

Improving strong coupling determinations from hadronic tau decays

Miguel Benitez-Rathgeb



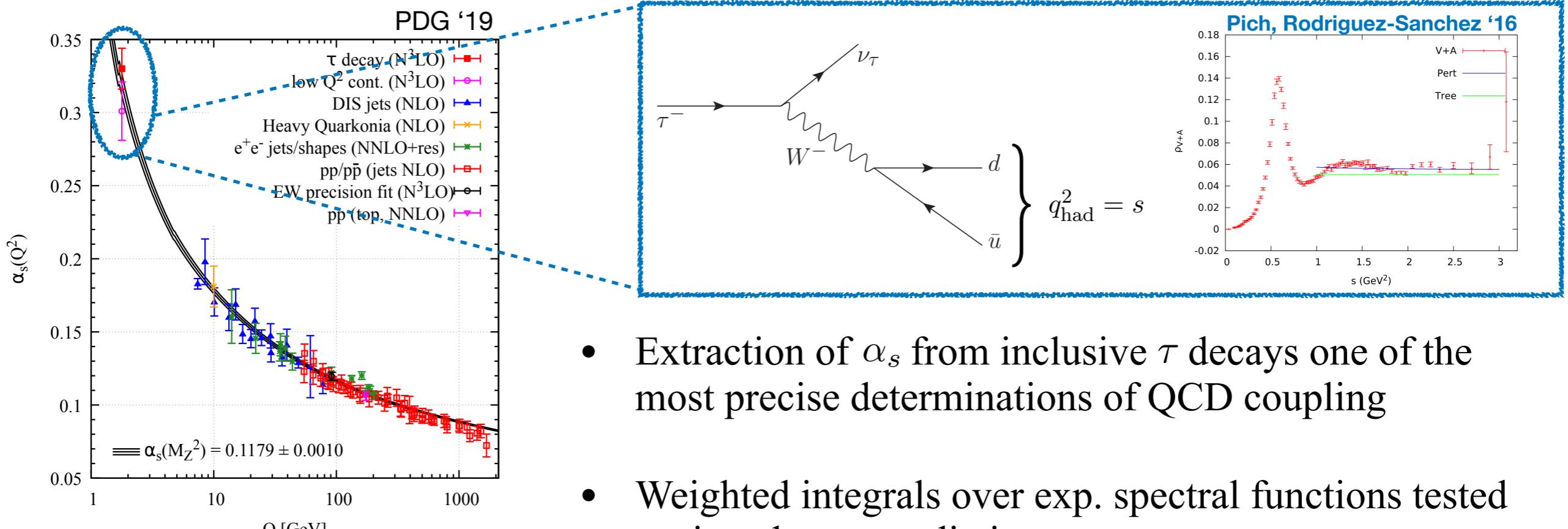
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MBR, Boito, Hoang, Jamin, arXiv:2202.10957 (JHEP 07 (2022) 016)
MBR, Boito, Hoang, Jamin, arXiv:2207.01116 (JHEP 09 (2022) 223)

Strong coupling determinations



$$R_{V+A}^{(w)}(s_0) \sim 12\pi^2 \int_0^{s_0} \frac{ds}{s_0} w(s/s_0) \rho_{V+A}(s) \quad \rho(s) = \frac{1}{\pi} \text{Im}[\Pi(s + i\epsilon)]$$

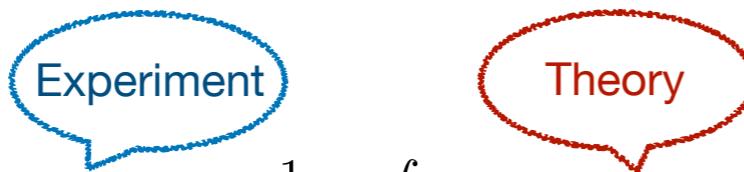
- For $w(x) = (1-x)^2(1+2x)$

$$R_{V+A}^{(w)}(m_\tau^2) \simeq \Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})$$

Hadronic tau decays: Theory

Massless correlation function: $\Pi_{\mu\nu}(p) \equiv i \int dx e^{ipx} \langle \Omega | T\{j_\mu(x) j_\nu(0)^\dagger\} | \Omega \rangle$
 (Only consider vector correlator)

Use Finite Energy Sum Rule (FESR) to relate experiment and theory
 (Cauchy's theorem)



$$\frac{1}{s_0} \int_0^{s_0} ds w(s) \rho(s) = -\frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) \Pi(s) \sim \delta_W^{\text{tree}} + \delta_W^{(0)}(s_0) + \sum_{d \geq 4} \delta_{W,V/A}^{(d)}(s_0) + \delta_{W,V/A}^{(\text{DV})}(s_0)$$

Perturbative contribution:
 (Coefficients known up to $\mathcal{O}(\alpha_s^4)$)

$$\Pi^{(0)}(s) \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^n \sum_{k=0}^{n+1} c_{n,k} \ln^k \left(\frac{-s}{\mu^2} \right)$$

$$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = 1.640$$

$$c_{3,1} = 6.371 \quad \text{Gorishnii, Kataev, Larin, '91} \quad \text{Surguladze&Samuel, '91}$$

$$c_{4,1} = 49.076 \quad \text{Baikov, Chetyrkin, Kühn, '08}$$

$$c_{5,1} = 280 \pm 140 \quad \text{Beneke, Jamin, '08} \quad \text{Boito, Masjuan, Oliani, '18} \quad \text{Caprini, '19}$$

OPE contribution: $\Pi^{\text{OPE}} \sim \sum_{d=4}^{\infty} \frac{\langle \mathcal{O}_d \rangle}{s^{d/2}}$

