PY411 / 506
Computational Physics 2

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Fluid Dynamics

- Now turn to fluid dynamics
  - Many particles together acting under conservation of mass and momentum
  - Simple cases are incompressible fluids without friction ("Newtonian fluids")

- Consider a volume V inside a fluid. The mass m inside that volume is the integral of the density rho :

$$m = \int \rho \, dV$$

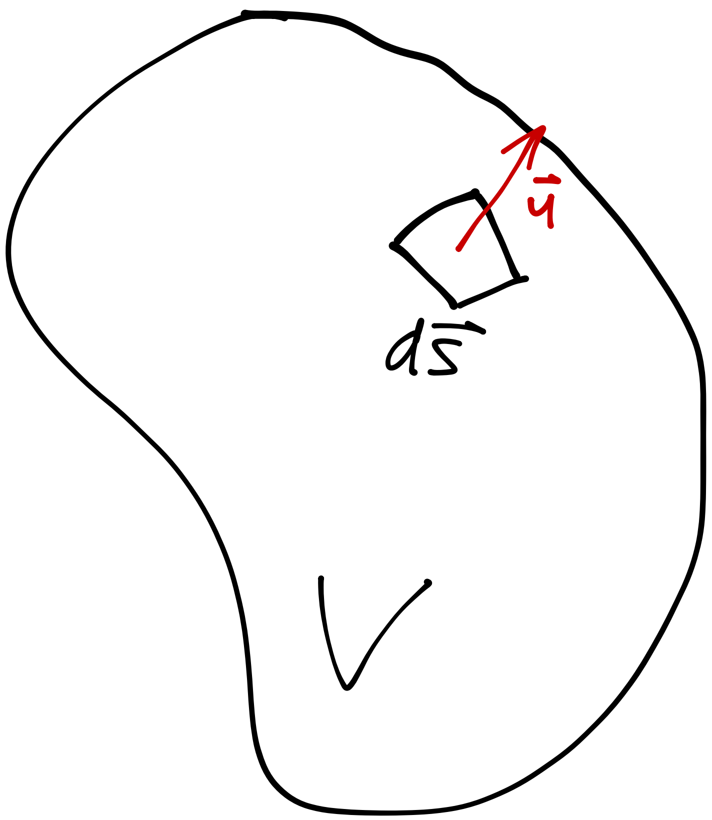
Fluid Dynamics

- The change in mass is determined by the rate of change at the surface

$$\frac{d}{dt}\int \rho\, dV = -\int d\vec{S}\cdot\rho\vec{u}$$

- Then apply the divergence theorem:

$$\int d\vec{S}\cdot\rho\vec{u} = \frac{d}{dt}\int \rho\, dV = \int dV =; \vec{\nabla}\cdot\rho\vec{u}$$



- So we then obtain the continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u} = 0$$

Fluid Dynamics

- Now we can apply Newton's second law (conservation of momentum):

$$\rho \frac{d\vec{u}}{dt} = \vec{F}$$

- and then we have:

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}$$

Change in fluid velocity at a fixed point in space

Change in fluid velocity due to motion of fluid from neighboring points (Advection)

Fluid Dynamics

- What about friction?
  - Add force of gravity g
- Introduce viscous forces:
  - Dynamic viscosity coefficient μ
  - Bulk viscosity coefficient ξ
  - Pressure p
- Then total force is:

$$\vec{F} = \mu\nabla^2\vec{u} + (\mu + \xi)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$$

- Then we get the Navier-Stokes equation with dynamic viscosity $\nu = \mu/\rho$

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{g} - \frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{u}$$

Fluid Dynamics

- If we look at this in 1-d, this is :

$$\frac{\partial u}{\partial t} + \left(u \frac{\partial u}{\partial x} \right) = g - \frac{1}{\rho}\frac{\partial p}{\partial t} + \nu \frac{\partial^2 u}{\partial x^2}$$

- If g = 0 and pressure p = 0 (or gradient = 0), this is Burgers' equation!

$$\frac{\partial u}{\partial t} + \left(u \frac{\partial u}{\partial x} \right) = \nu \frac{\partial^2 u}{\partial x^2}$$

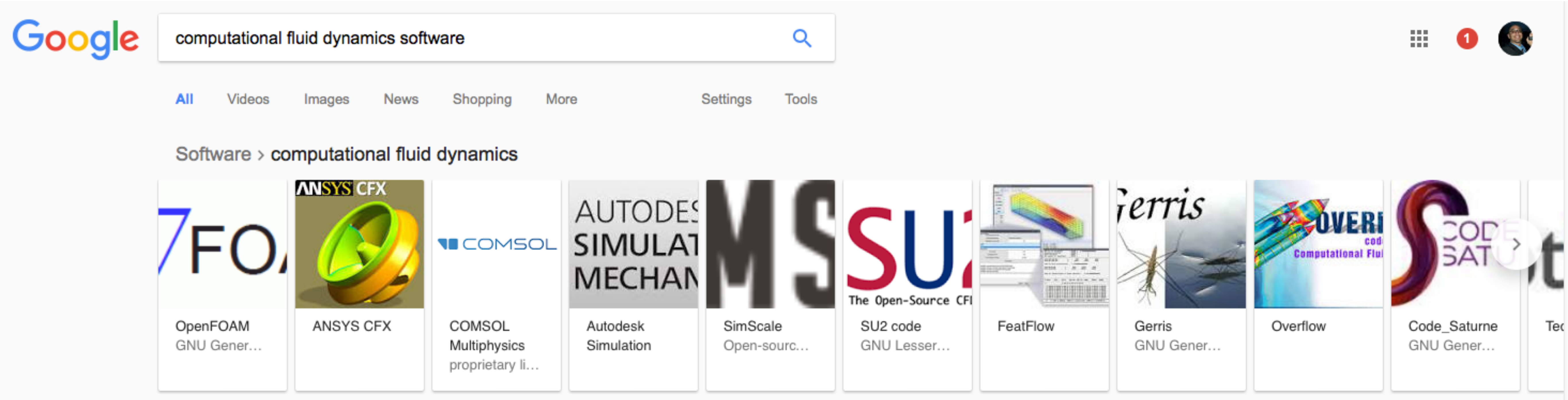
- So shock propagation is a special case of the Navier-Stokes dynamics

