

The role of Padé approximants in the context of the MuonE experiment

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Introduction

Introduction

$$\mu_\mu = g_\mu \frac{q}{2m_\mu} S$$

anomalous magnetic moment

$$a_\mu = \frac{g_\mu - 2}{2}$$

Dirac equation: $g_\mu = 2$

Experiment: $g_\mu > 2$

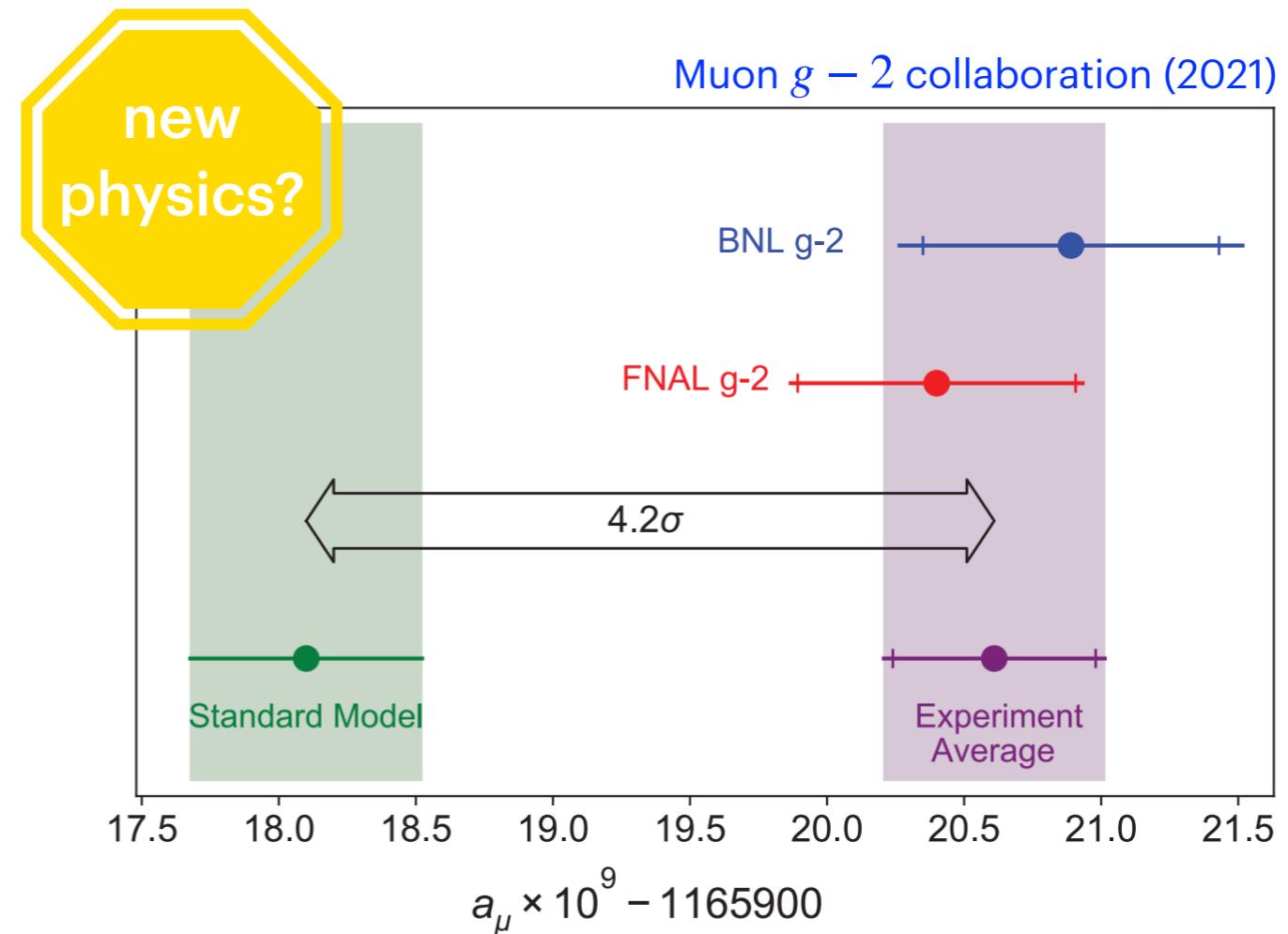
corrections due to Quantum Field Theory

Introduction

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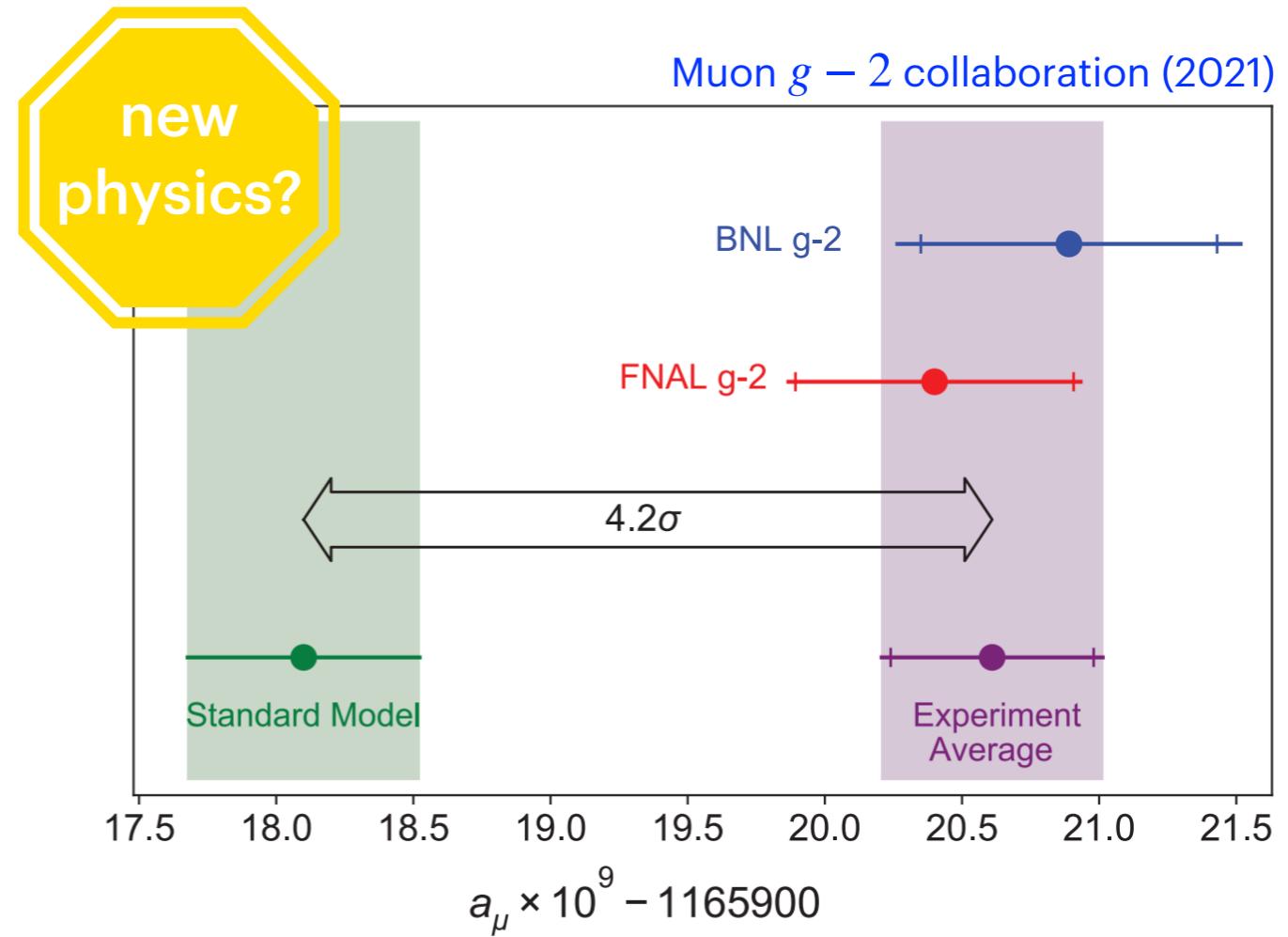


