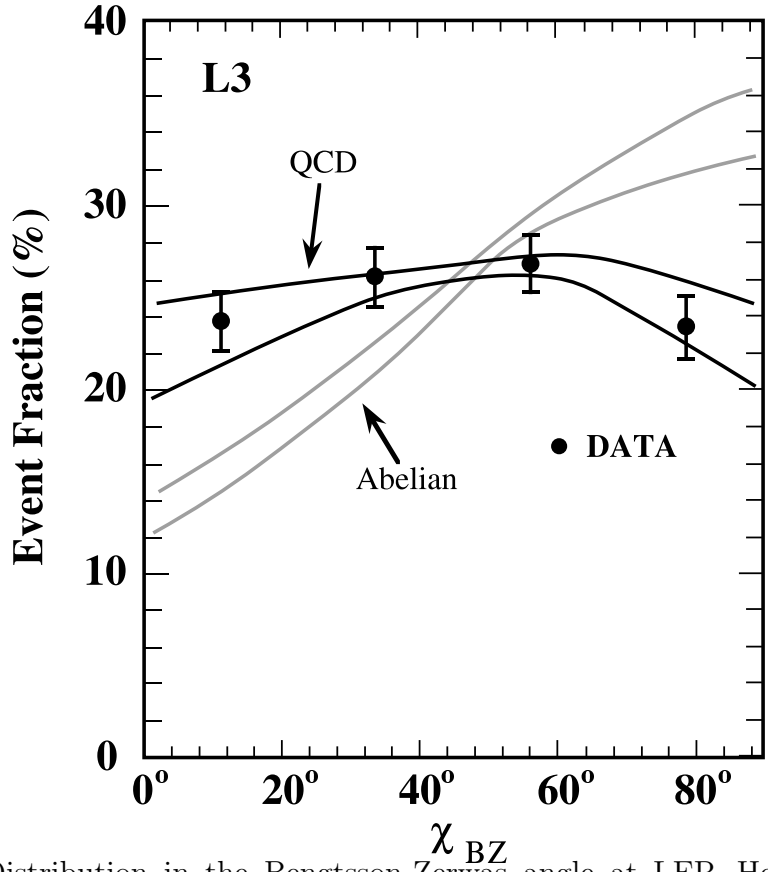

Figure 6: Distribution in the Bengtsson-Zerwas angle at LEP. Here the Abelian model includes only four-quark final states.

Run the collision on the peak of the Z and set a minimum invariant mass for the jets of $m_{jj} > 10$ GeV. Plot the angle between the planes identified by the two lowest and to highest energy jets:

$$\cos\chi_{BZ} = \frac{(\mathbf{p}_1 \times \mathbf{p}_2)\cdot(\mathbf{p}_3 \times \mathbf{p}_4)}{|\mathbf{p}_1 \times \mathbf{p}_2||\mathbf{p}_3 \times \mathbf{p}_4|}\,. \tag{39}$$

Comparison should be made with the plots of Ref. `http://arxiv.org/abs/hep-ph/9503354`, where various implementation of the abelian models are discussed. Our web implementation include the emission of abelian gluons as possible partons leading to jets.

4 Evolution and DIS

4.1 Test

1. Derive the formula for $d^2\sigma/dxdQ^2$ in terms of the structure functions.
2. Derive how a Lorentz transformation acts on the p^+ and p^- component of a four vector in the light-cone coordinates.
3. What is the purpose of the Breit frame? Can you explain how a DIS event looks like in this frame?
4. What has aymptotic freedom to do with the parton model?
5. What is the physical meaning of the Callan-Gross relation? Why? (You might want to calculate the scattering amplitude $eq \to eq$ for scalar quarks).
6. Scaling is indeed violated. How and by what?
7. In the NLO calculation for $\gamma^* q \to q$ all divergences cancel, except for ...? What is the nature of these left-over divergences? Are these divergences universal?
8. Take the explicit form of the splitting functions given below and ignore $\delta(1-x)$ terms and the $()_+$ distributions. Set the color factors C_F, T_R, C_A to one. Prove they satisfy the SUSY relation
$$p_{gq} + p_{qq} = p_{qg} + p_{gg} \tag{40}$$
9. Explain the idea of factorization. How do we exactly get rid of the large logs? What is the role of universality?
10. What is the strategy to be followed to make a prediction in QCD? What is the part to be calculated by theorists and the one measured by experimentalists?