Fluid Dynamics

- What about the full case?
  - It’s a $1M question
  - (No, really, it’s a Millennium Prize question).
  - http://www.claymath.org/millennium-problems

Fluid Dynamics

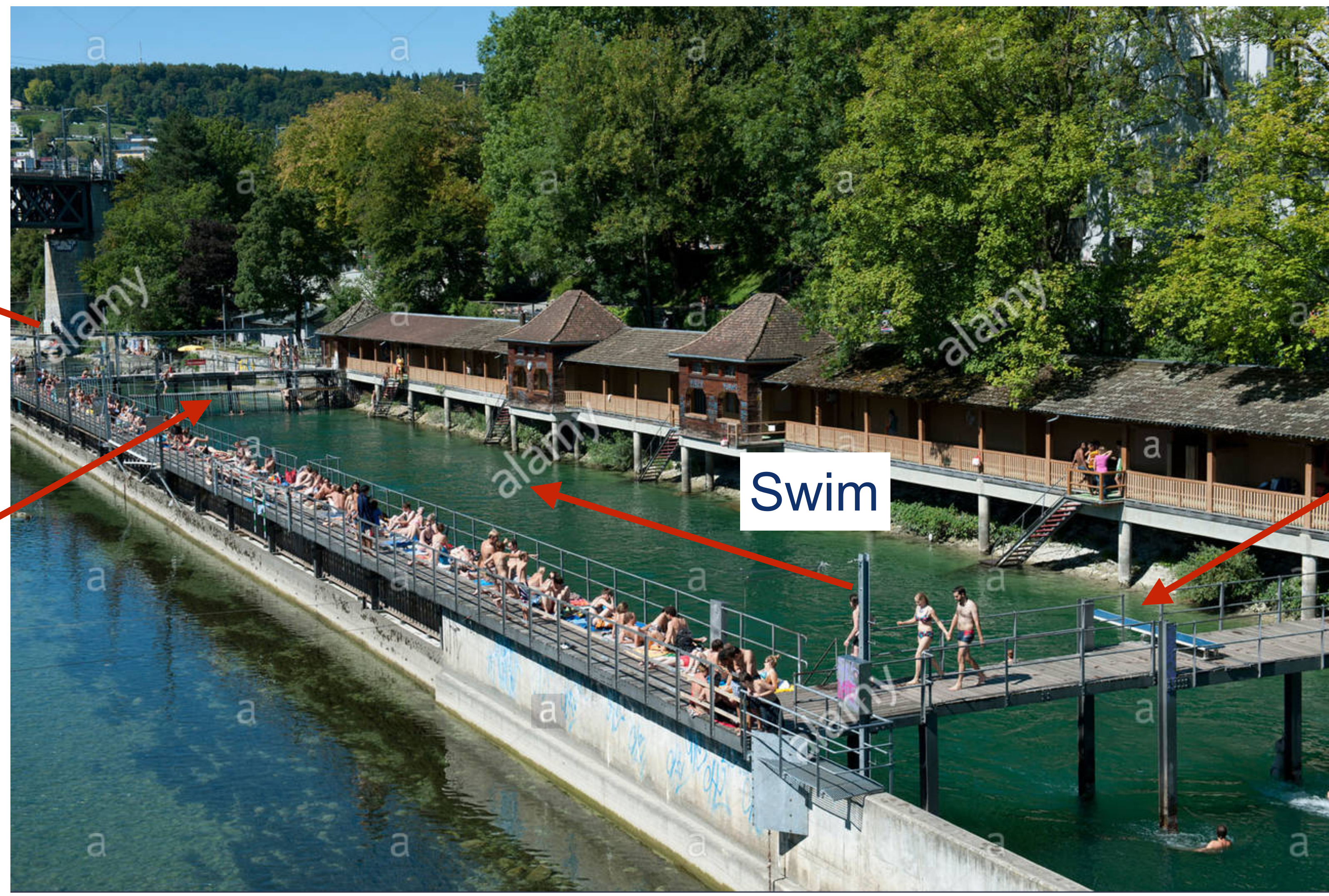
- OK, well, what about simple-ish solutions?
  - Even then, still pretty hard
  - Commercial software in abundance:



  - Open source examples:
    - https://people.sc.fsu.edu/~jburkardt/py_src/navier_stokes_2d_exact/navier_stokes_2d_exact.html
    - http://lorenabarba.com/blog/cfd-python-12-steps-to-navier-stokes/

Fluid Dynamics

- Let’s look at one special case of flow through a long tube with constant pressure, like the flow in a river.



Limmat river in Zurich, CH

Jupyter!