Shifman, Vainshtein, Zakharov, '78

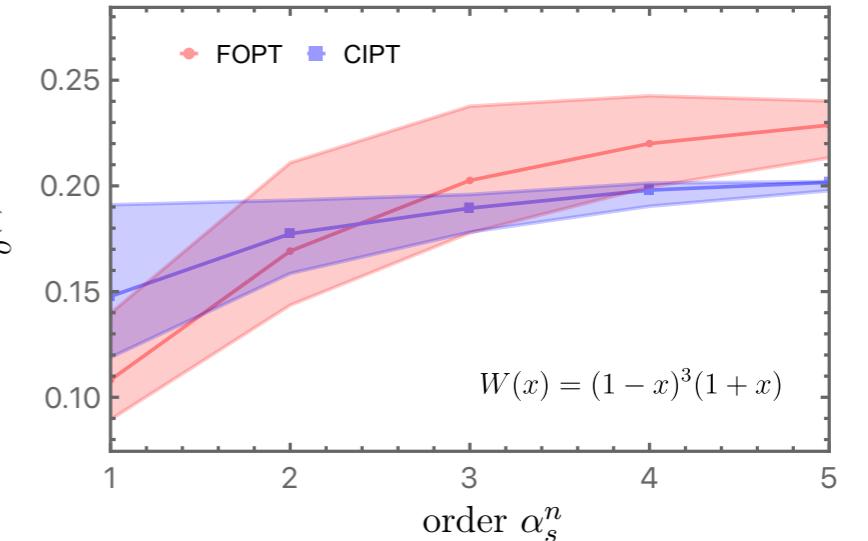
FOPT and CIPT

Fixed-Order Pt. Th. (FOPT): Fixed ren. scale $(x = s/s_0)$

$$\delta_W^{(0),\text{FO}}(s_0) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi} \right)^n \sum_{k=1}^n k c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W(x) \ln^k(-x)$$

Contour-Improved Pt. Th. (CIPT): Running ren. scale $\delta_W^{(0)}$

$$\delta_W^{(0),\text{CI}}(s_0) = \sum_{n=1}^{\infty} c_{n,1} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} W(x) \left(\frac{\alpha_s(-xs_0)}{\pi} \right)^n$$



One of the dominant sources of theoretical uncertainty: prescription for setting ren. scale,
FOPT/CIPT predictions not compatible

CIPT consistently higher than FOPT

	Pich et al. '16	Boito et al. '21
FOPT [$\alpha_s(m_\tau^2)$]	0.317(14)	0.308(7)
CIPT [$\alpha_s(m_\tau^2)$]	0.336(17)	0.324(9)

Motivation for a new scheme

Hoang, Regner '20 '21

Conclusions from Hoang, Regner:

1. CIPT/FOPT have different IR sensitivity \Rightarrow NP corrections differ
2. CIPT inconsistent with standard OPE in presence of IR renormalons in Π (or D)

↳ Asymptotic separation \equiv Discrepancy of CIPT to FOPT at asymptotically large orders

$$\sim \left(\frac{\Lambda_{\text{QCD}}^d}{s_0^{d/2}} \right) \text{ for an } \mathcal{O}(\Lambda_{\text{QCD}}^d) \text{ IR renormalon}$$

3. Effect dominated by GC renormalon

→ Starting Point of this work:

CIPT can be (largely) cured when the GC renormalon is subtracted

Analogy: quark pole mass \rightarrow short-distance mass (MSR, PS, RS, 1S)

Renormalon-Free Gluon Condensate Scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Original OPE approach (' $\overline{\text{MS}}$ ' OPE):

Barred quantities $\equiv C$ -scheme coupling (Boito, Jamin, Miravillas, '16)

$$\frac{1}{4\pi^2} [1 + D(s)] \equiv -s \frac{d}{ds} \Pi(s)$$

$$D(s) \sim \sum_{\ell} c_{\ell} \bar{a}^{\ell}(s) + \# \frac{\langle \bar{G}^2 \rangle}{s^2} + \dots$$

GC renormalon contribution



$$\langle \bar{G}^2 \rangle = \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$$

$$\bar{a}_Q \equiv \frac{\beta_0 \bar{\alpha}_s(Q^2)}{4\pi}$$

Cancellation of renormalon numerically by renormalon behavior of GC

GC renormalon in QCD

$$B[\hat{D}](u) \sim \frac{N_g}{(2-u)^{1+4\hat{b}_1}}$$

$\xrightarrow{\text{GC renormalon contr. to pert. series}}$

$$c_{\ell} \sim r_{\ell}^{(4,0)} \bar{a}_Q^{\ell} \sim N_g \frac{(\ell-1)!}{2^{\ell}} \bar{a}_Q^{\ell}$$

$$\hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

$$r_{\ell}^{(4,0)} = \left(\frac{1}{2}\right)^{\ell+4\hat{b}_1} \frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)}$$

Renormalon-Free Gluon Condensate Scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Define IR-subtracted scheme for GC

In analogy to RS scheme for quark mass advocated in Pineda '01

$$\langle \bar{G}^2 \rangle \equiv \langle G^2 \rangle(R^2) - R^4 \sum_{\ell=1} N_g r_\ell^{(4,0)} \bar{a}_R^\ell$$

IR factorization scale

$$\bar{a}_R = \frac{\alpha_s(R^2) \beta_0}{4\pi}$$

Leads to modified GC contribution $\longrightarrow B[\hat{D}^{\text{RF}}(-Q^2)](u) \sim N_g \int_0^\infty du \left[\frac{e^{-\frac{u}{\bar{a}_Q}}}{(2-u)^{1+4\hat{b}_1}} - \frac{R^4}{Q^4} \frac{e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} \right]$

Ambiguity free!