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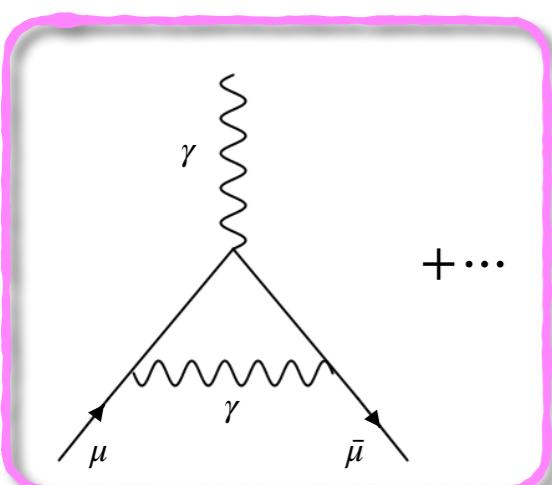
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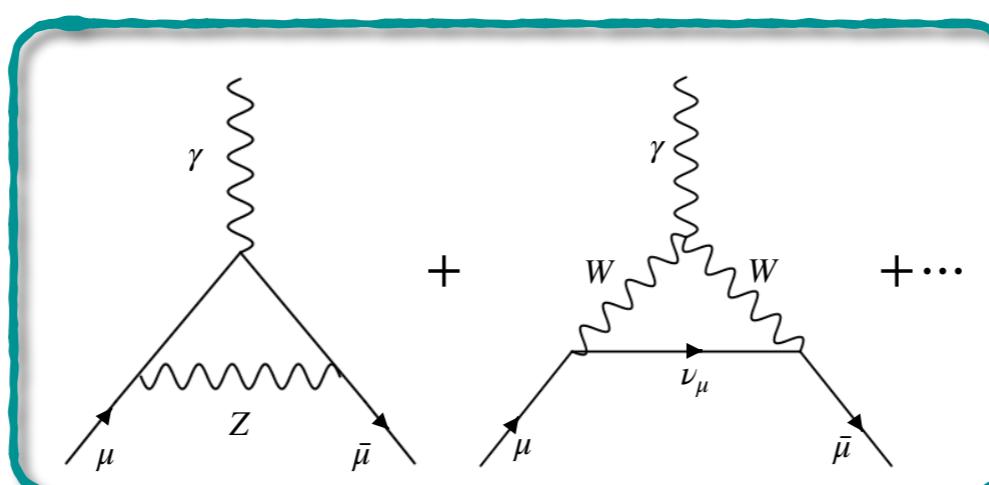
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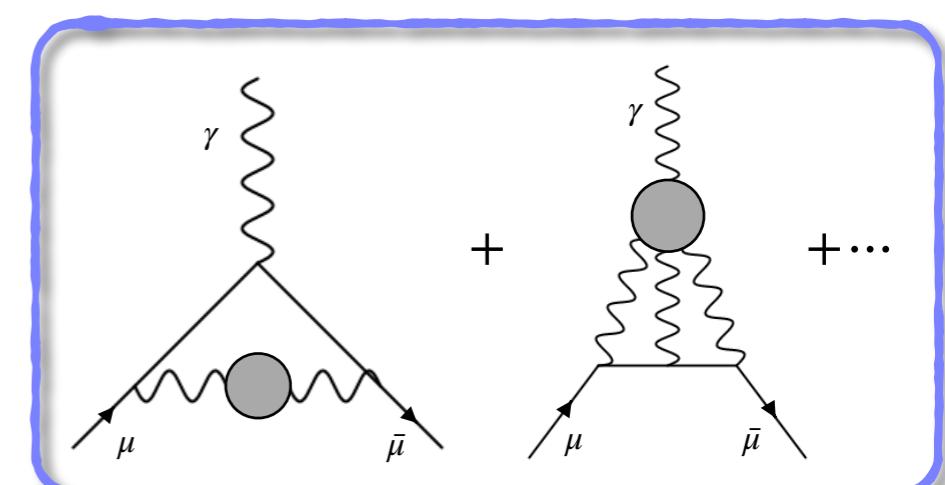
QED



