Impact on EWPOs of the $\alpha_{QED}(m^2_Z)$ uncertainty
and possible mitigation

context: on 14 July there is a **MiniWorkshop: parametric uncertainties: α_em**
https://indico.cern.ch/event/1173700/
where the possible calculations and improvements on this important effect
are discussed.

It is notable that it can be measured in FCC-ee from the slope of A_{FB} ($\mu\mu$) vs $\sqrt{s}$ around the Z pole (Janot)
This is new and specific to the circular Higgs/EW factory projects.

In view of the meeting I have prepared a table of the effect of $\Delta\alpha$ on important EW observables, compared with the expected statistical precisions of FCC-ee, as presented
by Juan Alcaraz, at the FCC physics workshop of 2022.
https://indico.cern.ch/event/1066234/contributions/4708127/

Table 1. Dependence of selected precision measurements at FCC-ee upon the uncertainty on $\alpha_{QED}(m_Z^2)$, or on $\sin^2\theta_W^{eff}$. Experimental data have been compiled from [3].

Observable	present value ± error	FCC-ee Stat.	from $\alpha_{QED}(m_Z^2)$	Comments
m_Z (keV)	91186700 ± 2200	4	N.A.	Input
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$1/\alpha_{QED}(m_Z^2)(\times 10^3)$	128952 ± 14	3	N.A.	Input; from $A_{FB}^{\mu\mu}$ off peak
$\sin^2\theta_W^{eff}(\times 10^6)$	231480 ± 160	1.5	10.5	from $A_{FB}^{\mu\mu}$ and $A_{FB}^{pol,\tau}$ at Z peak Possible alternative input
m_W (MeV)	80350 ± 15	0.250	0.547	
Γ_ℓ (keV)	83985 ± 86	0.2	0.53	stat. based on muon pair statistics ρ parameter
R_ℓ^Z $(\times 10^3)$	20767 ± 25	0.06	0.17	ratio of hadrons to leptons quark and lepton universality determination of $\alpha_{QCD}(m_Z^2$
R_b $(\times 10^6)$	216290 ± 660	0.3	0.42	ratio of $b\bar{b}$ to hadrons N.P. coupled to 3d generation
Γ_Z (keV)	2495200 ± 2300	4	27	From Z line shape scan Beam energy calibration

How this was calculated

History: at the time of LEP (1986-1989) I participated in the definition of ‘effective’ parameters representing definite blocks of corrections **(4)** in collaboration with e.g. Bryan Lynn and others.
Basically LEP corrections were driven by terms that looked like this:

$$\mathrm{M_Z^2} = \frac{\pi\alpha(\mathrm{M_Z^2})}{\sqrt{2}\mathrm{G_F}(1+\Delta\rho)(1+\Delta_{3Q})\sin^2\theta_{\mathrm{W}}^{\mathrm{eff}}\cos^2\theta_{\mathrm{W}}^{\mathrm{eff}}};$$

$$\Gamma_\ell = \frac{\mathrm{G_F M_Z^3}}{24\sqrt{2}\pi}[1+\Delta\rho]\left[1+(\frac{g_{V\ell}}{g_{A\ell}})^2\right](1+\frac{3}{4}\frac{\alpha}{\pi});$$

$$\Gamma_b = \Gamma_d(1+\delta_{vb});$$

$$\mathrm{M_W^2} = \frac{\pi\alpha(\mathrm{M_Z^2})}{\sqrt{2}\mathrm{G_F}(1-\Delta r^{ew})(1-\frac{\mathrm{M_W^2}}{\mathrm{M_Z^2}})}.$$

$$g^f_{L,R} = \sqrt{\rho^f}\,[I^f_{L,R} - Q^f\,\kappa^f\,sin^2\theta_{\mathrm{W}}^{\mathrm{eff}}]$$

related to S and T parameters
$S \propto \alpha(\Delta_{3Q} - \Delta_{3Q}{}^{SM})$
$T = \alpha\ (\Delta\rho - \Delta\rho^{SM})$ (see Peskin Takeuchi ‘90)
δ_{vb} also important!
running of α_{QED} explicit, same for $\sin^2\theta_w{}^{eff}$ and m_W

old stuff, see e.g. my ICHEP Warsaw presentation in 1994
https://inspirehep.net/literature/377153