4.2 Exercises

4.2.1 Splitting functions

Calculate the splitting functions for $q \to qg$ and $g \to q\bar{q}$. (To be completed...)

4.2.2 DGLAP resums towers of logs

Show that the DGLAP equations resum a full tower of logarithms of Q^2.

4.2.3 Evolution

As discussed in the lecture the parton distributions do not scale as in the naïve parton model but rather are expected to exhibit the scaling violations predicted by QCD. The structure of the evolution is determined by the DGLAP equation, whose basic ingredients are the splitting functions.

(a) The plus-prescription is defined by

$$\int_0^1 dx f(x) g_+(x) \equiv \int_0^1 dx [f(x) - f(1)] g(x) \,. \tag{41}$$

Show that

$$\left(\frac{1+z^2}{1-z}\right)_+ = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \tag{42}$$

and

$$\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ = \frac{z}{(1-z)_+} + \frac{1}{2}z(1-z) + \frac{11}{12}\delta(1-z) \,. \tag{43}$$

(b) The splitting functions are

$$\begin{aligned}
P_{qq}(z) &= C_F \left(\frac{1+z^2}{1-z}\right)_+ && (44)\\
P_{qg}(z) &= T_R \left(z^2 + (1-z)^2\right) && (45)\\
P_{gq}(z) &= C_F \frac{1+(1-z)^2}{z} && (46)\\
P_{gg}(z) &= 2C_A \left[\left(\frac{z}{1-z} + \frac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z} + \frac{1}{2}z(1-z)\right] - \frac{2}{3}n_f T_R \delta(1-z) && (47)
\end{aligned}$$

The anomalous dimensions are given by the moments of the splitting functions,

$$\begin{aligned}
\gamma_{ij}(N, \alpha_S) &= \sum_{n=0}^{\infty} \gamma_{ij}^{(n)}(N) \left(\frac{\alpha_S}{2\pi}\right)^{n+1} \,, && (48)\\
\gamma_{ij}^{(0)}(N) &= \int_0^1 dz z^{N-1} P_{ij}(z) \,. && (49)
\end{aligned}$$

Show that

$$\begin{aligned}
\gamma_{qq}^{(0)}(N) &= C_F \left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N} \frac{1}{k}\right] && (50)\\
\gamma_{qg}^{(0)}(N) &= T_R \left[\frac{2+N+N^2}{N(N+1)(N+2)}\right] && (51)\\
\gamma_{gq}^{(0)}(N) &= C_F \left[\frac{2+N+N^2}{N(N+1)(N-1)}\right] && (52)\\
\gamma_{gg}^{(0)}(N) &= 2C_A \left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - 2\sum_{k=2}^{N} \frac{1}{k}\right] - \frac{2}{3} n_f T_R \,. && (53)
\end{aligned}$$

(c) Now consider the evolution of the singlet quark distribution

$$\Sigma(x) = \sum_i q_i(x) + \bar{q}_i(x) \tag{54}$$

which mixes with the gluon distribution via the evolution equations. In terms of moments with evolution variable $t = \log(Q^2/\Lambda^2)$ we have

$$\frac{d}{dt}\Sigma(N) = \frac{\alpha_S(t)}{2\pi}\left[\gamma_{qq}(N)\Sigma(N) + 2n_f\gamma_{qg}(N)g(N)\right] \tag{55}$$

$$\frac{d}{dt}g(N) = \frac{\alpha_S(t)}{2\pi}\left[\gamma_{gq}(N)\Sigma(N) + \gamma_{gg}(N)g(N)\right] \tag{56}$$

Verify that for $N = 2$ there are two eigenvalues to the above evolution equation and the corresponding anomalous dimensions are $\lambda_\pm = 0, -(16/9 + n_f/3)$ and find the corresponding eigenfunctions.