More convenient to work with scale invariant GC

$$\langle \bar{G}^2 \rangle \equiv \langle G^2 \rangle^{\text{RF}} - R^4 \sum_{\ell=1}^n N_g r_\ell^{(4,0)} \bar{a}_R^\ell + N_g \bar{c}_0(R^2)$$

\downarrow

R-dependence cancels

$$\bar{c}_0(R^2) \equiv R^4 \text{ PV} \int_0^\infty \frac{du e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}}$$

Treated as ‘tree level’ contribution

Expand consistently in $\alpha_s(\mu^2)$ at a common renormalization scale μ

Borel model study

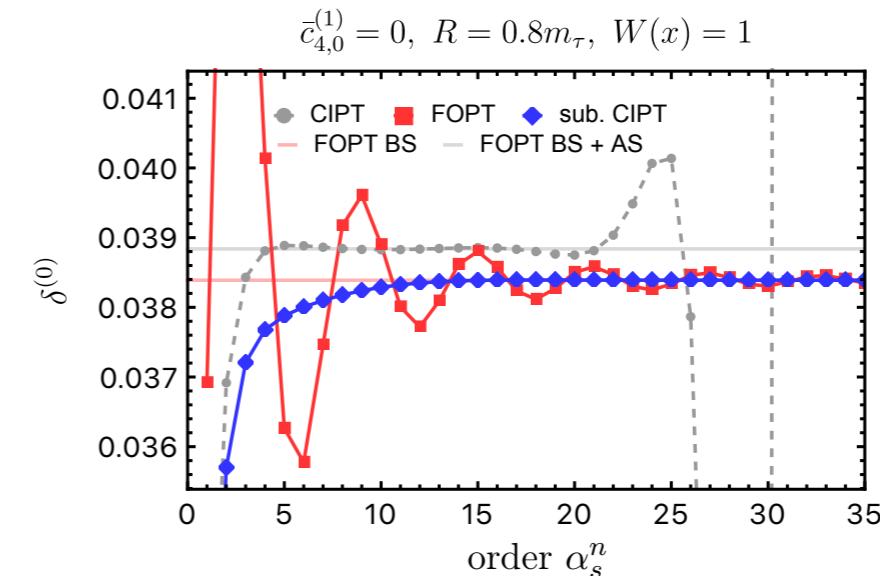
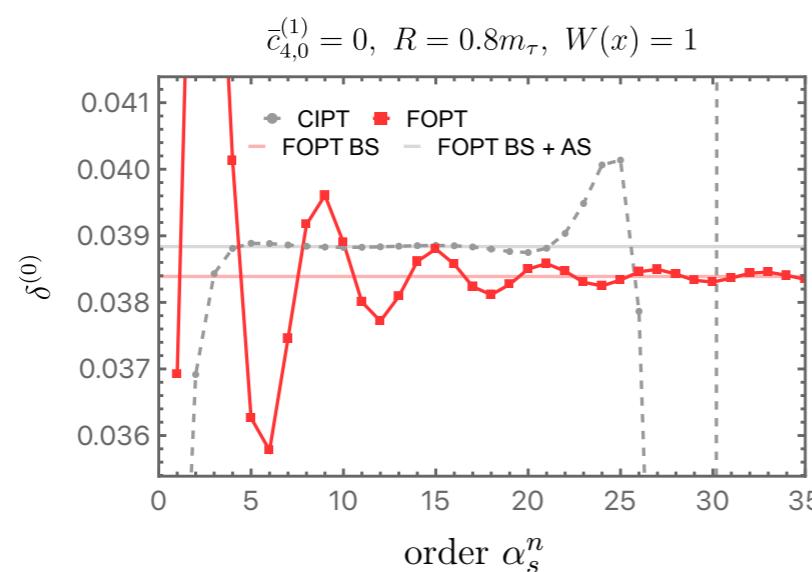
MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Take pure GC renormalon model

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}}$$

$$\longrightarrow \hat{D}^{\text{model}}(s) = \sum_{\ell=1}^{\infty} r_{\ell}^{(4,0)} \bar{a}^{\ell}(-s) \quad N_g = 1$$

Use GC suppressed moment



CIPT not consistent with standard OPE

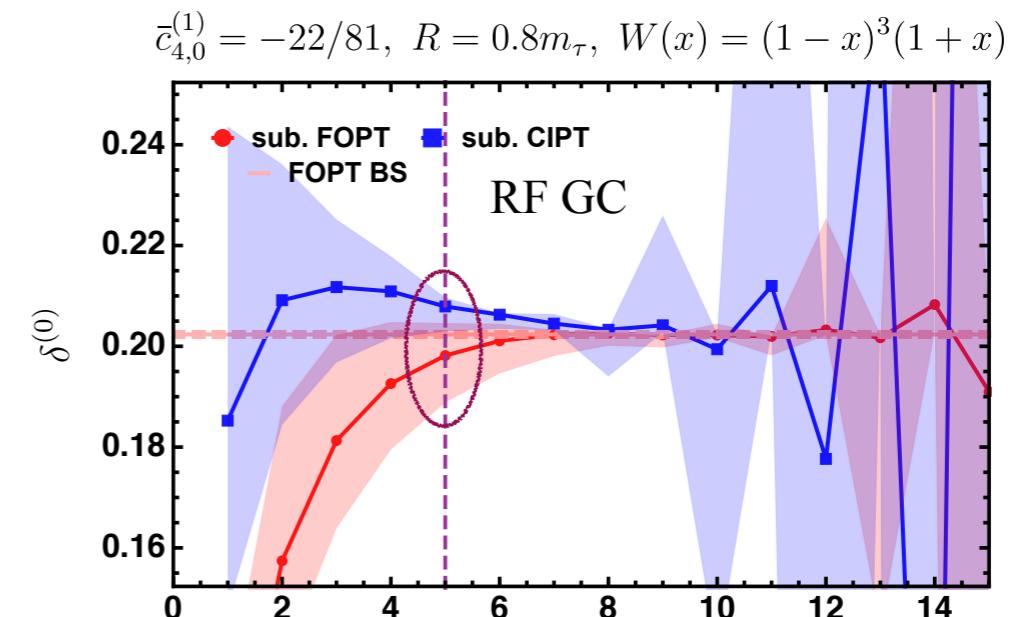
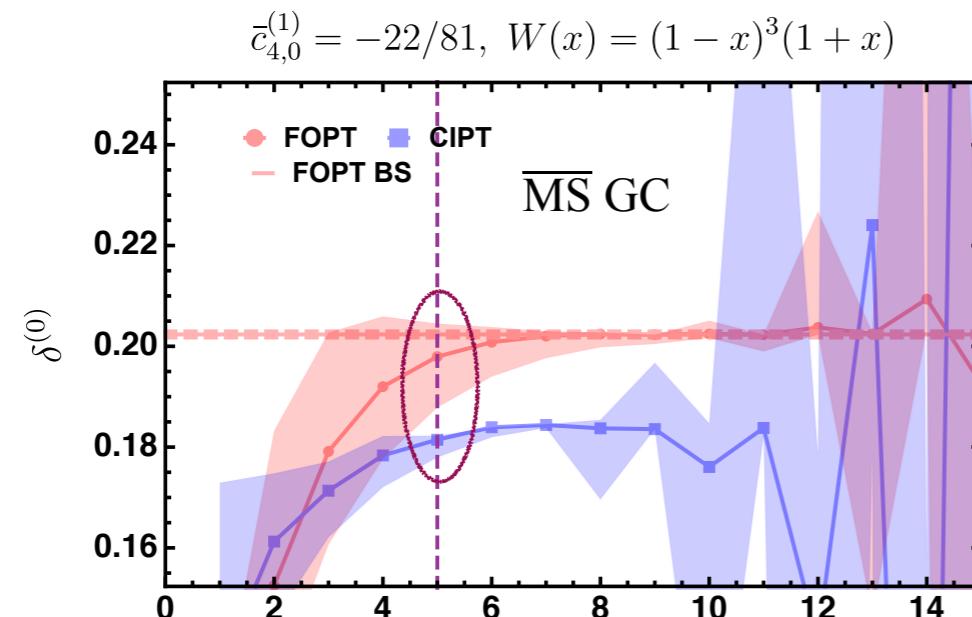
CIPT cured (+ better convergence than FOPT)