A very nice set of equations appears in the 2005 LEP physics report arXiv:0509008

Electroweak predictions for Z → ff observables can be obtained by using the coupling formulae

$$g^f_{L,R} = \sqrt{\rho^f}\,[I^f_{L,R} - Q^f \kappa^f \sin^2\theta^{\rm eff}_{\rm W}]$$

(I_R = 0 for right-handed fermions)
$g_L = ½ (g_V + g_A)$ $g_R = ½ (g_V - g_A)$
$g_V = (g_L + g_R)$ $g_A = (g_L - g_R)$

for practical reasons the effective weak mixing angle is defined from the leptonic couplings:

$\sin^2\theta^{\rm lept}_{\rm eff} = ¼\,(1 - g_v/g_A)^{\rm charged\ lepton}$ and $\rho^{\rm charged\ lepton} \equiv (1 + \Delta\rho)$

For neutrinos and quarks small correction factors are necessary for non-universal corrections (vertex and box diagrams).
In the SM, these additional corrections bear no sensitivity to running alphaQED.

No sensitivity to heavy physics in SM either, except for the b-quark where the correction is sensitive to top and Higgs boson masses.
They bear sensitivity to other new physics such as heavy partners, such as stop mass or Heavy Neutrinos.

It is important to note that the running of $\alpha_{\rm QED}$ appears at zeroth order
only in the prediction for **$\sin^2\theta_w^{\rm eff}$ and m_W, and with the same correction factor $\Delta\alpha$**

$$\alpha_{\rm QED}(m_Z^2) = \frac{\alpha_{\rm QED}(0)}{1 - \Delta\alpha}$$

it is also possible to establish a relation that eliminates $\Delta\alpha$ between $\sin^2\theta_w^{\rm eff}$ and m_W

$$\sin^2\theta^{\rm eff}_{\rm W} = \mathcal{K}_w\left(1 - \frac{m^2_{\rm W}}{m^2_{\rm Z}}\right)$$

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m_W (MeV)	80350 ± 15	**0.250**	**0.547**	
Γ_ℓ (keV)	83985 ± 86	**0.2**	**0.53**	stat. based on muon pair statistics ρ parameter
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m_W (MeV)	80350 ± 15	**0.250**	**0.547**	**0.078**	
Γ_ℓ (keV)	83985 ± 86	**0.2**	**0.53**	**0.076**	stat. based on muon pair statistics ρ parameter
R_ℓ^Z $(\times 10^3)$	20767 ± 25	**0.06**	**0.17**	**0.025**	ratio of hadrons to leptons quark and lepton universality determination of $\alpha_{QCD}(m_Z^2$
R_b $(\times 10^6)$	216290 ± 660	**0.3**	**0.42**	**0.06**	ratio of $b\bar{b}$ to hadrons N.P. coupled to 3d generation
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from inspecting the table one can observe the following:

1. Even with the direct measurement of α_{QED} **(m_Z)** at FCC-ee (3 10^{-5}) the resulting uncertainty on the prediction of all observables (except the neutrino partial widths) is larger than the expected statistical uncertainty.

2. The observable that is most affected is **$\sin^2\theta_w^{eff}$**

3. Since all sensitivity to $\Delta\alpha$ is contained in the correction to **$\sin^2\theta_w^{eff}$**

it is conceivable to eliminate most of the $\Delta\alpha$ uncertainty in the prediction of the other observables by using the measured value of $\sin^2\theta_w^{eff}$ in the SM prediction, thus sharpening the predictive power of these other measurements.

➔ **By using the measured value of $\sin^2\theta_w^{eff}$ rather $\Delta\alpha$ in for the predictions, the $\Delta\alpha$ parametric errors on all quantities becomes smaller than the expected FCC-ee accuracy.**

for several observables (such as R_{lepton}) this procedure would also removes the top mass uncertainty

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Conclusion

1. Of course is is extremely important to calculate and measure experimentally α_{QED} **(m_Z)** as precisely as possible.

2. and to measure $\sin^2\theta_w^{eff}$ as accurately as possible.

3. In order to enhance the precision of the other EWPOs (including the W mass) it would be very useful to be able to implement the SM prediction of the other EWPOs including the measured value of $\sin^2\theta_w^{eff}$, since this is an efficient way to eliminate their dependence on $\Delta\alpha$