electroweak



hadronic

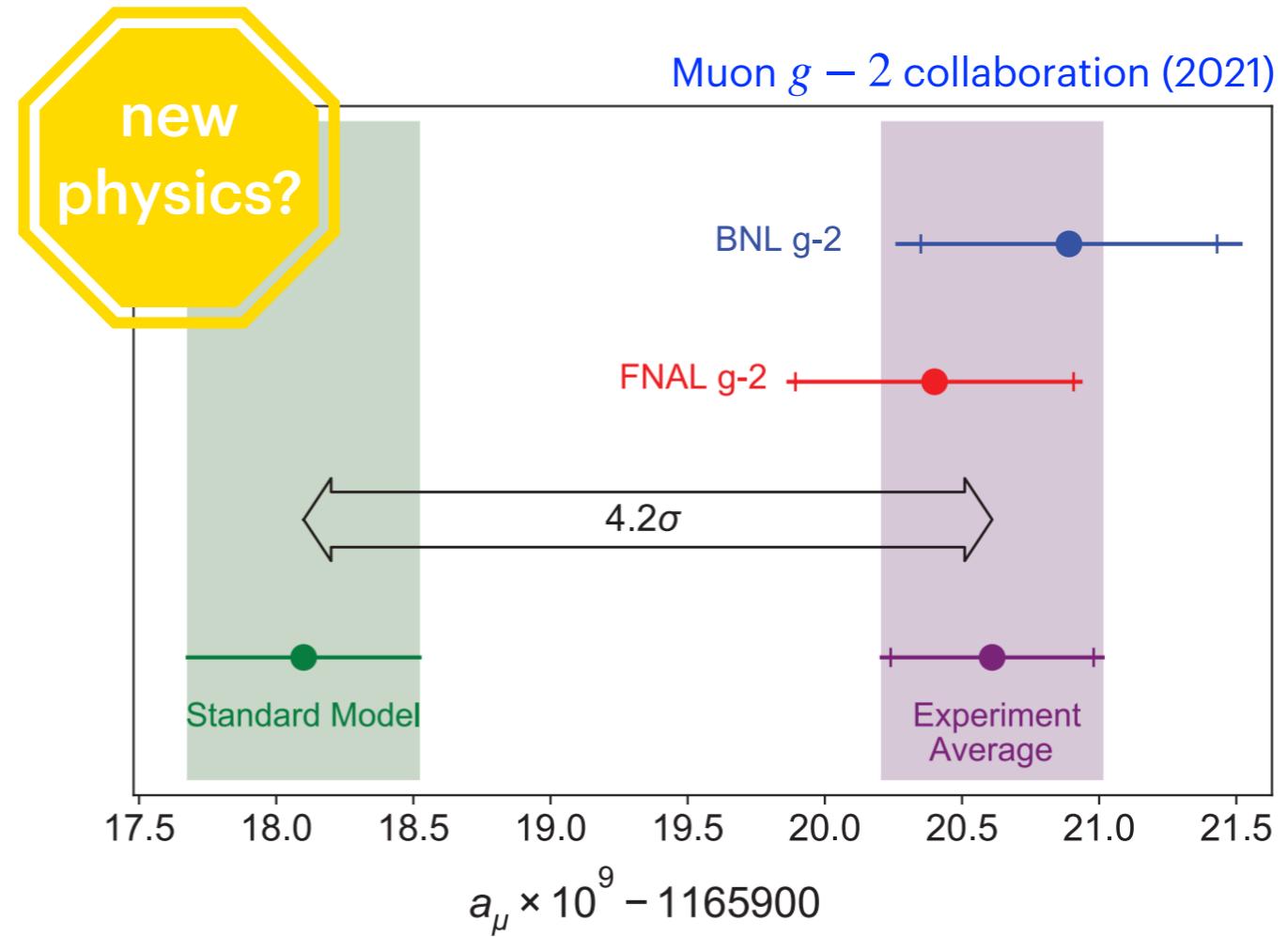


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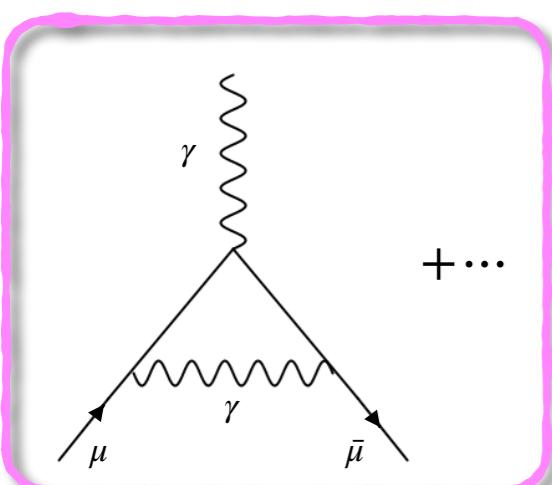
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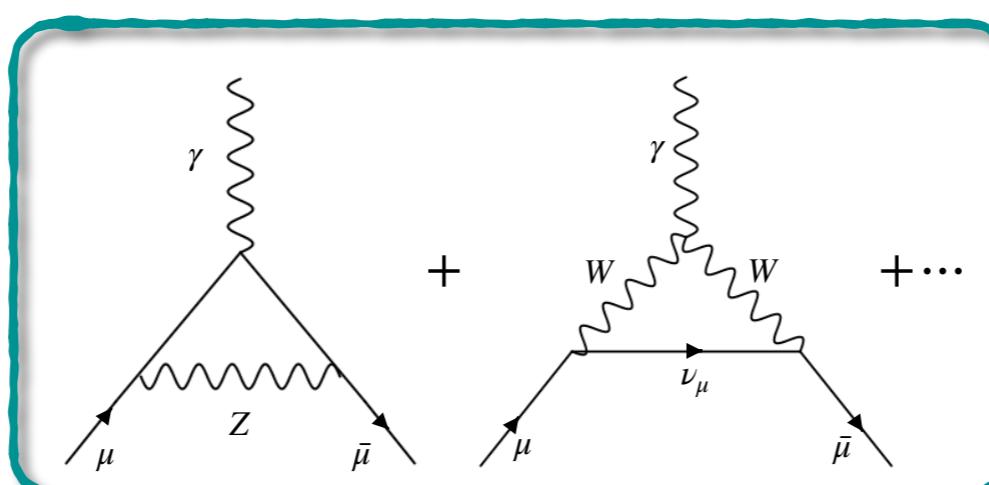
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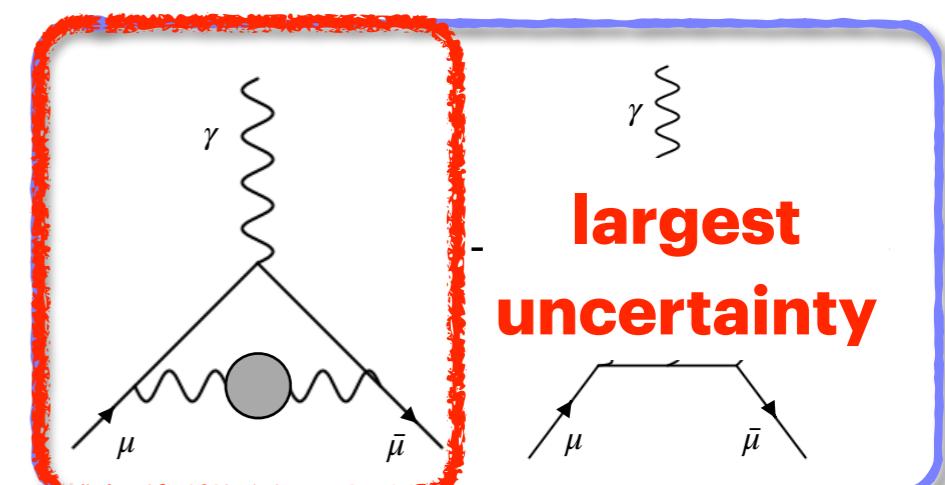