Renormalon-Free Gluon Condensate Scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

Use Multi-Renormalon model (MRM) ($N_g = 0.64$)

Beneke, Jamin '08



Observations:

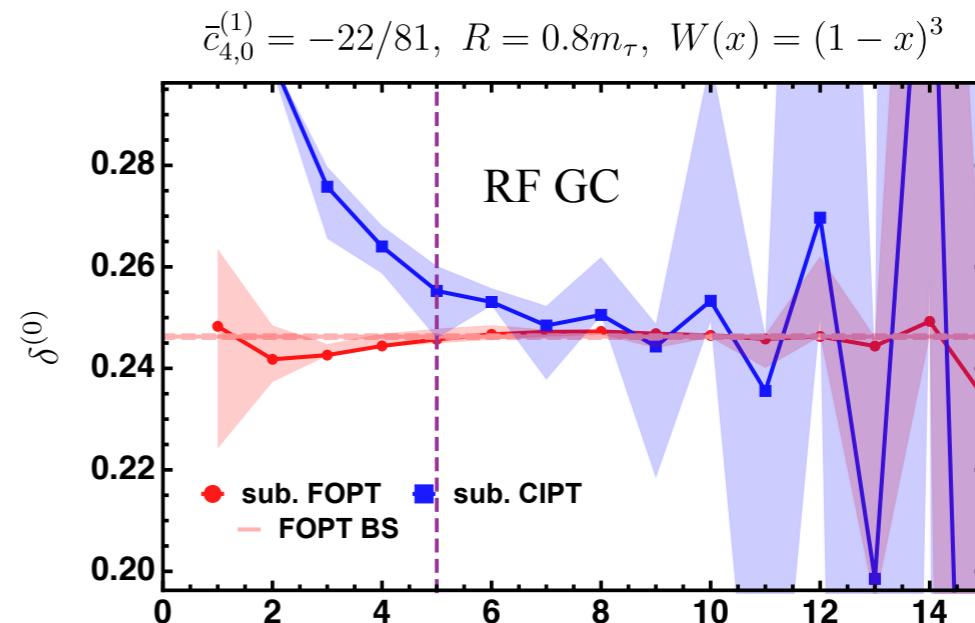
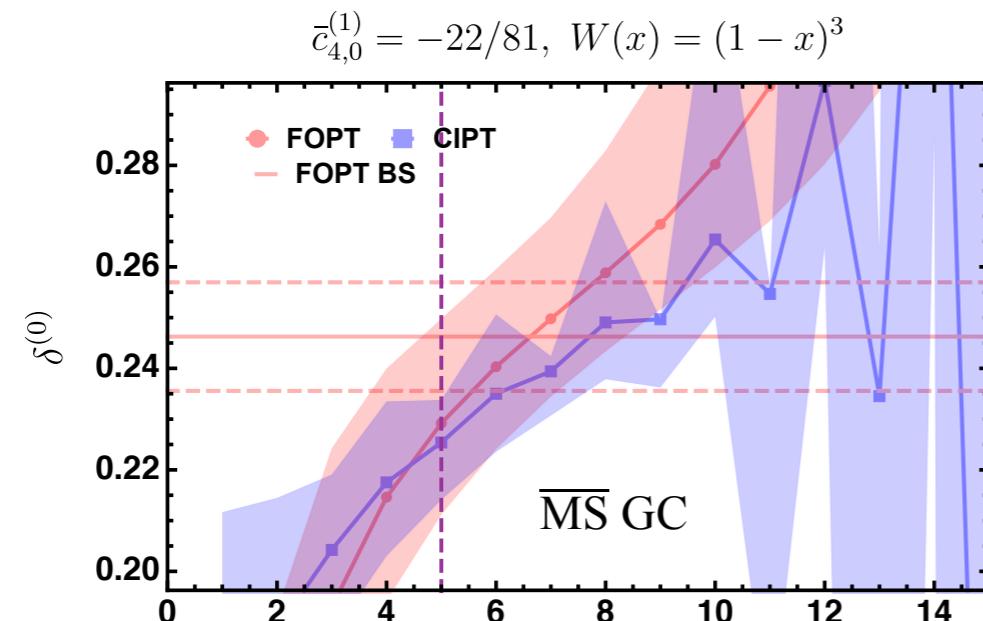
- Discrepancy between CIPT/FOPT removed in RF GC scheme
- CIPT/FOPT consistent within ren. scale variation errors already at $\mathcal{O}(\alpha_s^5)$

