The currents

\[ J_m = \bar{\Psi} \gamma_m \psi \quad , \quad J^5_m = \bar{\Psi} \gamma_m \gamma_5 \psi \]

satisfy

\[ \partial^m J_m = 0 \quad , \quad \partial^m J^5_m = 2iM \bar{\Psi} \gamma_5 \psi - \frac{g^2}{16\pi^2} \epsilon^{mnpq} F_{mn} F_{pq} \]

The last term is the quantum anomaly. Even if they are both classically conserved for \( M = 0 \), there is no regularization preserving both the vector and the axial conservation.
This explain why the $\eta'$ meson is not a pseudo-Goldstone
for $U(2)_L \times U(2)_L = SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A \to
SU(2)_V \times U(1)_B$. Indeed,

$$\begin{align*}
J^U(1)_m &= \bar{u}\gamma_m\gamma_5 u + \bar{d}\gamma_m\gamma_5 d \\
\partial^m J^U(1)_m &= 2i(m_u\bar{u}u + m_d\bar{d}d) - \frac{3g^2}{16\pi^2}\epsilon^{mnpq} F^A_{mn} F^A_{pq}
\end{align*}$$

Another manifestation of the axial anomaly is $\pi^0 \to \gamma\gamma$.

Define the $SU(2)$ currents

$$J^a_m = \bar{q}\gamma_m\tau^a q \quad , \quad J^5_m = \bar{q}\gamma_m\gamma_5\tau^a q$$

Pions are Goldstone's $\Leftrightarrow \langle |J^5_m(x)|\pi^b(p)\rangle = -ip_m f_\pi \delta^{ab} e^{-ipx}$.

Axial isospin currents have no QCD anomalies, but $J^5_m$ has an electromagnetic anomaly.
\[ \partial^m J_m^{53} = -\frac{e^2}{32\pi^2} \epsilon^{mnpq} F_{mn} F_{pq} \]

\( \pi^0 \to \gamma\gamma \) is related to the axial \( U(1)_A \) anomaly.

\[ \Rightarrow \Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{64\pi^3} \frac{m^3}{f^2}, \text{ agreement with experiment.} \]
- For gauge symmetries, if present, they generate inconsistencies, since it would violate gauge invariance of the theory:

\[ \delta \mathcal{L} \sim \alpha_A \partial^m J_m^A \]
The corresponding currents are of chiral type

\[ J_m^A = \bar{\Psi} \gamma_m \gamma_5 T^a \Psi = \bar{\Psi}_R \gamma_m T^a \Psi_R - \bar{\Psi}_L \gamma_m T^a \Psi_L \]

and its divergence is proportional to

\[ \partial_m J_m^A \sim \frac{g_A g_B}{16\pi^2} A^{ABC} \epsilon^{mnpq} F_{mn}^B F_{pq}^C, \]

where the anomaly coeff. that has to vanish is

\[ A^{ABC} = tr (\{T^A, T^B\} T^C)_L - tr (\{T^A, T^B\} T^C)_R = 0, \]

where the trace is taken over all the fermions. For the SM, the only possible anomalies are (Homework:) \( SU(2)_L^2 U(1)_Y, U(1)_Y^3 \) and \( SU(3)_c^2 U(1)_Y \). The results in the SM are
\[ tr \left( \frac{\tau^a}{2}, \frac{\tau^b}{2} \right) Y_L = \frac{1}{2} \delta^{ab} (trY)_L = 3 \times (N_c \times \frac{1}{3} - 1) = 0 , \]
\[ tr \left( \{Y, Y\}Y \right)_{L-R} = \cdots = 6(-2N_c + 6) = 0 \]
\[ tr \left( \{\lambda^A, \lambda^B\}Y \right)_{L-R} = \frac{1}{3} \delta^{AB} (trY)_{L-R} = \cdots = 0 \]

- Anomaly cancelation happens precisely for \( N_c = 3 \) !
- Provides a deep connection between quarks and leptons in the SM, hint towards Grand Unified Theories?

Strong constraint on new chiral particles.

Homework: fourth lepton generation \( l_4, E_R \) alone is inconsistent.
Similar diagrams generate new terms in the SM lagrangian from the redefs. of quarks we did to get the CKM matrix:

\[
\mathcal{L}_\theta \sim \theta \frac{g^2}{16\pi^2} \epsilon^{mnpq} \text{Tr}(F_{mn} F_{pq})
\]

The gluonic term violates CP and unless \( \theta < 10^{-9} \), it generates a neutron dipole moment in conflict with exp. data \( \rightarrow \) the strong CP problem.