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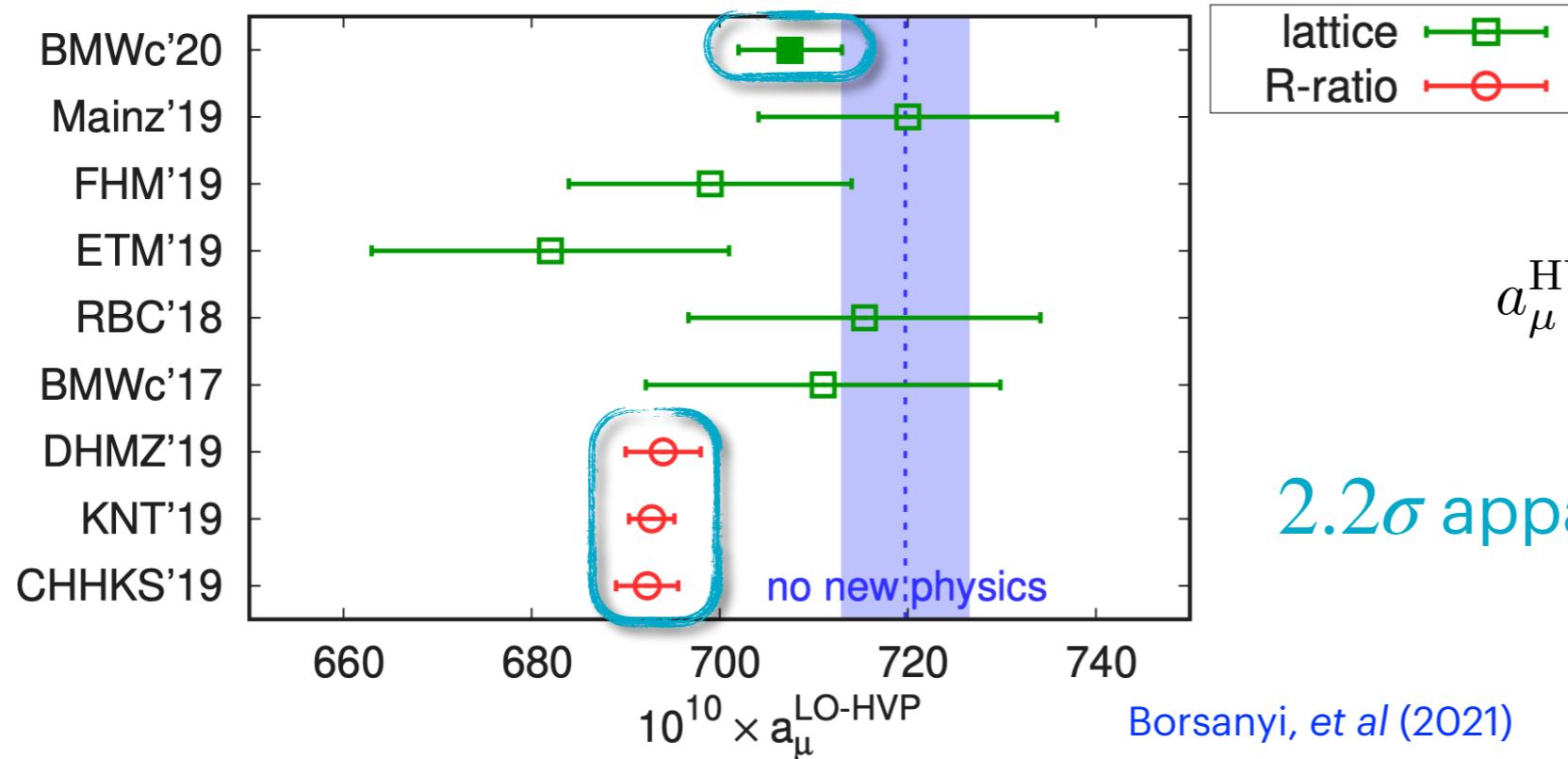
HVP, LO contribution

Leading order (LO) term of hadronic vacuum polarization (HVP) contribution to a_μ

**Usual
determination**

Lattice QCD

Cross section of $e^+e^- \rightarrow \text{hadrons}$

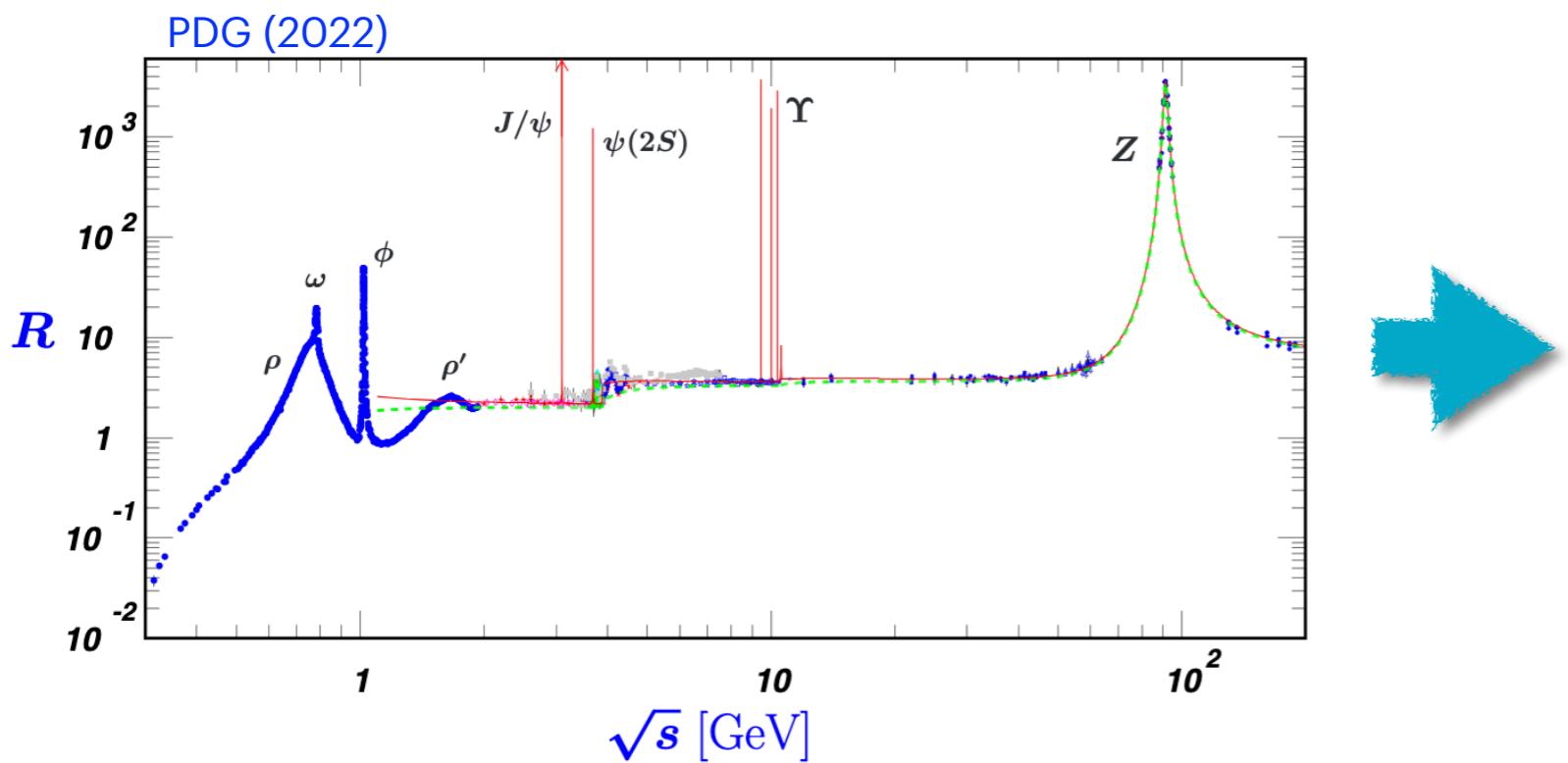


$$a_\mu^{\text{HVP, LO}} = \frac{\alpha^2}{3\pi^2} \int_{4m_\pi^2}^\infty \frac{ds}{s} K(s) R(s)$$