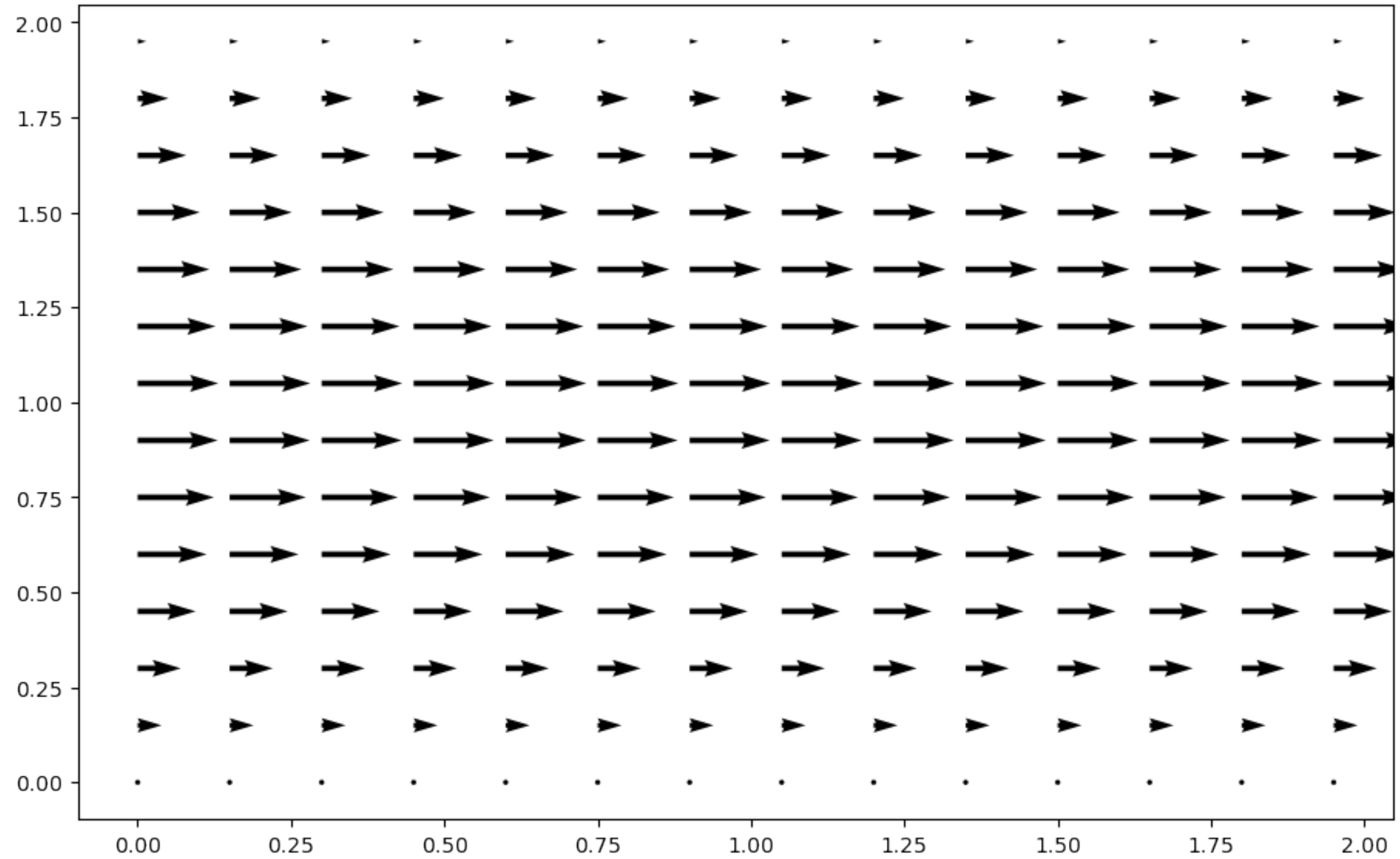
- Let’s play with some Jupyter now.

- We will use the example from here :
  - http://lorenabarba.com/blog/cfd-python-12-steps-to-navier-stokes/

- We will walk through it.

Fluid Dynamics

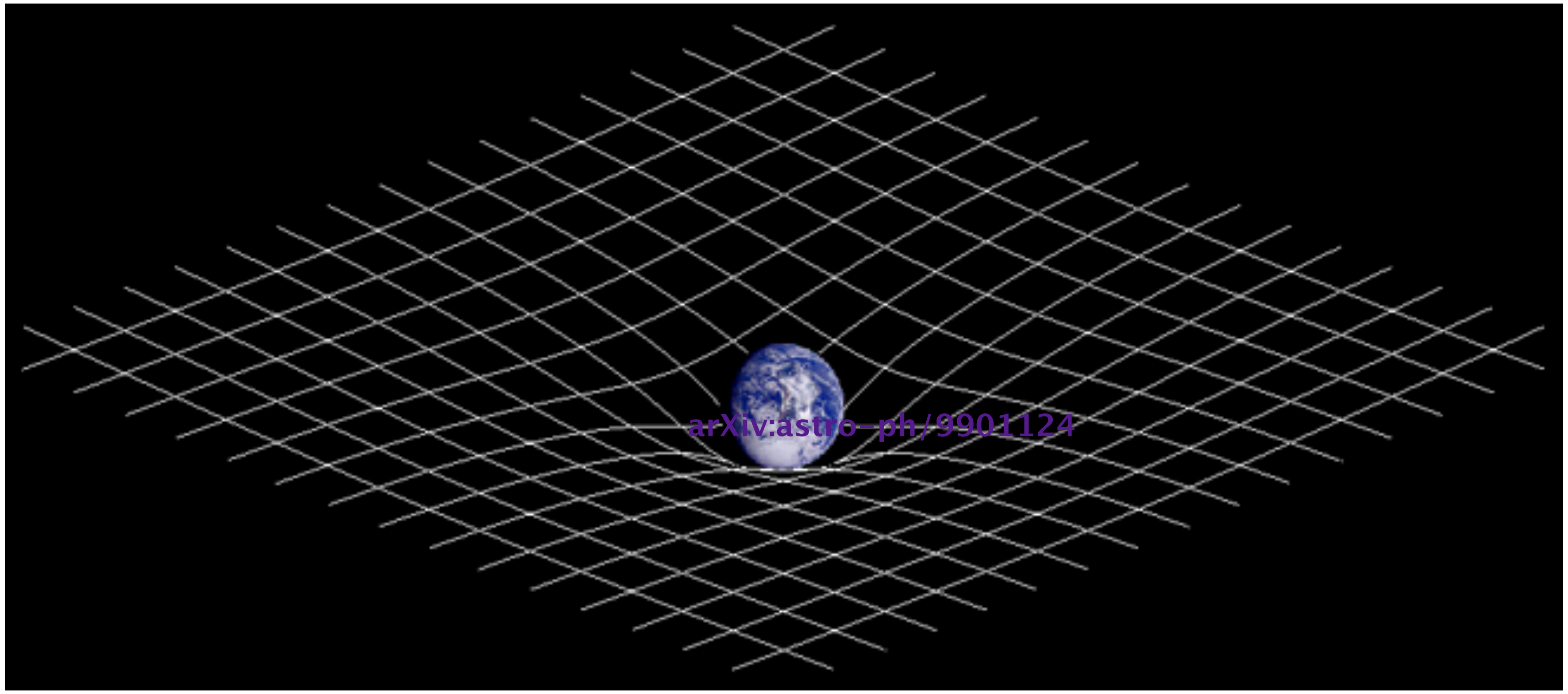
- The flow in the center is highest, and the flow at the edges is lowest:



Recall: General Relativity

- http://en.wikipedia.org/wiki/General_relativity

- Relates gravity to the curvature of space-time!



- Objects with mass or energy distort space-time, and this induces a gravitational field

Recall: General Relativity

- Space-time is a tensor
- So gravity is a tensor
- Einstein's equations :

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{8\pi G_{\rm N}}{c^4} T_{\mu\nu} - g_{\mu\nu} \Lambda$$

Stress-energy tensor (energy and momentum density of matter + radiation)

Newton's constant

Ricci tensor (gravitational force)

Curvature scalar

Speed of light

Cosmological constant

Metric tensor

Recall: General Relativity

- That's a huge set of nonlinear partial differential equations, and can be arbitrarily complicated ($T_{\mu\nu}$ has no constraint to its format)
- A few simple cases can be derived :
  - If spacetime is homogeneous and isotropic, this is the Robertson-Walker metric :

Cosmological scale factor

$$ds^2 \equiv \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] ,$$

  - Assuming that the matter+radiation behave like a uniform perfect fluid with density ρ and pressure p, this is the Friedmann-Lamaitre equations:

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N \rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} , \qquad \frac{\ddot{R}}{R} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda c^2}{3} .$$

Hubble parameter : H(t0) = 72 km/s/Mpc at present time

Recall: Hubble's Law

- Hubble used this equation to determine a linear relationship :

$$rK + X\cos\alpha\cos\delta + Y\sin\alpha\cos\delta + Z\sin\delta = v\ ,$$

Distance (r)
Constants (K, X, Y, Z)
Longitude (α)
Latitude (δ)
Velocity (v)

- Plotting the data :