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Determination of the norm of the GC

MBR, Boito, Hoang, Jamin, 2207.01116

Determine norm in three different ways:

→ Varying the Borel Model

Beneke, Jamin '08

Use conformal mapping methods

Lee '12 Caprini, Fischer '09

Optimal subtraction approach

Borel Model approach:

Beneke, Jamin '08

$$B[\hat{D}(s)]_{\text{mr}}(u) = b^{(0)} + b^{(1)}u + \frac{2\pi^2}{3} \frac{N_g [1 - \frac{22}{81}\bar{a}(-s)]}{(2-u)^{1+4\hat{b}_1}} + \frac{N_6}{(3-u)^{1+6\hat{b}_1}} + \frac{N_{-2}}{(1+u)^{2-2\hat{b}_1}}$$

$$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = 1.640$$

$$c_{3,1} = 6.371$$

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Result we obtain using this method:

$$N_g = 0.64 \pm 0.27$$

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Conformal mapping method:

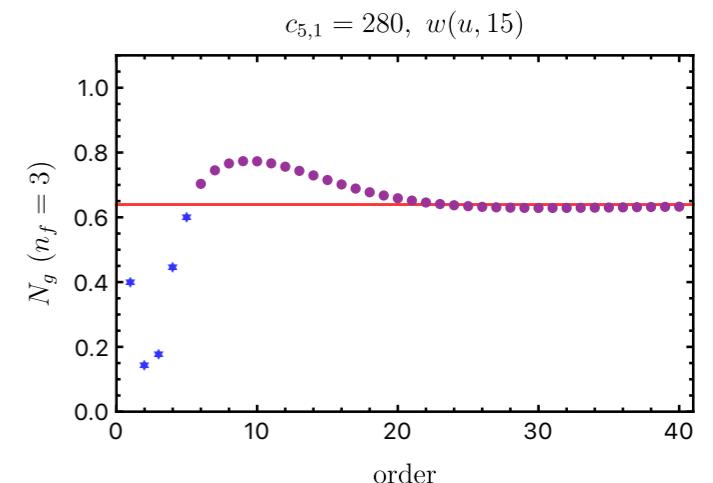
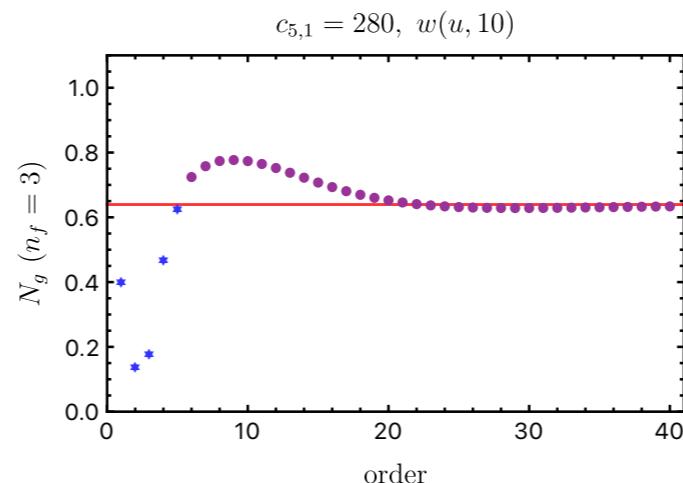
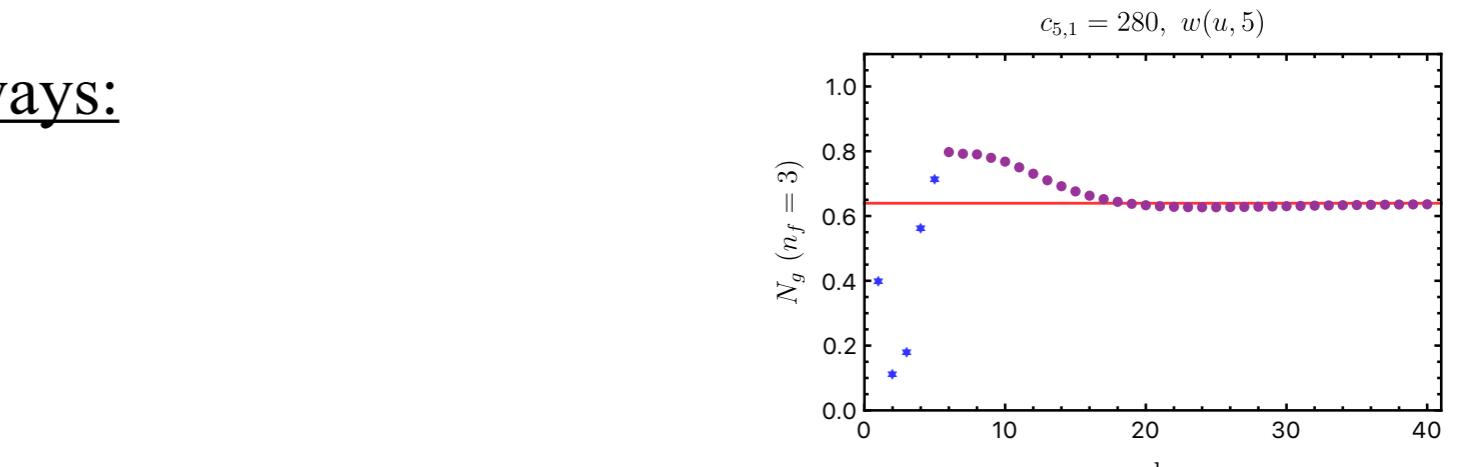
Lee '12 Caprini, Fischer '09

Consider the function

$$\tilde{B}(u) \equiv \frac{3(2-u)^{1+4\hat{b}_1}}{2\pi^2} B[\hat{D}(s)](u)$$

Idea: Apply conformal transformations such that
u=2 closest singularity to the origin in the w -plane

$$w(u, p) = \frac{\sqrt{1+u} - \sqrt{1-\frac{u}{p}}}{\sqrt{1+u} + \sqrt{1-\frac{u}{p}}} \rightarrow N_g = \tilde{B}(w(2, p))$$