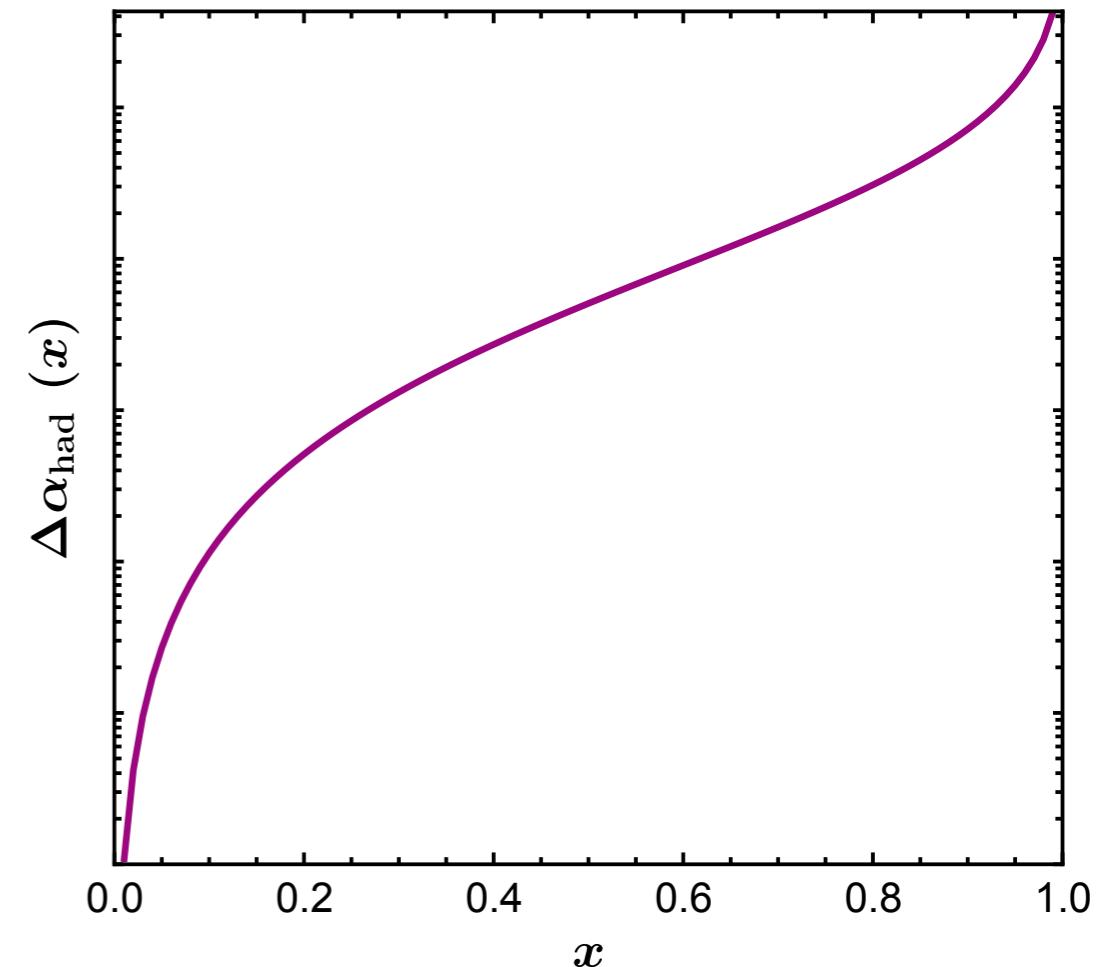
2.2 σ apart

MUonE Experiment

time-like

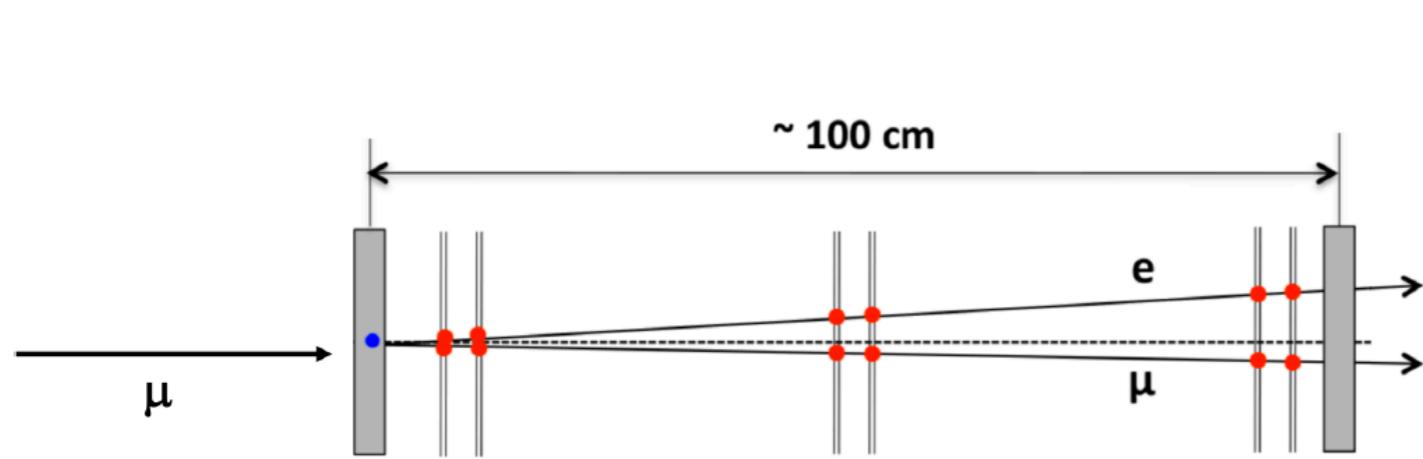


space-like



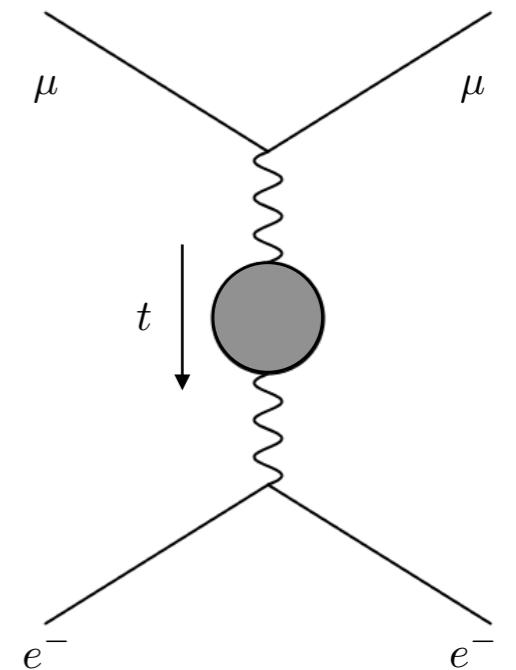
$$t = -\frac{x^2 m_\mu^2}{1-x}$$

MUonE Experiment



**space-like
region**

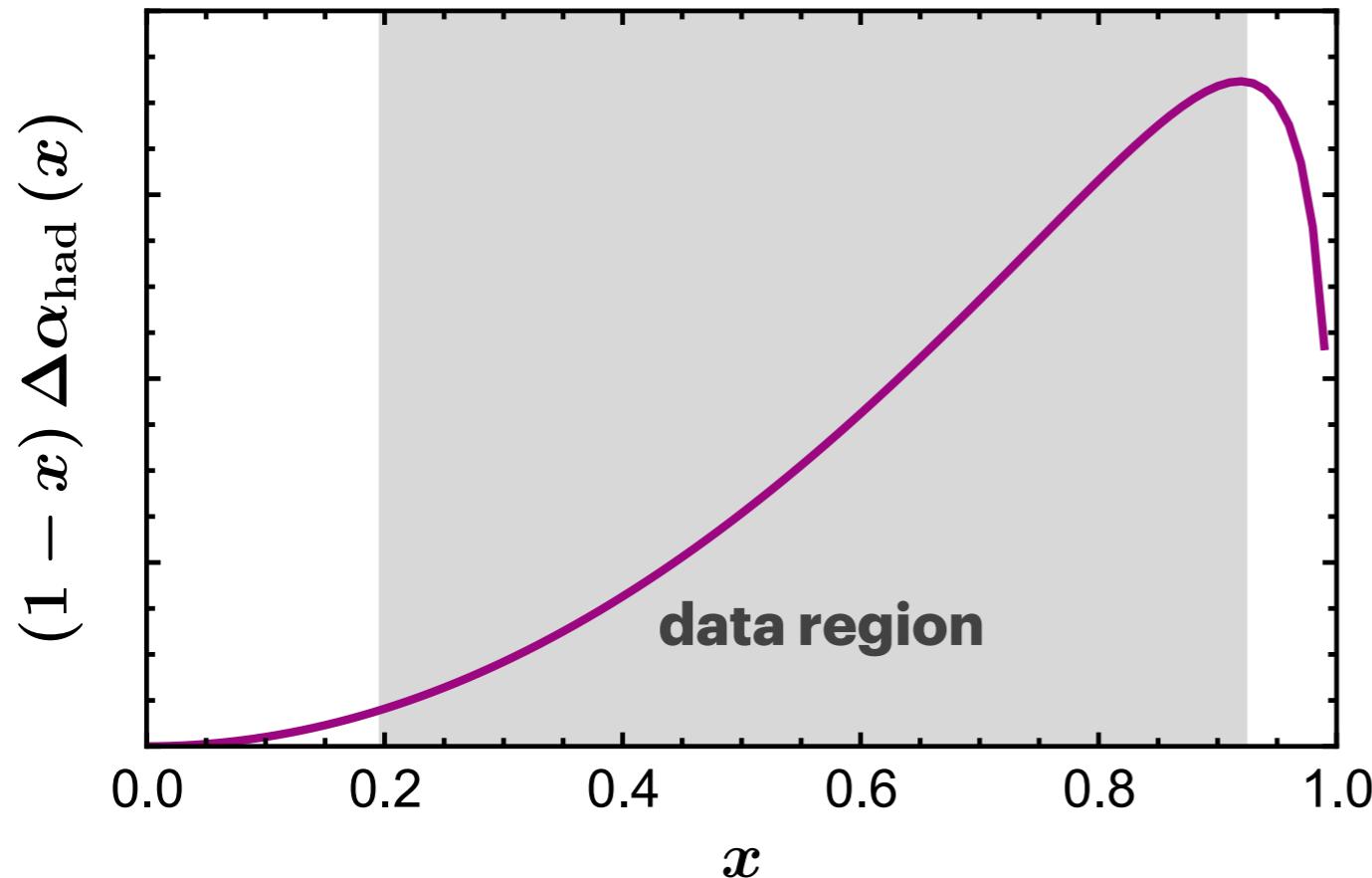
$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{\pi} \int_0^1 dx (1-x) \Delta\alpha_{\text{had}}(t)$$



$$t = -\frac{x^2 m_\mu^2}{1-x}$$

- Single scattering process: $\mu + e^- \rightarrow \mu + e^-$
- Measure the scattering angles
- Determine the hadronic running of fine structure constant $\Delta\alpha_{\text{had}}$ — proportional to cross-section

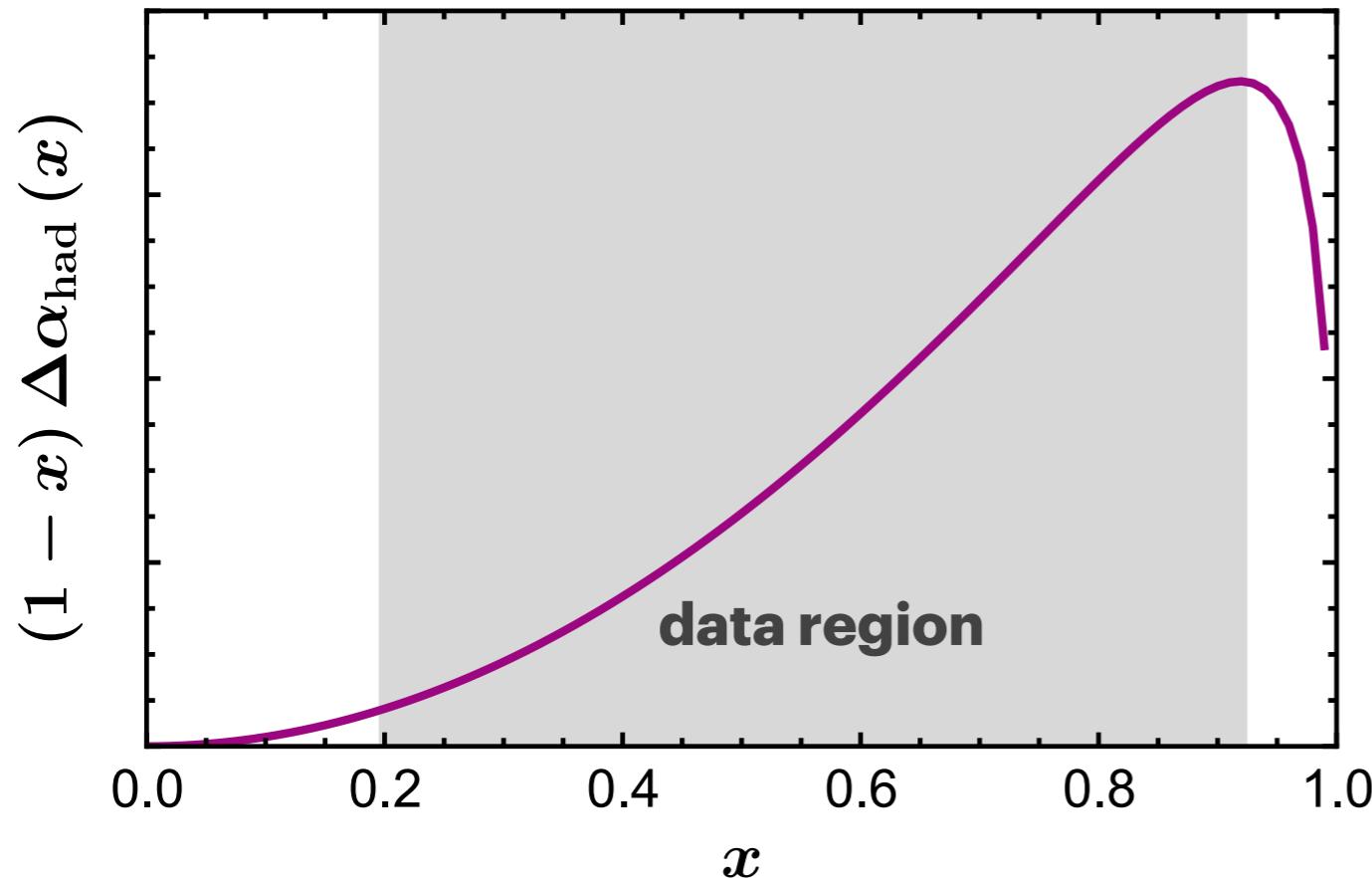
Motivation



Problem

Finding a reliable method to fit the data + good extrapolation outside data region