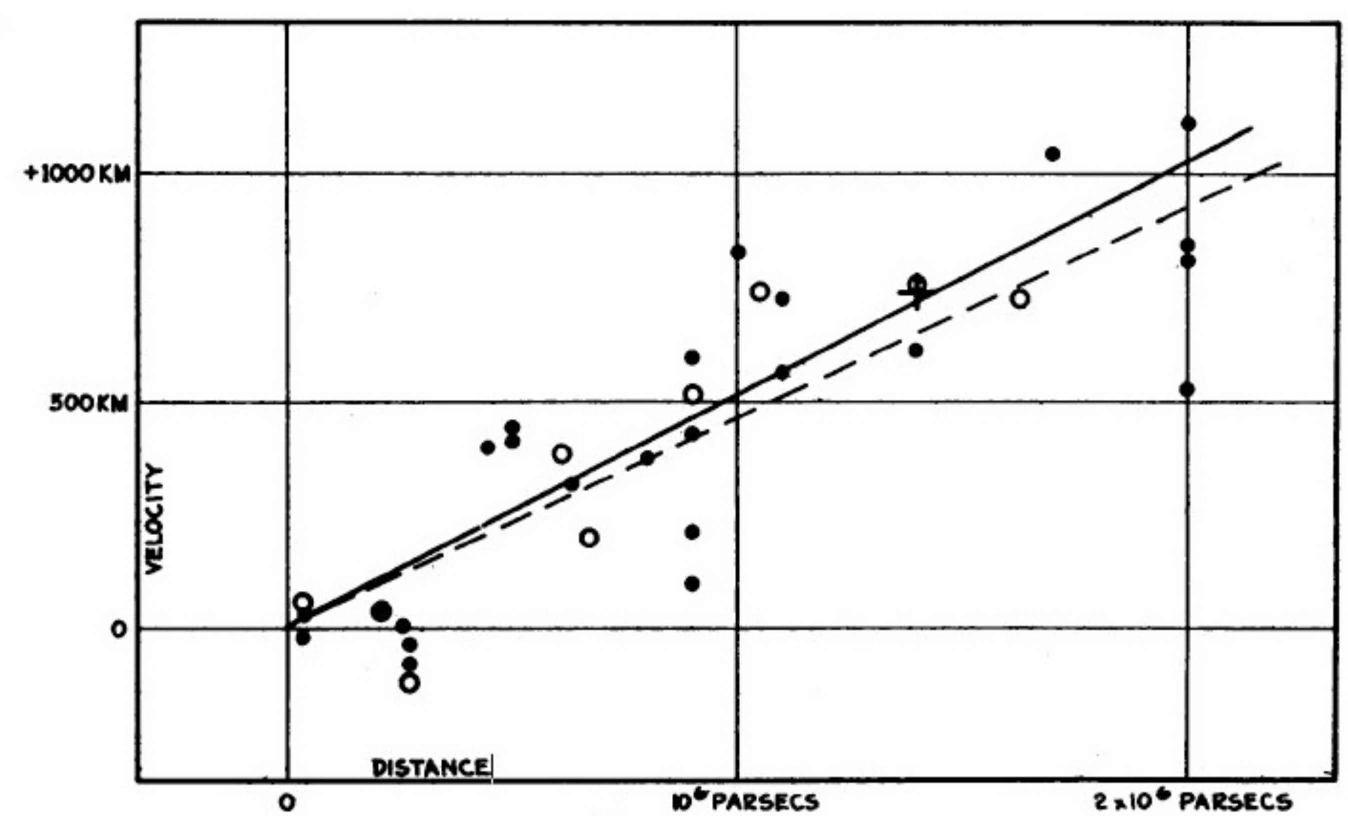


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

General Relativity

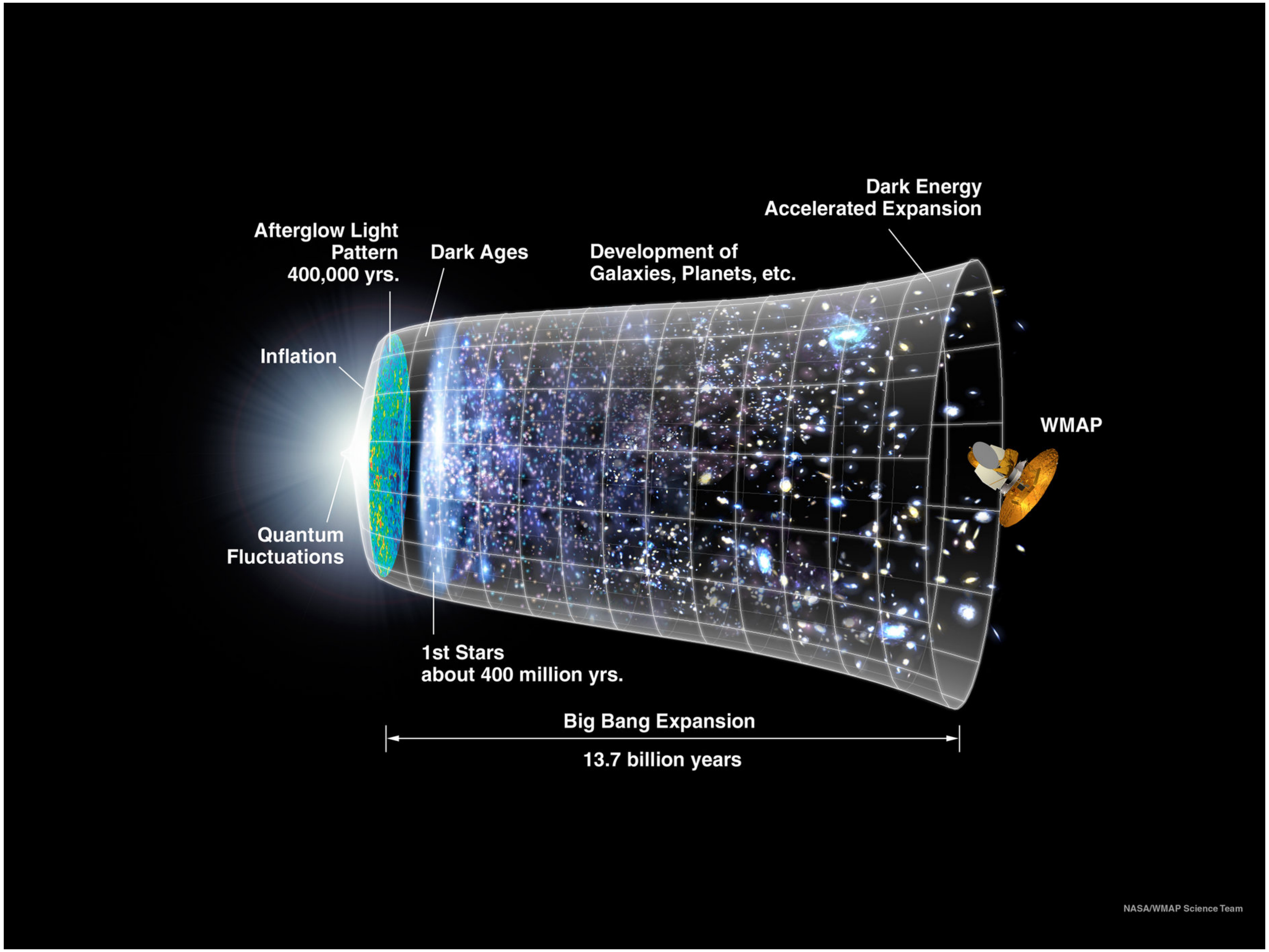
- Now in a position to calculate some GR numerically instead of just analyzing the data

- Some approaches approximate matter as perfect fluid evolving according to GR
  – Morally equivalent equations to Navier-Stokes, so similar techniques can be used for solutions

- We will investigate inflation
    - For an overview : arXiv:astro-ph/9901124

General Relativity

- Cosmic inflation



General Relativity

- Go back to our equations:

Constants

Density

Pressure

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_{\mathrm{N}}\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3}\,, \qquad \frac{\ddot{R}}{R} = -\frac{4\pi G_{\mathrm{N}}}{3}(\rho + 3p) + \frac{\Lambda c^2}{3}\,.$$

Cosmological scale factor

Hubble parameter : H(t0) = 72 km/s/Mpc at present time

- Values of k:
  - k = 1: closed 3-sphere
  - k = 0: flat
  - k = -1: open 3-hyperboloid

General Relativity

- Eliminate Lambda, use conservation of mass+energy, and combine the two equations, you get:

$$\dot{\rho} = -3H(\rho + 3p)$$

- Further simplify the math by scaling mass density and pressure to include all of the constants in the system:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

$$\dot{H} + H^2 = \left(\frac{\ddot{R}}{R}\right)^2 = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

General Relativity

- If we consider a perfect fluid, the pressure is linearly dependent on the density, so you get $p = w\rho$ for some constant w
- If space is completely flat, and we set c=1, we have k = 0, so