Results for N_g we obtain using this method:

	$w(u, 5)$	$w(u, 10)$	$w(u, 15)$
$\mathcal{O}(\alpha_s^4)$	0.57	0.47	0.45
$\mathcal{O}(\alpha_s^5)$	0.72 ± 0.24	0.63 ± 0.17	0.60 ± 0.15

Determination of the norm of the GC

MBR, Boito, Hoang, Jamin, 2207.01116

Determine norm in three different ways:

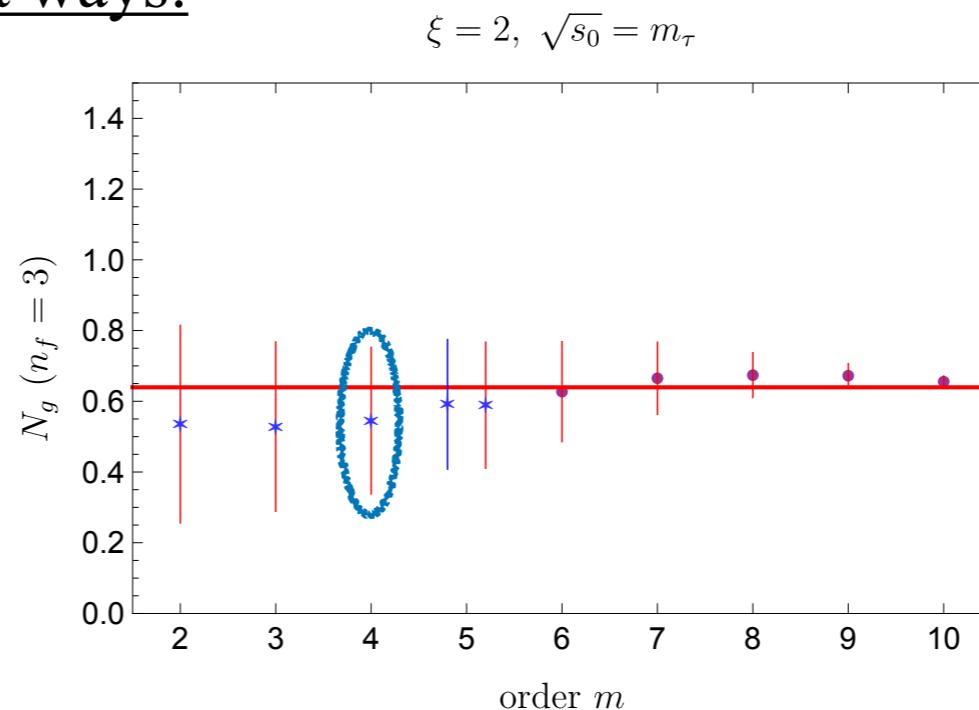
Varying the Borel Model

Beneke, Jamin '08

Use conformal mapping methods

Lee '12 Caprini, Fischer '09

→ Optimal subtraction approach



- Converges to correct N_g for MRM
- Use known $\mathcal{O}(\alpha_s^4)$ result
- Final result: $N_g = 0.57 \pm 0.23$

Optimal subtraction approach:

- Construct a suitable χ^2 - type function
- Improvements from RF GC scheme provide quantitative measure for χ^2 - function

$$\chi_m^2(N_g) = \chi_{m,\text{GCS}}^2(N_g) + \chi_{m,\text{GCE}}^2(N_g)$$

Small discrepancy for five
GC suppressed moments

Good convergence of five
GC enhanced moments

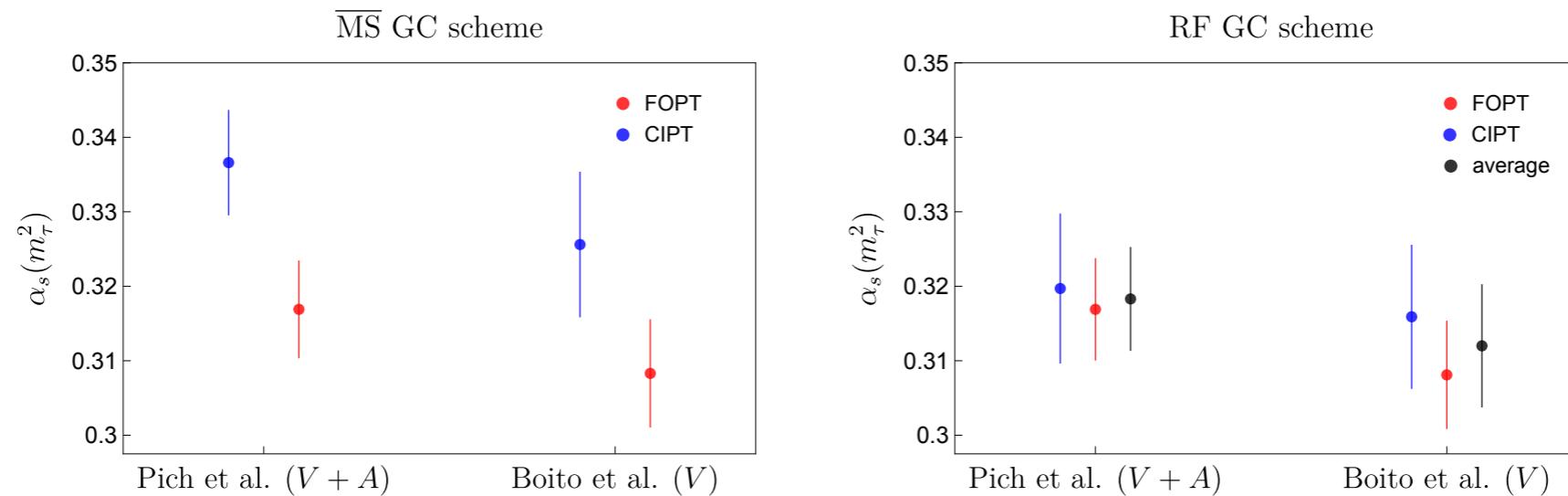
The RF GC scheme in strong coupling determinations