Motivation



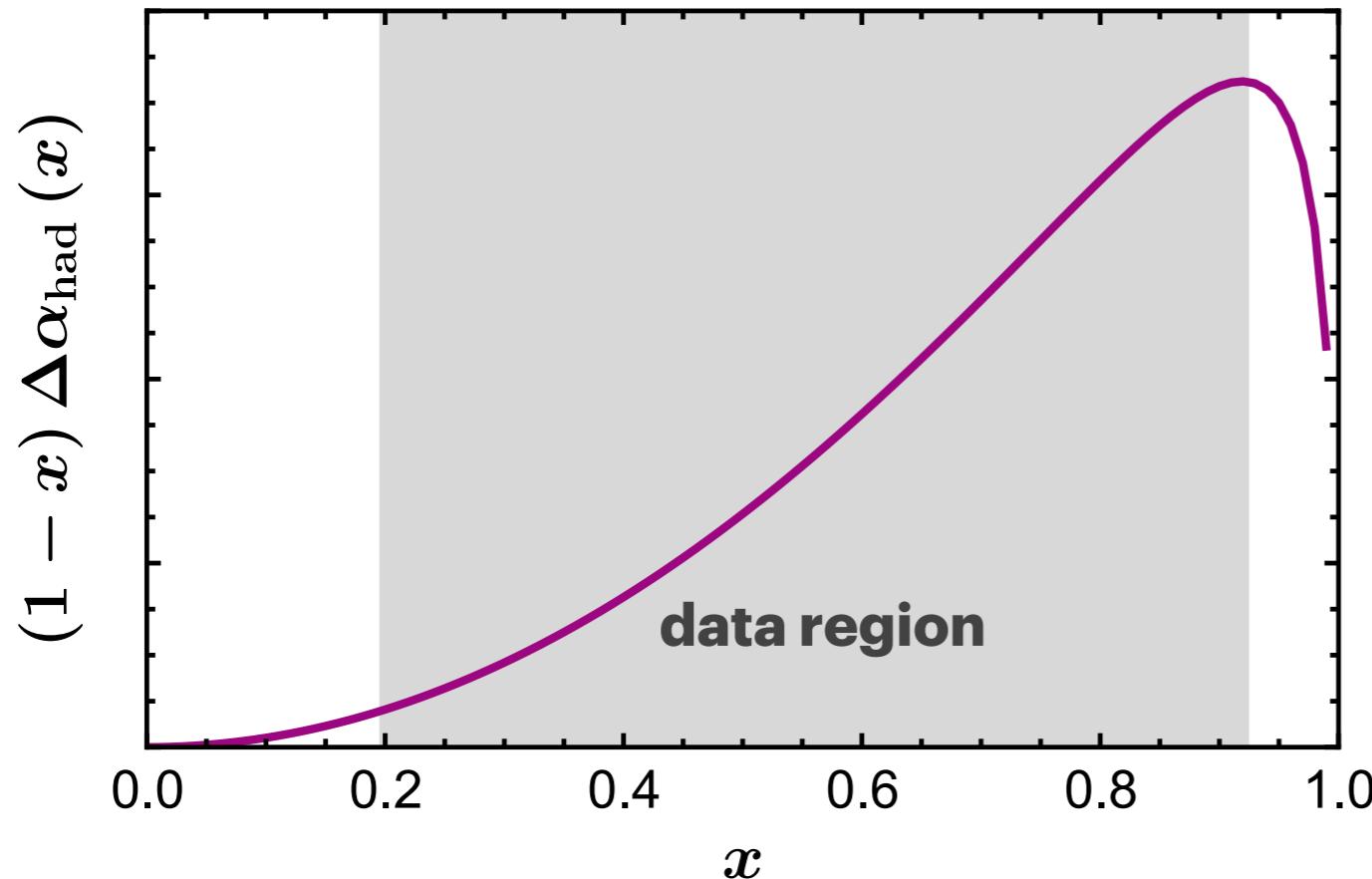
Problem

Finding a reliable method to fit the data + good extrapolation outside data region

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{q_0 + q_1 z + \cdots + q_M z^M}{1 + r_1 z + \cdots + r_N z^N}$$

Motivation



- Already applied in similar contexts:

- ❖ Masjuan, Peris (2009) [arXiv:0903.0294 \[hep-ph\]](https://arxiv.org/abs/0903.0294)
- ❖ Golterman, Maltman, Peris (2014) [arXiv:1405.2389 \[hep-lat\]](https://arxiv.org/abs/1405.2389) [arXiv:1512.07555 \[hep-lat\]](https://arxiv.org/abs/1512.07555)
- ❖ Chakraborty, Davies, de Oliveira, Koponen, Lepage, van de Water (2016)
- ❖ Aubin, Blum, Chau, Golterman, Peris, Tu (2017) [arXiv:1601.03071 \[hep-lat\]](https://arxiv.org/abs/1601.03071)
- ❖ Masjuan, Sanchez-Puertas (2017) [arXiv:1701.05829 \[hep-ph\]](https://arxiv.org/abs/1701.05829)

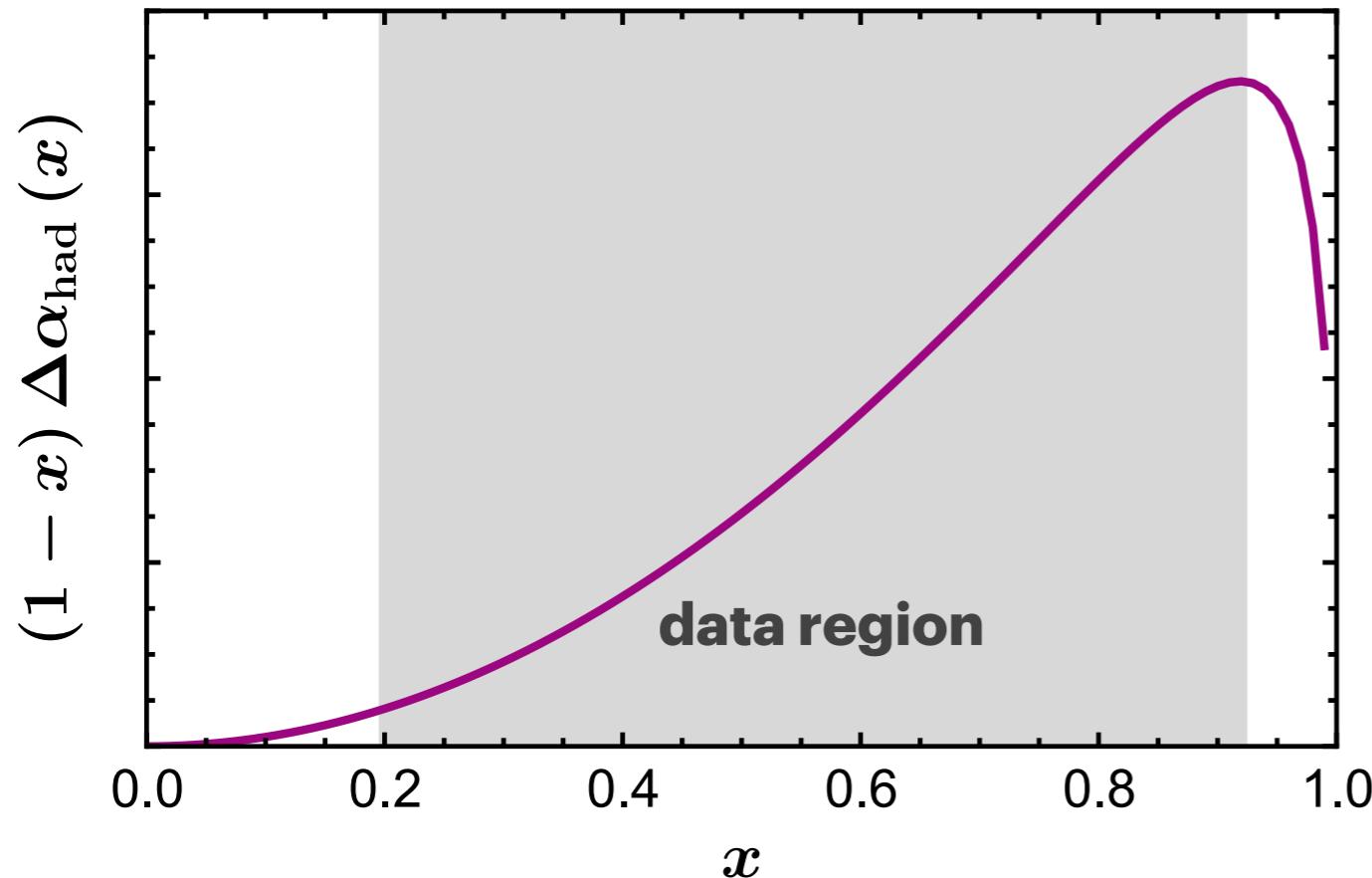
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Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{q_0 + q_1 z + \cdots + q_M z^M}{1 + r_1 z + \cdots + r_N z^N}$$

Motivation



Problem

Finding a reliable method to fit the data + good extrapolation outside data region

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{q_0 + q_1 z + \cdots + q_M z^M}{1 + r_1 z + \cdots + r_N z^N}$$

Advantages of PAs

- Systematic and model-independent method
- Partial reconstruction of analytical (physical) properties
- Efficient approximation
- Possible to provide a convergence error

Stieltjes Functions and Padés

Stieltjes function

$$f(z) = \int_0^\infty \frac{d\phi(u)}{1+zu} \quad \phi(u) \text{ is a measure in } u \in [0, \infty)$$

$$f(z) = \sum_{i=0}^{\infty} f_i (-z)^i, \quad f_i = \int_0^\infty u^i d\phi(u)$$

Stieltjes series

$$\begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \vdots & \vdots & & \vdots \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n} \end{vmatrix} > 0 \quad \begin{matrix} m \geq 0 \\ n \geq 0 \end{matrix}$$

determinant condition

- $\Delta\alpha_{\text{had}}(t)$ is a Stieltjes function in $t \in (-\infty, 0]$ Aubin, Blum, Golterman, Peris (2012)

$$\Delta\alpha_{\text{had}}(t) = \sum_{i=1}^{\infty} a_i t^i \quad \xrightarrow{\hspace{1cm}} \quad 0 < a_i < a_{i+1}, \quad i \in \mathbb{N}$$

Padé's Theory

Stieltjes function

$$f(z) = \int_0^\infty \frac{d\phi(u)}{1+zu} \quad \phi(u) \text{ is a measure in } u \in [0, \infty)$$