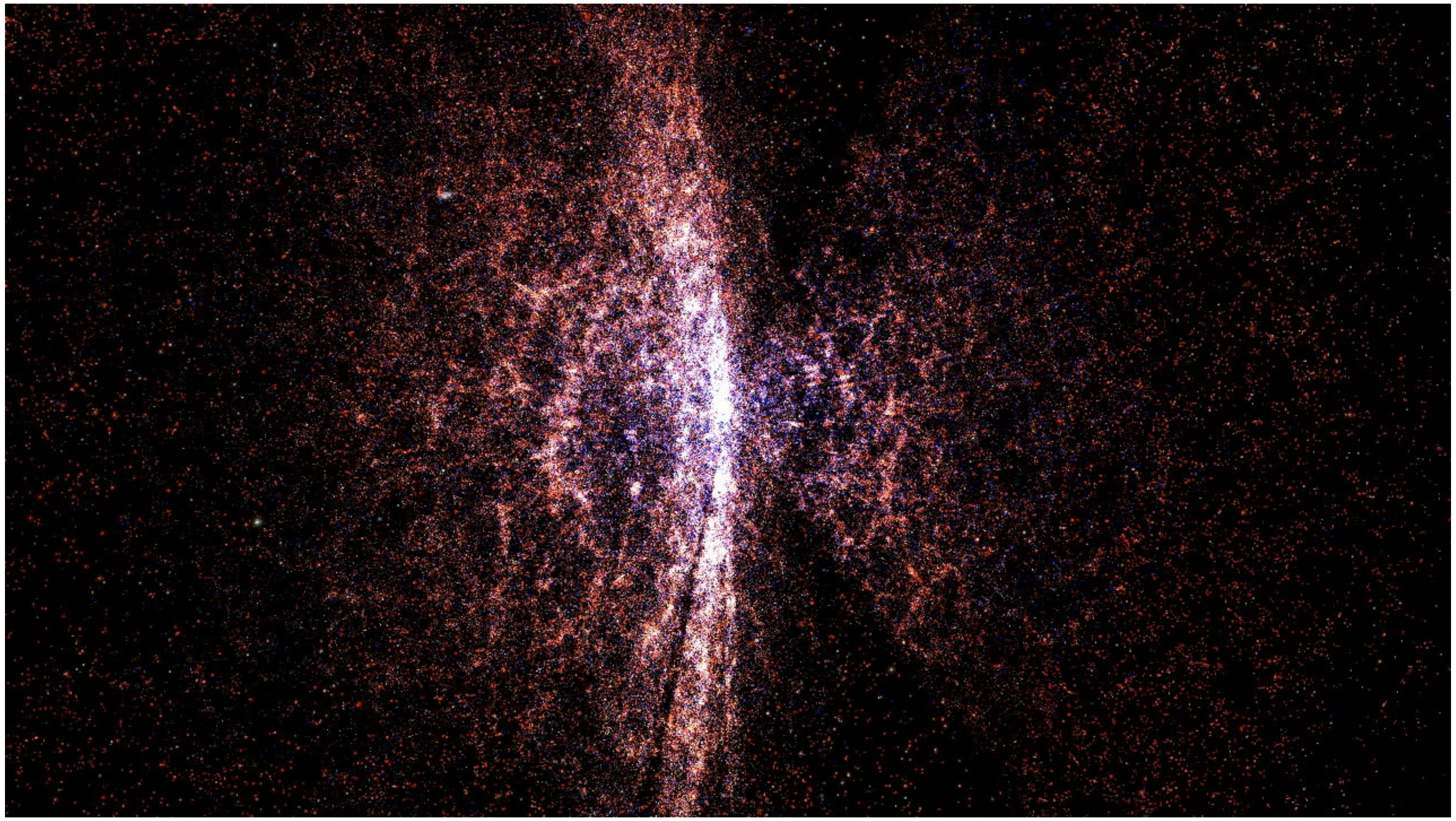
$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\dot{H} + H^2 = \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+3w)\rho$$

- You will solve this for your homework problem!

N-body simulations

- Consider N interacting particles with long range force acting between them (i.e. gravity)
  - Structure formation of galaxies, stars, superclusters, etc
- Recall our previous attempts at the 3-body problem
  - Will now extend to N bodies



Sloan Digital Sky Survey (NASA)