MBR, Boito, Hoang, Jamin, 2207.01116

Strategy:

Repeat in detail two state-of-the-art determination methods in the RF GC scheme:

- ➡ Truncated OPE approach **Pich, Rodriguez-Sanchez '16**
- ➡ Duality violation model approach **Boito, Golterman, Maltman, Peris, Rodrigues, Schaaf '21**



Observations:

- In contrast to original $\overline{\text{MS}}$ GC scheme determinations we obtain consistent results in the RG GC scheme
- CIPT becomes consistent with FOPT
- Additional errors from IR factorization scale variation as well as uncertainty related to norm of GC have minor impact on error of the strong coupling

Summary:

- Introduction of RF GC scheme
- In RF GC scheme, size of discrepancy between FOPT and CIPT strongly reduced
- In original ‘ $\overline{\text{MS}}$ ’ GC scheme, CIPT not compatible with standard OPE
- In RF GC scheme, inconsistency of CIPT w.r.t. standard OPE largely ‘cured’
- Strong coupling extractions based on FOPT/CIPT in RF GC scheme lead to compatible results in contrast to using original $\overline{\text{MS}}$ GC scheme

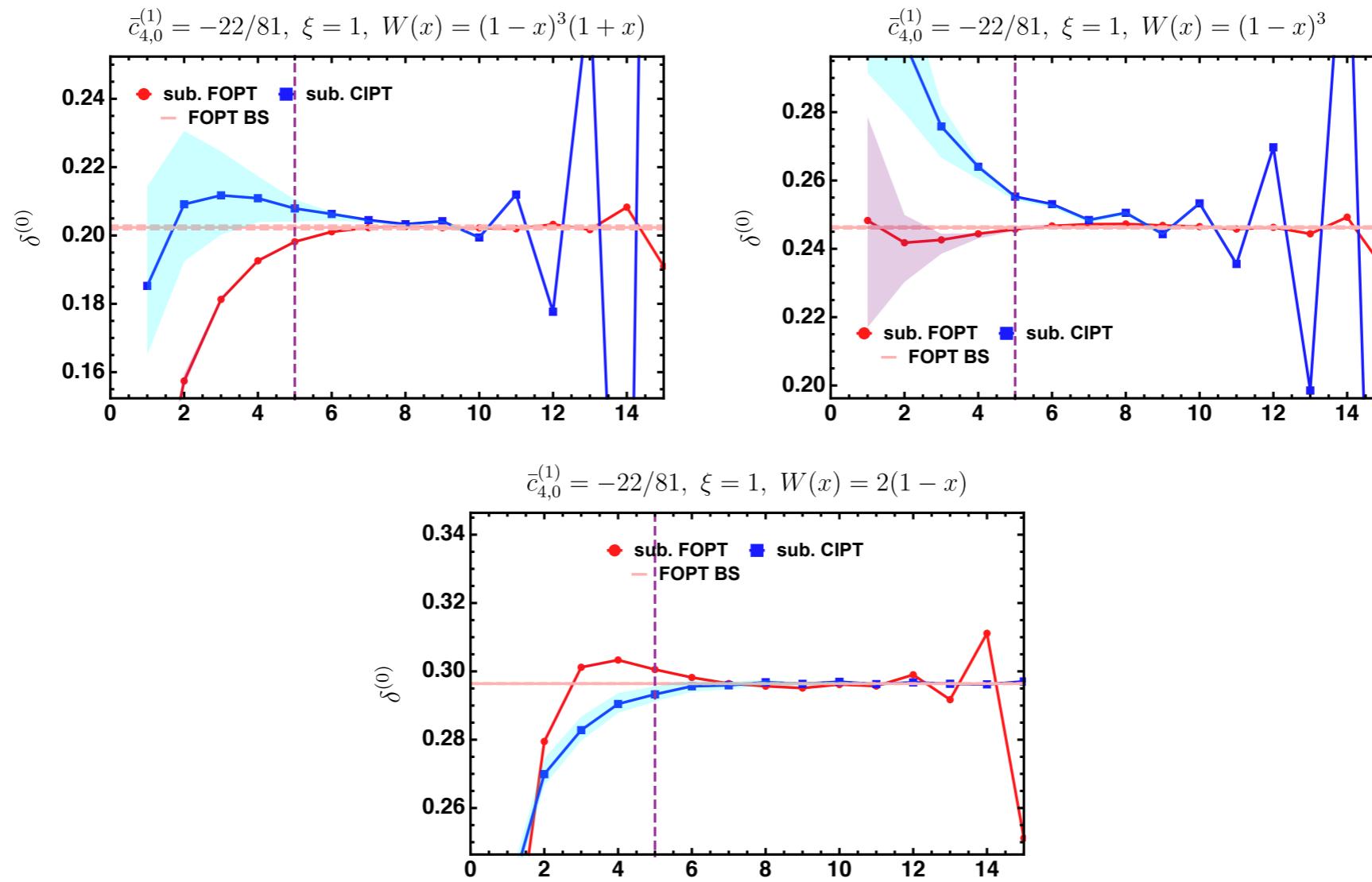
Back-up

IR factorization scale variation in the RF GC scheme

MBR, Boito, Hoang, Jamin, JHEP 07 (2022) 016

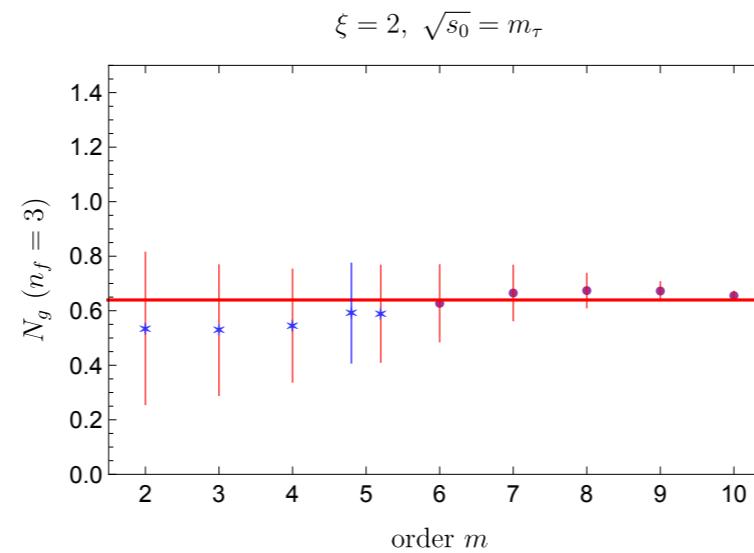
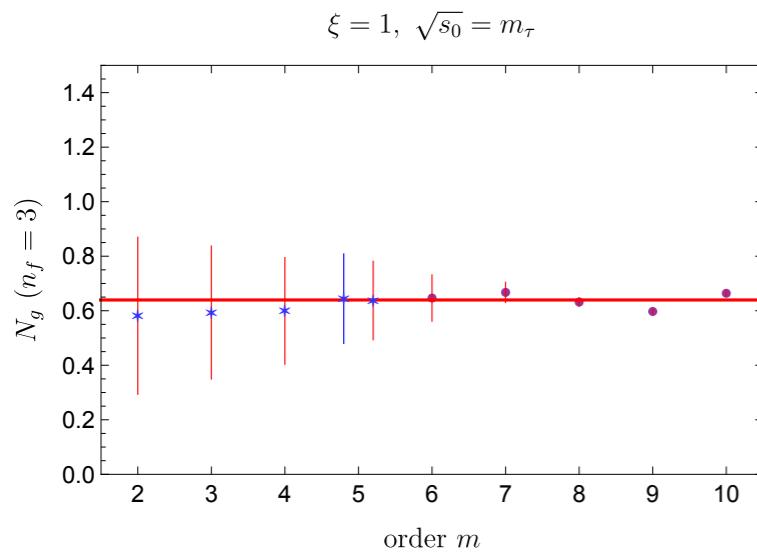
Use MRM ($N_g = 0.64$)

Beneke, Jamin '08



Results for GC norm from Optimal subtraction approach

MBR, Boito, Hoang, Jamin, 2207.01116



Conformal mapping approach:

	$w(u, 5)$	$w(u, 10)$	$w(u, 15)$
$\mathcal{O}(\alpha_s^4)$	0.57	0.47	0.45
$\mathcal{O}(\alpha_s^5)$	0.72 ± 0.24	0.63 ± 0.17	0.60 ± 0.15

$$w(u, p) = \frac{\sqrt{1+u} - \sqrt{1-\frac{u}{p}}}{\sqrt{1+u} + \sqrt{1-\frac{u}{p}}}$$

Optimal subtraction approach:

	$\sqrt{s_0} = m_\tau$	$\sqrt{s_0} = 3 \text{ GeV}$
$m = 4 (\xi = 1)$	0.60 ± 0.20	0.51 ± 0.17