(d) Use the above result to find the momentum fractions carried by the quarks and gluons at truly asymptotic values of Q^2

$$\Sigma^{(2)} = \frac{1}{1 + 4\frac{C_F}{n_f}} \tag{57}$$

$$f_g^{(2)} = \frac{4C_F}{1 + 4\frac{C_F}{n_f}} \tag{58}$$

4.3 Gluon at small x

The evolution of the pdf's tends to build up the gluon distribution at small x, which will be important at the LHC. In the limit of small x and very large Q^2 the DGLAP equations are dominated by the small argument behaviour of the splitting functions P_{gg}.

(a) Verify that in this limit the gluon distribution $G(x,t) = xg(x,t)$ satifies

$$\frac{dG(x,t)}{dt} \simeq \frac{3\alpha_S(t)}{\pi}\int_x^1 \frac{dy}{y}G(y,t)\,. \tag{59}$$

(b) Now use the 1-loop form for α_S and change variables to $\tau = \log t$ and $\xi = 24/b_0 \log(1/x)$ to show that the approximate equation to solve is

$$\frac{d^2G}{d\xi^2} \simeq \frac{1}{2}G\,. \tag{60}$$

(c) Verify that at truly large values of both ξ and τ a solution is

$$G(\xi,\tau) \sim e^{\sqrt{2\xi\tau}}\,. \tag{61}$$

or

$$g(x,t) \sim \frac{1}{x}\exp\sqrt{\frac{48}{b_0}\log\left(\frac{t}{t_0}\right)\left(\frac{1}{x}\right)} \times xg(x,t_0)\,. \tag{62}$$

(d) Use the following (fictious) form of the gluon distribution

$$g(x, Q_0 = 5\,\text{GeV}) = \frac{420}{99}\frac{(1-x)^7}{x} \tag{63}$$

to study the enhancement for $Q = 100$ GeV at $x = 0.01$ ($\Lambda = 0.1$ GeV).

4.3.1 Q^2_{min} in the EPA$^{(*)}$

In the Equivalent Photon Approximation (EPA), the process $ep \to e + X$ is approximated by the collinear emission of an almost-on-shell photon which then scatters with the proton, $\gamma^* p \to X$,

$$d\sigma_{ep} = \sigma_{\gamma p} f_\gamma^{(e)}(y) dy \,, \tag{64}$$

Show that the photon distribution in the electron is given by

$$f_\gamma^{(e)}(y) = \frac{\alpha}{2\pi}\left[\frac{1+(1-y)^2}{y}\log\frac{q^2_{min}}{q^2_{max}} + 2m_e^2 y\left(\frac{1}{q^2_{max}} - \frac{1}{q^2_{max}}\right)\right] \tag{65}$$

where

$$q^2_{max} = -\frac{m_e^2 y^2}{1-y} \tag{66}$$

$$q^2_{min} = q^2_{max} - E^2(1-y)\theta_c \tag{67}$$

with θ_c the maximum allowed value for θ, which depends on the geometry of the detector.

4.3.2 Soft cones$^{(*)}$

A soft function for a an emitter quark i, a soft gluon k, and a spectator anti-quark j is defined as

$$W_{(i)} \equiv \frac{1}{2}\left[\frac{\cos\theta_{jk} - \cos\theta_{ij}}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} + \frac{1}{1-\cos\theta_{jk}}\right] . \tag{68}$$