One of possible solutions is the axion \( a \). If:

- there is a new \( U(1)_{PQ,\text{spont}} \) broken global symmetry, pseudo-Goldston boson \( a \), symmetry breaking scale \( f \).
- which has triangle anomalies \( U(1)_{PQ}SU(3)_c^2 \)
then the anomaly generates new couplings

\[
\frac{g^2}{16\pi^2} \frac{a(x)}{f} \epsilon^{mnpq} \text{Tr}(F_{mn} F_{pq}) \to \theta_{\text{eff}} = \theta + \frac{a}{f}
\]

Non-perturbative QCD effects then generate an axion potential

\[
V \sim \Lambda^4_{QCD} \left[ 1 - \cos \left( \frac{a(x)}{f} + \theta \right) \right].
\]

The minimum is then for

\[
\theta_{\text{eff}} = 0, \quad \text{and the axion mass } m_a \sim \frac{\Lambda^2_{QCD}}{f}.
\]

Axions were intensively searched since the 80’s. They are also present in most SUSY and string extensions of the SM.
Axion searches and constraints:
Comment: the anomaly is actually a total derivative:

\[ \epsilon^{mnpq} \ Tr(F_{mn} F_{pq}) = \partial^m K_m , \]

where

\[ K_\mu = 2\epsilon_{\mu\nu\alpha\beta} \left( A^\nu a \partial^\alpha A^{\beta a} + \frac{1}{3} f^{abc} A^\nu a A^{\alpha b} A^{\beta c} \right) , \]

Despite this, classical configurations generate effects like theta angle, B and L number nonconservation.

6.1.1 Perturbativity bounds

The RGE for the Higgs self-coupling in the SM is

\[ 16\pi^2 \frac{d\lambda}{d \ln \mu} = 24\lambda^2 - (3g'2 + 3g^2 - 12h_t^2) \lambda + \frac{3}{8}(g'^4 + 2g^2g'^2 + 3g^4) - 6h_t^4 + \cdots, \]

where \( \cdots \) denote smaller Yukawas. In the large Higgs mass limit \( \lambda \gg g^2, h_t^2 \), it reduces to

\[ \frac{d\lambda}{\lambda^2} = \frac{3}{2\pi^2} d \ln \mu \rightarrow \frac{1}{\lambda(\mu)} = \frac{1}{\lambda(\Lambda)} + \frac{3}{2\pi^2} \ln \frac{\Lambda}{\mu}. \]

This can be interpreted in two alternative ways:
i) If the Higgs mass is known, SM has a Landau pole (non-pert. regime) $\lambda(\Lambda) \gg 1$ for

$$\Lambda = v \ e^{\frac{2\pi^2}{3\lambda}} = v \ e^{\frac{4\pi^2 v^2}{3 M_h^2}}$$

ii) Converesely, asking for perturbativity up to scale $\Lambda$ (say $M_{GUT}$), we obtain an upper bound on the Higgs mass (homework)

$$M_h^2 \leq \frac{4\pi^2 v^2}{3 \ln \frac{\Lambda}{v}}.$$
### 6.1.2 Stability bounds

SM has another instability in the small Higgs mass limit, since $\lambda$ can become negative at high-energy.

If $\lambda \ll h_t^2$, the leading RGE’s are

$$16\pi^2 \frac{d\lambda}{d\ln \mu} = -6h_t^4, \quad 16\pi^2 \frac{dh_t}{d\ln \mu} = \frac{9h_t^3}{2}$$

which integrate to (homework)

$$\lambda(\mu) = \lambda(\lambda) + \frac{3h_t^4(\Lambda)}{8\pi^2} \ln \frac{\Lambda}{\mu} \left( \frac{9h_t^2(\Lambda)}{16\pi^2} \ln \frac{\Lambda}{\mu} \right),$$

$$h_t^2(\mu) = \frac{h_t^2}{1 + \frac{9h_t^2(\Lambda)}{16\pi^2} \ln \frac{\Lambda}{\mu}}.$$
This can be interpreted in two ways:

i) For a fixed, known value of the Higgs mass: take $\mu = v$. Then, new physics should show up before the scale $\Lambda$ where $\lambda(\Lambda) = 0$

$$\Lambda \leq v e^{\frac{8\pi^2 \lambda}{3h^4}} = v e^{\frac{4\pi^2 M^2_h}{3h^4 v^2}}$$

ii) For a fixed $\Lambda$, we get a lower bound on the Higgs mass (homework)

$$M^2_h \geq \frac{3h_i^4 v^2}{4\pi^2} \ln \frac{\Lambda}{v} = \frac{3m_i^4}{\pi^2 v^2} \ln \frac{\Lambda}{v}$$
These theoretical Higgs mass limits are summarized in the following plot.
- 6.2. \( W W \) scattering and unitarity.