$m = 4 (\xi = 2)$	0.54 ± 0.21	0.55 ± 0.11
$m = 5 (\xi = 1)$	0.64 ± 0.16	0.50 ± 0.19
$m = 5 (\xi = 2)$	0.59 ± 0.18	0.52 ± 0.15

➡ Envelope of $\mathcal{O}(\alpha_s^4)$ results our final result

$$N_g = 0.57 \pm 0.23$$

Construction of the Optimal subtraction approach

MBR, Boito, Hoang, Jamin, 2207.01116

Perturbative series in RF GC scheme

$$\delta_{w,m}^{(0),\text{FO/CI}}(N_g, s_0; \alpha_s(s_0)) = \sum_{n=0}^m r_{w,n}^{\text{FO/CI}}(N_g, R^2, \xi; s_0, \alpha_s(s_0))$$

Consider five GC suppressing (GCS) and enhancing (GCE) moments separately

$$\chi_{m,\text{GCS}}^2(N_g) = \sum_i \left(\delta_{w_i,m}^{(0),\text{CI}}(N_g) - \delta_{w_i,m}^{(0),\text{FO}}(N_g) \right)^2$$

$$\chi_{m,\text{GCE}}^2(N_g) = \sum_i \left(r_{w_i,m}^{\text{FO}}(N_g) - r_{w_i,m-1}^{\text{FO}}(N_g) \right)^2$$

Moments:

$w_1(x) = 1 - 3x^2 + 2x^3$	$w_3(x) = 1 - 5x^4 + 4x^5$
$w_2(x) = 1 - 4x^3 + 3x^4$	$w_4(x) = 1 - 6x^5 + 5x^6$
$w_5(x) = 1 - 7x^6 + 6x^7$	

GCS



Vary IR factorization scale between
 $0.7\sqrt{s_0} \leq R \leq \sqrt{s_0}$ to obtain
error estimate for norm of GC

$$w_6(x) = \frac{3}{2}(1-x)^2 = \frac{3}{2} - 3x + \frac{3x^2}{2}$$

$$w_7(x) = (1-x)^2 \left(\frac{13}{12} + \frac{5x}{3} \right) = \frac{13}{12} - \frac{x}{2} - \frac{9x^2}{4} + \frac{5x^3}{3}$$

$$w_8(x) = \frac{1}{2}(1-x)^2(5-8x) = \frac{5}{2} - 9x + \frac{21x^2}{2} - 4x^3$$

$$w_9(x) = (1-x)^2 \left(\frac{3}{2} + x - 3x^2 + x^3 \right) = \frac{3}{2} - 2x - \frac{7x^2}{2} + 8x^3 - 5x^4 + x^5$$

$$w_{10}(x) = (1-x) \left(1 - \frac{x^3}{2} + \frac{3x^4}{4} \right) = 1 - x - \frac{x^3}{2} + \frac{5x^4}{4} - \frac{3x^5}{4}$$

GCE

Final χ^2 - type function: $\chi_m^2(N_g) = \chi_{m,\text{GCS}}^2(N_g) + \chi_{m,\text{GCE}}^2(N_g)$

Sum GCS and GCE χ^2 - type functions and average over results \rightarrow final value at each m