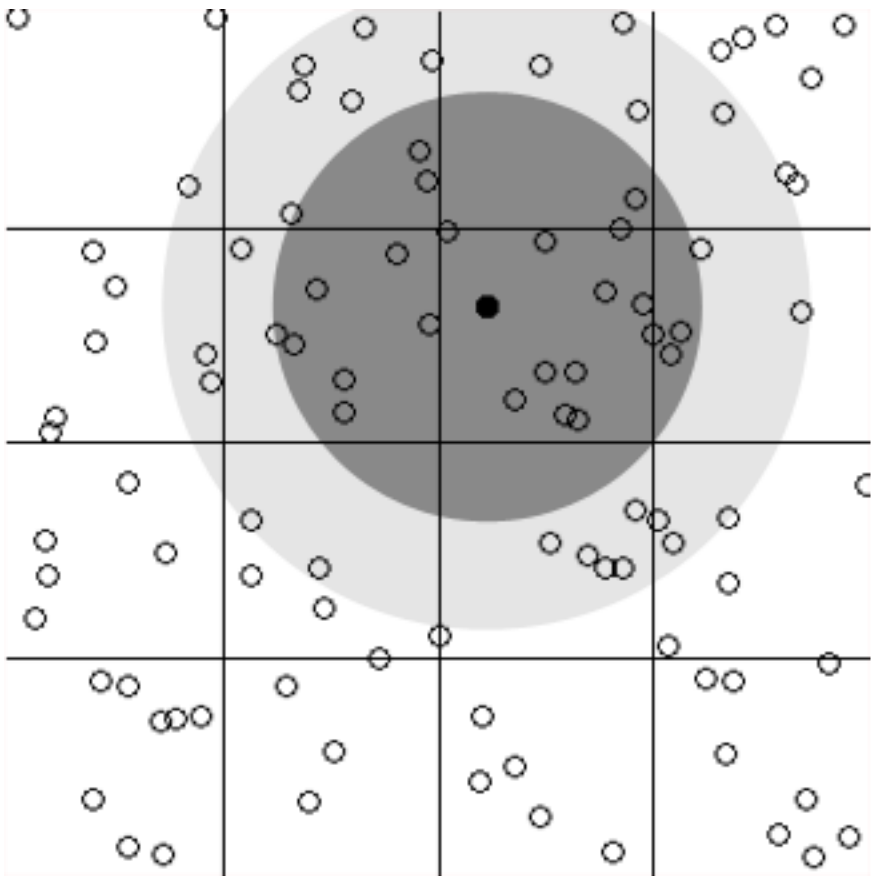
N-body simulations

- Long-range interaction very inconvenient here
- Force goes like

$$F \sim \frac{1}{r^2}$$

- But surface area goes like

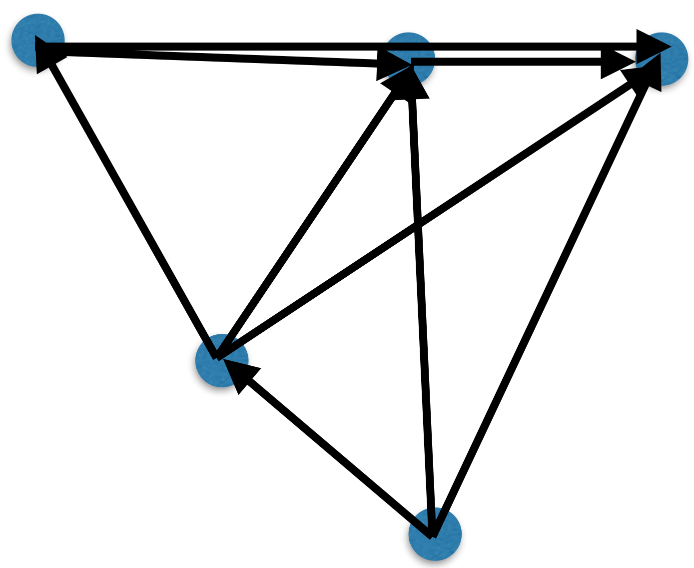
$$A \sim 4\pi r^2$$



- Thus the number of particles at distance r times the strength of the force is basically constant
  – Cannot truncate!

N-body simulations

- If we calculate the forces between N bodies, we therefore need to compute N(N-1)/2 forces each iteration



$$\vec{F}_i = \sum_j \vec{F}_{ij} = -\sum_j \frac{G m_i m_j}{r^3} \vec{r}$$

Simple, but computationally intractable

N-body simulations

- Try the next level of sophistication in our ODE solvers
  - Currently evolve position and velocity (according to velocity and acceleration, respectively)
  - How about evolving acceleration!?
  - Use the JERK!

N-body simulations

- Try the next level of sophistication in our ODE solvers
  - Currently evolve position and velocity (according to velocity and acceleration, respectively)
  - How about evolving acceleration!?
  - Use the JERK!

Not this jerk

This jerk

$$\vec{\mathcal{J}}_{ij} = \dot{\vec{a}}_{ij} = \frac{M_i}{r_{ij}^3}\left[\vec{v}_{ij} - 3\frac{\vec{v}_{ij}\cdot\vec{r}_{ij}}{r_{ij}^2}\vec{r}_{ij}\right]$$

N-body simulations

- First let's look at simple systems where we actually solve the ODEs for all particles
- Going to use third-party software:
  - https://www.ids.ias.edu/~piet/act/comp/algorithms/starter
  - Instructions:
  - https://github.com/rappoccio/PHY410/blob/master/Lecture35/README.md

- Can't animate and solve the ODEs in real time for any large-ish number of N (like, 30).

Works, but computationally intractable!

N-body simulations

- Code uses a "predictor-corrector" algorithm:
  - Predicts position and velocity at next time steps

$$\vec{r}_p = \vec{r} + \vec{v}\delta t + \frac{1}{2}\vec{a}\delta t^2 + \frac{1}{6}\vec{\mathcal{J}}\delta t^3$$

$$\vec{v}_p = \vec{v} + \vec{a}\delta t + \frac{1}{2}\vec{j}\delta t^2$$

  - Computes acceleration and jerk for those predictions using Taylor series

$$\vec{k} \equiv \frac{1}{2}\vec{a}''\delta t^2 = 2(\vec{a} - \vec{a}_p) + \delta t(\vec{\mathcal{J}} - \vec{\mathcal{J}}_p)$$

$$\vec{l} \equiv \frac{1}{2}\vec{a}'''\delta t^3 = -3(\vec{a} - \vec{a}_p) - \delta t(2\vec{\mathcal{J}} - \vec{\mathcal{J}}_p)$$

N-body simulation

- Then corrects position and velocity with the acceleration and jerk:

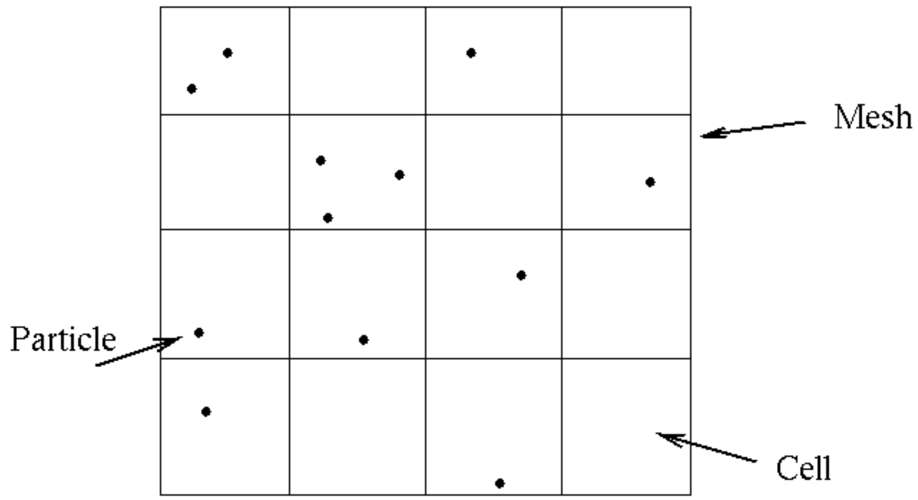
$$\vec{r}_c = \vec{r}_p + \left(\frac{1}{12}\vec{k} + \frac{1}{20}\vec{l} \right) \delta t^2$$

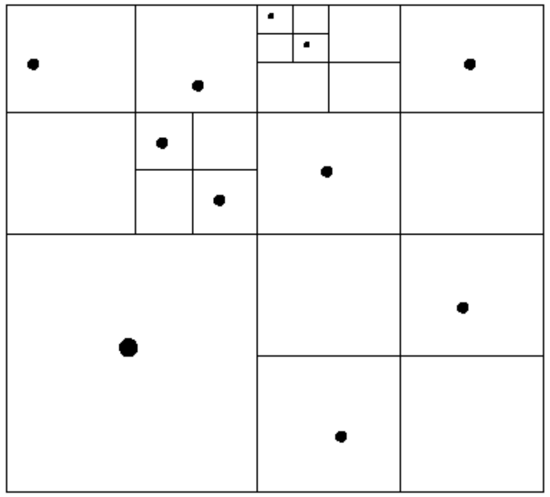
$$\vec{v}_c = \vec{v}_p + \left(\frac{1}{3}\vec{k} + \frac{1}{4}\vec{l} \right) \delta t$$

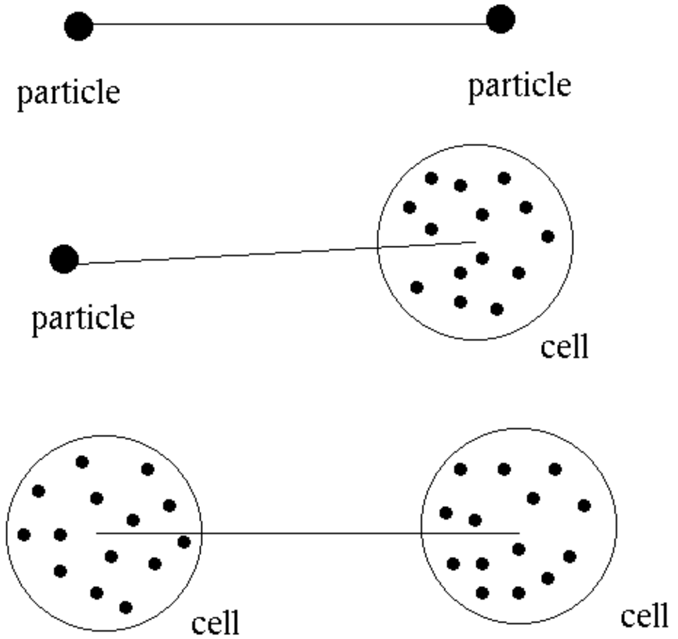
- Their code also calculates the “least collision time”, i.e. the smallest time between interactions of two objects.
  - Used to adjust the time step, similar to ARK4

N-body simulations

- We need some kind of approximation then:
  - Particle mesh
    - Create a 3-d lattice, approximate forces from them
    - Good for uniform configurations
    - O(M log(M)) (M = num grid points)
  - Trees
    - Partition space into a hierarchy of cubes
    - Compute particle-particle interactions for close interactions
    - Compute particle-cell or cell-cell interactions for far interactions
    - O(N log(N)) (N = num particles)







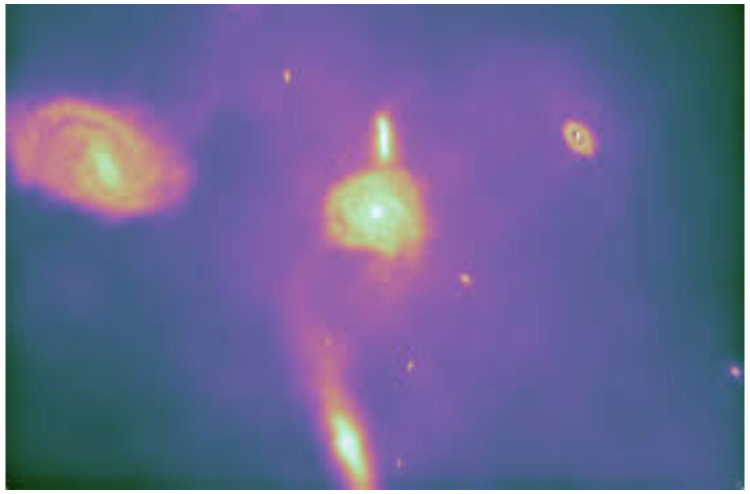
Figures from http://www.new-npac.org/projects/cdroms/cewes-1999-06-vol2/cps615course/nbody-materials/nbody-simulations.html

N-body simulations

- Fast multipole methods
  - Expands Green's function in a multipole
  - O(N) (N = num particles)
- Fluid dynamic approximations
  - Approximate by PDEs, not individual particles
  - Only applicable in certain situations
-

N-body simulations

- Lots of codes out there!
  - AREPO:
    - http://wwwmpa.mpa-garching.mpg.de/~volker/arepo/
  - GADGET:
    - http://wwwmpa.mpa-garching.mpg.de/gadget/



- Example of simulation from GADGET from a former Comp. Phys. student (Leigh Korbel) for his Master's project