Prove that by averaging over the azimuthal angle, one obtains a positive definite quantity with the following properties:

$$\int \frac{d\phi_{ik}}{2\pi} W_{(i)} = \frac{1}{1-\cos\theta_{ik}} \quad \text{if} \quad \theta_{ik} < \theta_{ij} \tag{69}$$

$$= 0 \quad \text{otherwise.} \tag{70}$$

Hint: An integral on a coplex contour is needed. Write $1 - \cos\theta_{jk} = a - b\cos\phi_{ik}$, where $a = 1 - \cos\theta_{ij}\cos\theta_{ik}$ and $b = \sin\theta_{ij}\sin\theta_{ik}$. Then define $z = \exp(i\phi_{ik})$ and rewrite the integral

$$I^{(i)} \equiv \int_0^{2\pi} \frac{\phi_{ik}}{2\pi}\frac{1}{1-\cos\theta_{jk}} = \int \frac{dz}{(z-z_+)(z-z_-)} \,, \tag{71}$$

where the integration is done over the unit circle. Once the expression for $z_\pm$ are found, one realizes that only one pole, $z = z_-$ can lie insider the unit circle, so

$$I^{(i)} = \sqrt{\frac{1}{a^2-b^2}} = \frac{1}{|\cos\theta_{ik} - \cos\theta_{ij}|} \,. \tag{72}$$

4.4 Web

4.4.1 PDF plots

Log onto the Durham on-line calculator and graphical display for the pdf, `http://durpdg.dur.ac.uk/hepdata/pdf3.html`. Plot the parton distributions, $xf(x,\mu^2)$ for $Q^2 = 10$ GeV^2 and the Bjorken $0.01 < x < 1.0$. Plot the error range of the gluon pdf for $Q^2 = 10000$ GeV^2. Which values of x are associated to the largest uncertainty?

5 Hadron hadron collisions

5.1 Exercises

5.1.1 Basic kinematics

The rapidity y and pseudo-rapidity η are defined as:

$$y = \frac{1}{2}\log\left(\frac{E+p_z}{E-p_z}\right) \tag{73}$$

$$\eta = -\log\left(\tan\left(\frac{\theta}{2}\right)\right) , \tag{74}$$

where the z direction is that of the colliding beams.

(a) Verify that for a particle of mass m

$$E = \sqrt{m^2+p_T^2} \quad \cosh y \tag{75}$$

$$p_z = \sqrt{m^2+p_T^2} \quad \sinh y \tag{76}$$

$$p_T^2 = p_x^2 + p_y^2 . \tag{77}$$

(b) Prove that $\tanh\eta = \cos\theta$.

(c) Consider a set of particles produced uniformly in longitudinal phase space

$$dN = C\frac{dp_z}{E} . \tag{78}$$

Find the distribution in η.

(d) Prove that rapidity equals pseudo-rapidity, $\eta = y$ for a relativistic particle $E \gg m$.

(e) Prove that for Lorentz transformation (boost) in the beam (z) directions, the rapidity y of every particle is shifted by a constant y_0, related to the boost velocity. Find the relation between β and y_0 for a generic boost:

$$E' = \gamma(E-\beta p_z) \tag{79}$$

$$p'_z = \gamma(p_z-\beta E) \tag{80}$$

$$p'_x = p_x \tag{81}$$

$$p'_y = p_y \tag{82}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{83}$$

(f) Consider a generic particle X of mass M (such as a Z boson or a Higgs) produced on shell at the LHC , with zero transverse momentum, $pp \to X$. Find the relevant values of x_1, x_2 of the initial partons that can be accessed by producing such a particle. Compare your results with that of Fig. 7, considering the scale $Q = M$.