- There are convergence theorems for PAs to Stieltjes functions
- Some convergence properties:
 - * poles of P_N^{N+k} , $k \geq -1$, are located in the positive real axis;
 - * Padé sequences uniformly converge to the original function;
 - * Padés act as bounds of the function

$$P_1^1(t) \leq P_2^2(t) \leq \dots \leq \Delta\alpha_{\text{had}}(t) \leq \dots \leq P_1^2(t) \leq P_0^1(t)$$

Padé's Theory

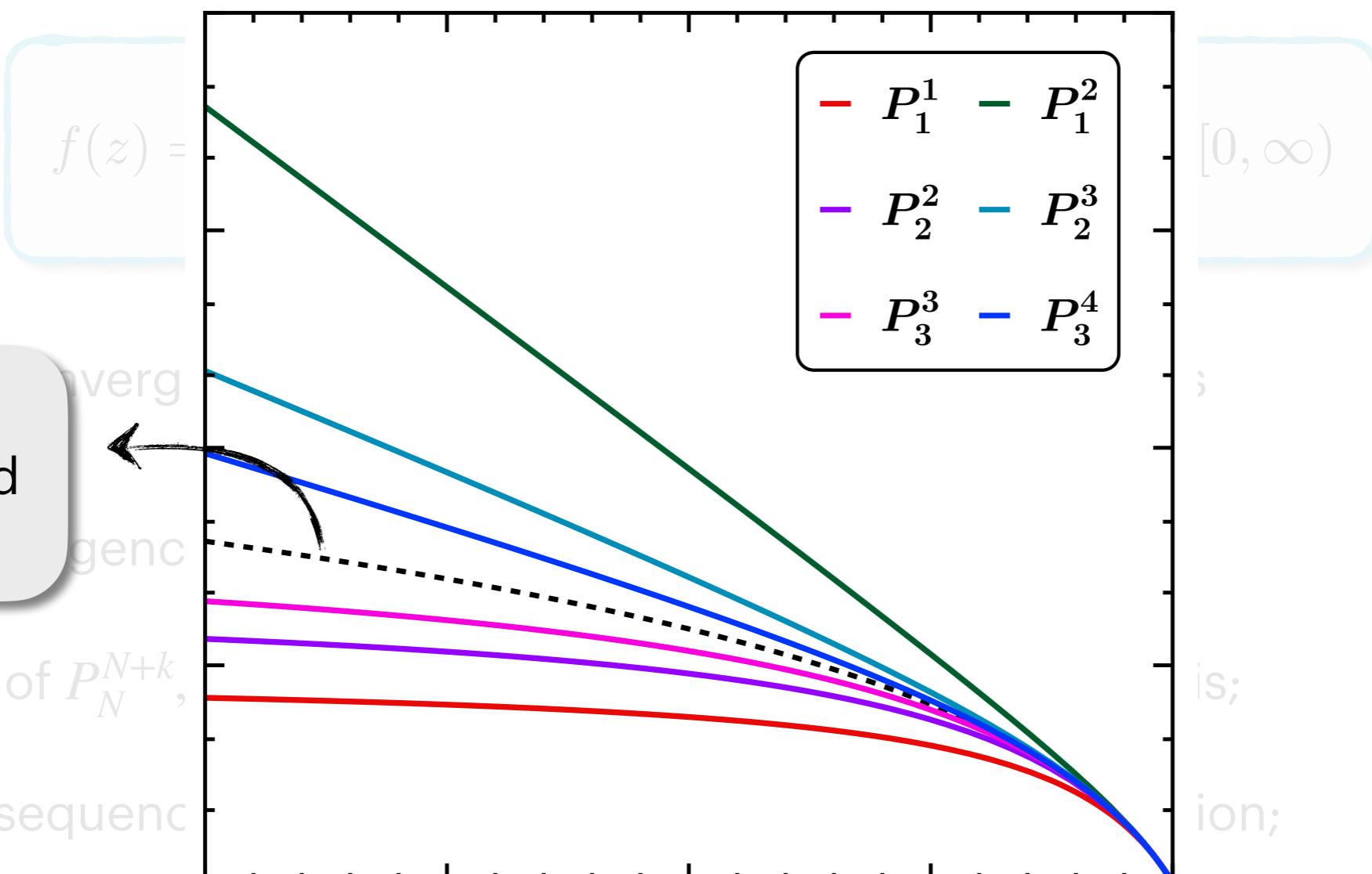
function approximated by PAs

* poles of P_N^{N+k} ,

* Padé sequences

* Padés act as bounds of the function

$$P_1^1(t) \leq P_2^2(t) \leq \dots \leq \Delta\alpha_{\text{had}}(t) \leq \dots \leq P_1^2(t) \leq P_0^1(t)$$



Method

Fitting Method

- Data generation – model for $\Delta\alpha_{\text{had}}(t)$ Greynat, de Rafael (2022)

$$(x, \alpha \Delta\alpha_{\text{had}}(x) \times 10^5)$$
$$0.2 \leq x \leq 0.93$$

30 equally spaced bins –
mean value of each bin

Fitting Method

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30 equally spaced bins –
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- Fitting function – example:

$$\Delta\alpha_{\text{had}}(t) = a_1 t + a_2 t^2 + \dots$$

$$P_1^1(t) = \frac{q_0 + q_1 t}{1 + r_1 t}$$

$$\approx q_0 + (q_1 - q_0 r_1)t + (q_0 r_1^2 - q_1 r_1)t^2 + \dots$$

$$P_1^1(t) = \frac{a_1^2 t}{1 - a_2 t}$$

Fitting Method

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- Fitting function – example:

$$P_1^1(t) = \frac{a_1^2 t}{1 - a_2 t} \quad \xrightarrow{\hspace{1cm}}$$
$$t = -\frac{x^2 m_\mu^2}{1 - x}$$

unknown Taylor
series of $\Delta\alpha_{\text{had}}(t)$

$$P_1^1(x) = -\frac{a_1^2 m_\mu^2 x^2}{a_1 - a_1 x + a_2 m_\mu^2 x^2} = -\frac{b_1 m_\mu^2 x^2}{1 - x + b_2 m_\mu^2 x^2}$$

$$b_1 = a_1 < 0 \quad b_2 = \frac{a_2}{a_1} > 1$$

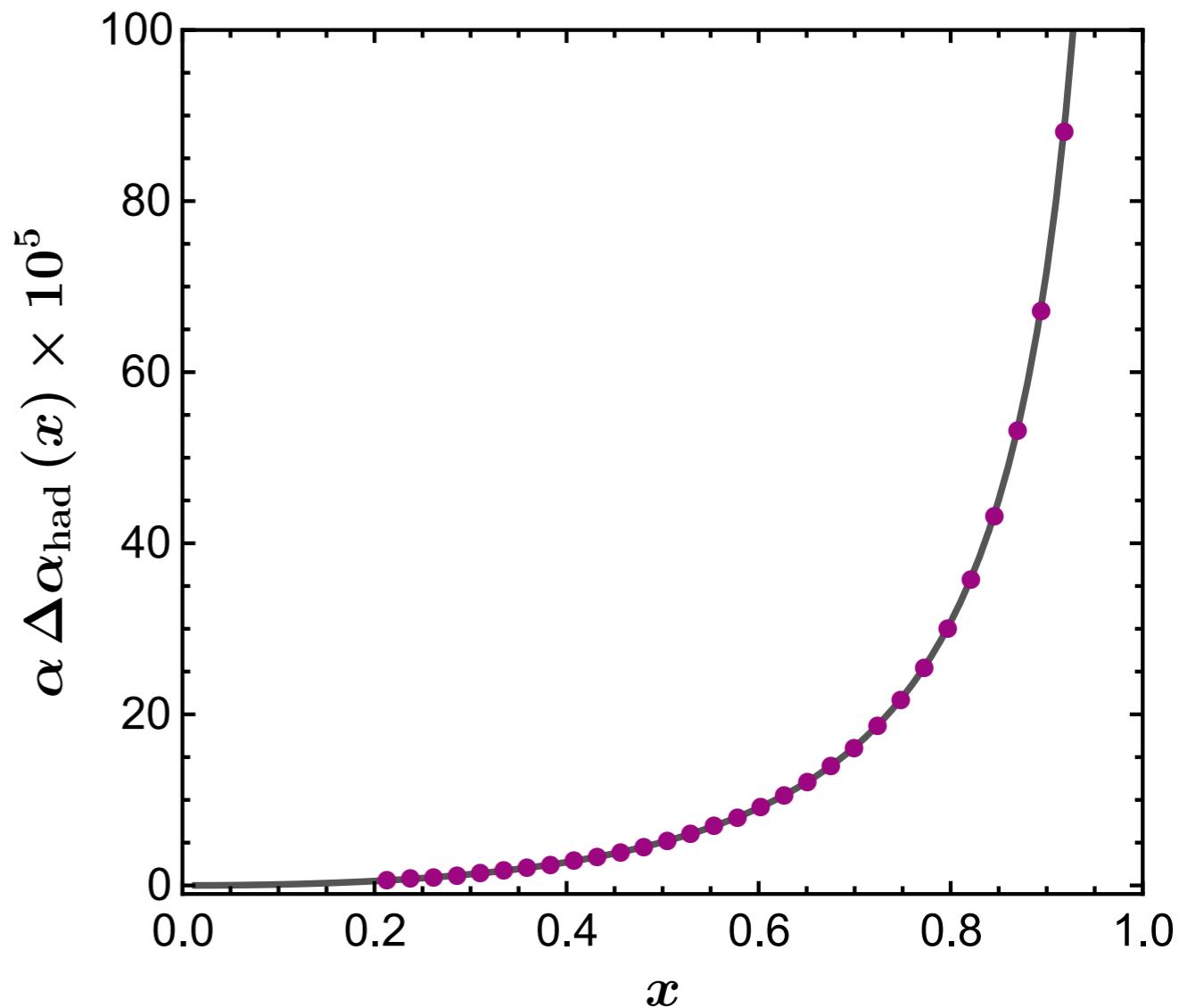
model independent constraints

- PA parameters: χ^2 minimization

Padés to the model

Ideal Scenario

- Data without fluctuations or errors — estimate method uncertainty
- PAs used to calculate $a_\mu^{\text{HVP,LO}}$ in the whole region of $x \in [0,1]$