Let us consider the longitudinal \( W_L W_L \rightarrow W_L W_L \) scattering

- (a)
- (b)
- (c)
- (d)
- (e)
For a massive gauge particle of momentum $k$ and mass $M_W$, $A_m = \epsilon_m e^{ikx}$, the three polarizations satisfy

$$\epsilon_m \epsilon^m = -1, \ k_m \epsilon^m = 0.$$ 
For $k^m = (E, 0, 0, k)$, they are

\[
\begin{align*}
\text{transverse: } & \quad \epsilon_1^m = (0, 1, 0, 0), & \epsilon_2^m = (0, 0, 1, 0), \\
\text{longitudinal: } & \quad \epsilon_L^m \approx \left(\frac{k}{M_W}, 0, 0, \frac{E}{M_W}\right) \sim \frac{k^m}{M} + \mathcal{O}\left(\frac{E}{M_W}\right).
\end{align*}
\]

Since the longitudinal polarization is proportional to the energy, we expect a tree-level amplitude behaving as

$$\mathcal{A} = \mathcal{A}^{(4)} (\frac{E}{M_W})^4 + \mathcal{A}^{(2)} (\frac{E}{M_W})^2 + \cdots$$

Actually, the diagrams a), b) and c) give $\mathcal{A} = g^2 (\frac{E}{M_W})^2$. On the other hand, unitarity constrains the amplitude to stay small enough at any energy.
Start with the unitarity of the S-matrix $S^\dagger S = 1$. Then

$$S = 1 + iA \quad \rightarrow \quad i(A - A^\dagger) + A^\dagger A = 0$$

Let us sandwich this eq. between a two-particle state $|i\rangle$:

$$i(A - A^\dagger)_{ii} + \sum_f |A_{fi}|^2 = 0$$

which is the optical theorem: the imaginary part of the forward amplitude of the process $i \rightarrow i$ is proportional to the total cross section of $i \rightarrow$ anything.

Let us decompose the scattering amplitude into partial waves
\[ A = \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) a_l , \]

where \( a_l \) are partial wave amplitudes of elastic scattering of two particles. Projecting (73) into the partial wave \( l \) gives \( \text{Im} \ a_l = |a_l|^2 . \) This is only possible if

\[ |\text{Re} \ a_l| \leq 1/2 , \ 0 \leq \text{Im} \ a_l \leq 1 \rightarrow |a_l|^2 \leq 5/4 , \]

which is the unitarity bound we were searching for.

• For the SM without the Higgs boson

\[ a_0 = \frac{g^2 E^2}{M_W^2} \rightarrow \text{unitarity breaks down for} \sqrt{s} \sim 1.2 \ TeV \]
With the Higgs boson, amplitudes d),e) cancel the raising energy term, such that

\[ a_0 = \frac{g^2 M_H^2}{4 M_W^2} \rightarrow \text{unitarity breaks down unless } M_H \leq 1.2 \text{ TeV} \]

By considering other channels, one get the stronger bound \( M_H \leq 800 \text{ GeV} \).

Interpretation:
- If LHC finds no Higgs with a mass \( M_H \leq 800 \text{ GeV} \), unitarity of S-matrix will be violated! New light degrees of freedom should exist in order to restore unitarity → the no-loose ”theorem” for LHC.
Most theories have a biased towards a light Higgs, since it provides a better fit for the SM precision tests.
Higgs and the hierarchy problem

Quantum corrections to the Higgs mass in the SM are quadratically divergent

$$\delta m_h^2 \simeq \frac{3\Lambda^2}{8\pi^2v^2}(4m_t^2 - 4M_W^2 - 2M_Z^2 - m_h^2)$$

- Diagrams

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In a theory including gravity or GUT’s, $\Lambda$ is physical mass scale $\Lambda = M_P, M_{GUT}$. It is then difficult to understand why

$$m_h^2 = (m_h^0)^2 + \frac{3\Lambda^2}{8\pi^2v^2}(4m_t^2 - 4M_W^2 - 2M_Z^2 - m_h^2) \sim v^2 << \Lambda^2$$

→ the hierarchy problem.
Latest news ("Lepton-Photon", August 2011): Both ATLAS+CMS exclude the SM Higgs at 95% CL for $145 \leq M_H \leq 446\ GeV$ except $288 - 296\ GeV$.

M. Peskin (LP2011) "There is therefore strong evidence that either:

- **Higgs is light**, compatible with electroweak precision tests and theoretical prejudice, or
- **the Higgs boson is very heavy** and strongly self-coupled". 
Can Standard Model be the final theory?

**NO**
- No neutrino masses at the renormalizable level (lect. Boris).
- Mysterious hierarchies in the quarks/lepton masses and mixings (lect. Yuval).
- Problem with the radiative stability of the electroweak scale ("the hierarchy problem").
- No accurate gauge coupling unification.

Last three problems ⇒ SUPERSYMMETRY?
- the strong CP problem.
- gravity not incorporated into a renormalizable framework ⇒ STRING THEORY?
- cosmological constant problem
\[ \Lambda \sim 10^{-4} \text{ eV}^4 \sim 10^{-120} \, M_P^4. \]

**YES**

- no signal of new physics yet... But if no SM higgs the next year, something else must replace it...