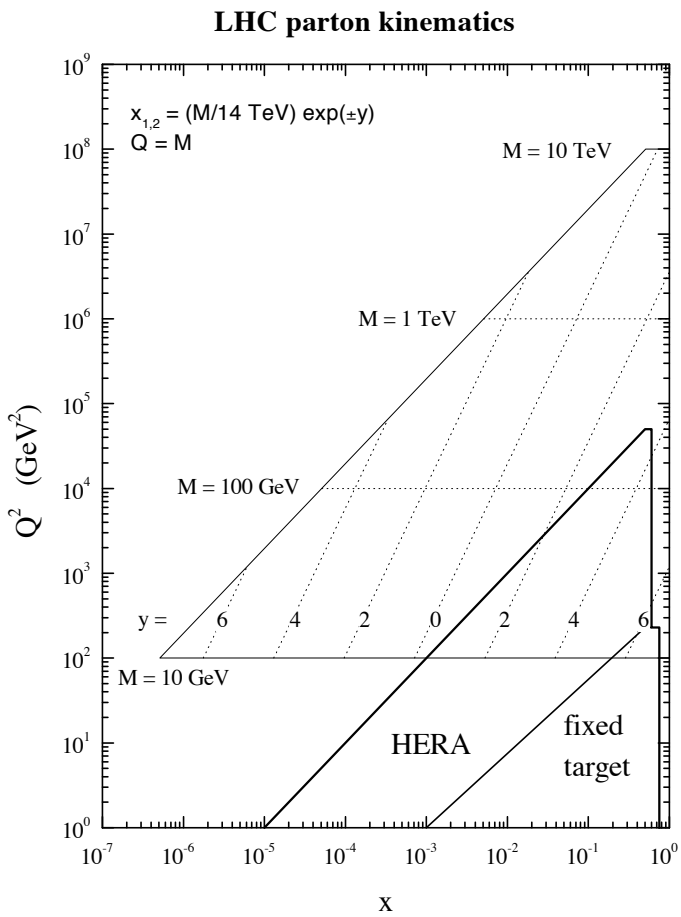

Figure 7: Range in x, Q accessible at the LHC.

5.1.2 Jet kinematics

At the LHC, partons in the incoming beams (beam energy E_b=7 TeV) collide with a momementum fraction $x_{1,2}$ and produce two jets with negligible mass, transverse momentum p_T and rapidities $y_{3,4}$.

(a) Show that

$$x_1 = \frac{p_T}{\sqrt{s}}(e^{y_3} + e^{y_4}), \qquad x_2 = \frac{p_T}{\sqrt{s}}(e^{-y_3} + e^{-y_4})\,. \tag{84}$$

(b) Show that the invariant mass of the jet-jet system is

$$M_{JJ} = 2p_T \cosh\left(\frac{y_3 - y_4}{2}\right)\,, \tag{85}$$

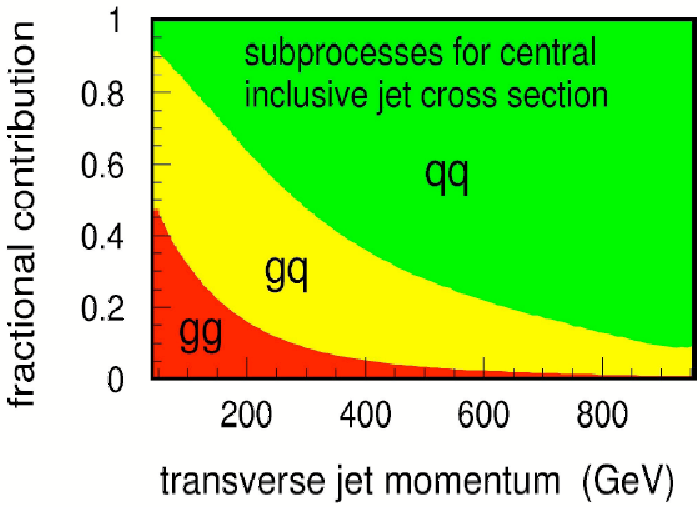

Figure 8: Plot showing the fraction of the jet E_T distribution initiated by different parton combinations.

and the centre-of-mass scattering angle is given by

$$\cos\theta^* = \tanh\left(\frac{y_3 - y_4}{2}\right) . \tag{86}$$

(c) Discuss the regions of $x_{1,2}, M_{JJ}$ and θ^* that can be studied at the LHC with a jet trigger of $p_T > 35$ GeV and $|y_{3,4}| < 3$.

5.2 MC simulations

5.2.1 Jet fraction from different parton combinations

Use `MadGraph/MadEvent` to obtain the relative contribution of $gg, qg + \bar{q}g, qq + q\bar{q}$ initial states to the jet E_T distribution as function of the E_T ($10 < E_T < E_{\max}/4$) at the Tevatron Run II ($p\bar{p}$ collisions at 1.96 TeV) and the LHC (pp collisions at 14 TeV). Compare with the results at the Tevatron, Run I shown in Fig. 8

5.2.2 Multijet production at the Tevatron$^{(*)}$

Use `MadGraph/MadEvent` to obtain the distributions of x_3, x_4 where $x_i = 2E_i/M_{3j}$ are the energy fraction of the jets normalized as $x_3 + x_4 + x_5 = 2$, with $x_3 > x_4 > x_5$, in three-jet events. Consider $p\bar{p}$ collisions at 1.8 TeV of c.m.s. Set the minimum p_T for the jets to 15 GeV, the maximum rapidity of 3.5 and the $\Delta R = 0.8$. Compare your results with the experimental data from CDF, Run I, shown in Fig. 9.

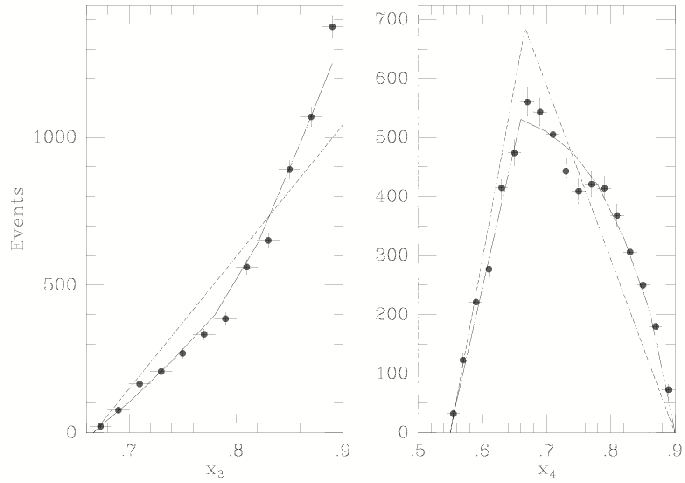

Figure 9: Distributions in the variables x_3 and x_4 in a sample of three jet events as measured by the CDF collaboration (Run I data), $p\bar{p}$ collisions at 1.8 TeV. The solid and dashed lines are the predicitons from QCD and phase space respectively.

5.2.3 $t\bar{t}$ production: Tevatron vs LHC

$t\bar{t}$ production at hadron collider come from both $q\bar{q}$ annhilation and gg fusion.

(a) Use the web interface of `MadGraph/MadEvent` to find the LO cross sections for $t\bar{t}$ production at Tevatron and LHC. Which initial parton contributions are dominating in the two cases?

(b) Use the web interface of `MadGraph/MadEvent` to find the cross sections for $t\bar{t} + 1j$ production at the LHC. Select events for which the jet has $p_T > 20$ GeV and $|\eta| < 4$ (Is a ΔR cut needed to have a finite cross section?). Estimate the cross section and compare it with the LO result for $t\bar{t}$. Is the result reasonable? What's going on? Explain.

5.2.4 W rapidity asymmetry at the Tevatron

The rapidity asymmetry $A_W(y)$ for $W^\pm$ production at a $p\bar{p}$ collider is defined as:

$$A_W(y) = \frac{d\sigma(W^+)/dy - d\sigma(W^-)/dy}{d\sigma(W^+)/dy + d\sigma(W^-)/dy} \,. \tag{87}$$

(a) Give an estimate of such asymmetry and show that it is proportional to the slope of $d(x)/u(x)$ evaluated at $x = M_W/\sqrt{s}$.

(b) Use the web interface of `MadGraph/MadEvent` to plot the rapidity distributions of the the charged leptons coming from $W^\pm$ decays at the Tevatron.