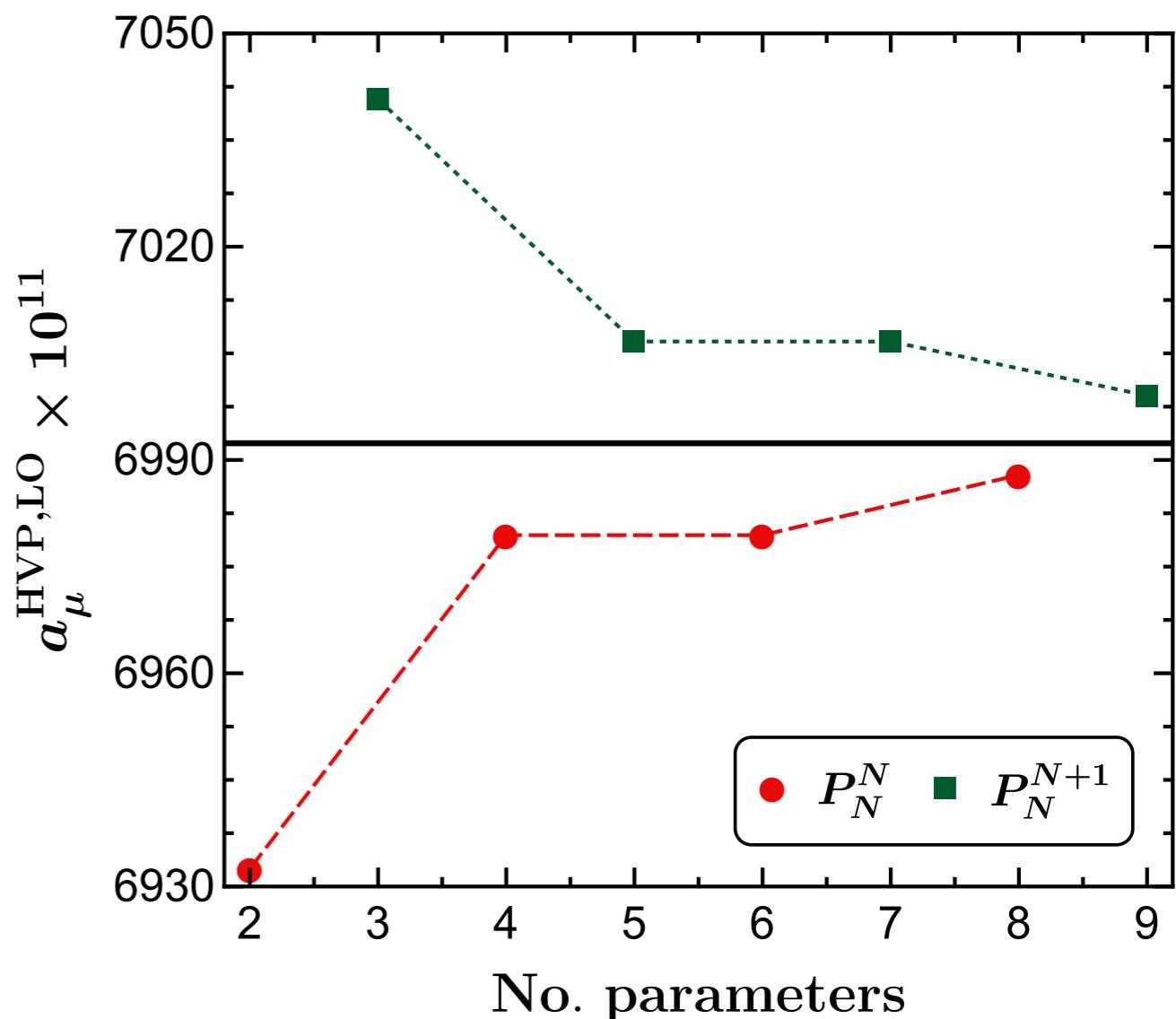


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Convergence pattern observed

PAs poles in t are all real and positive



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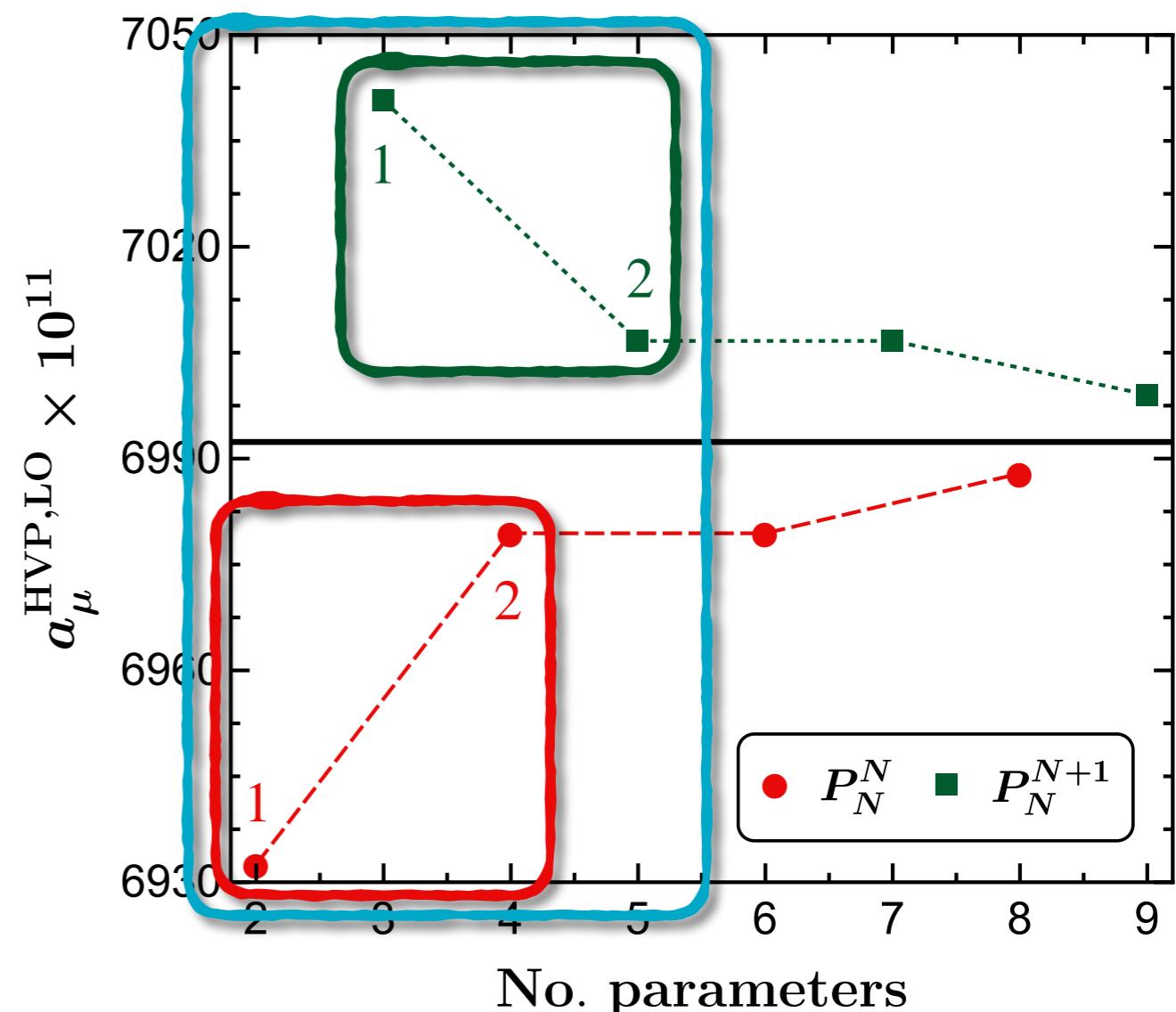
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- Final estimate — weighted mean

$$a_\mu^{\text{PAs}} = 6991 \times 10^{-11} \quad 0.02\% \text{ off}$$



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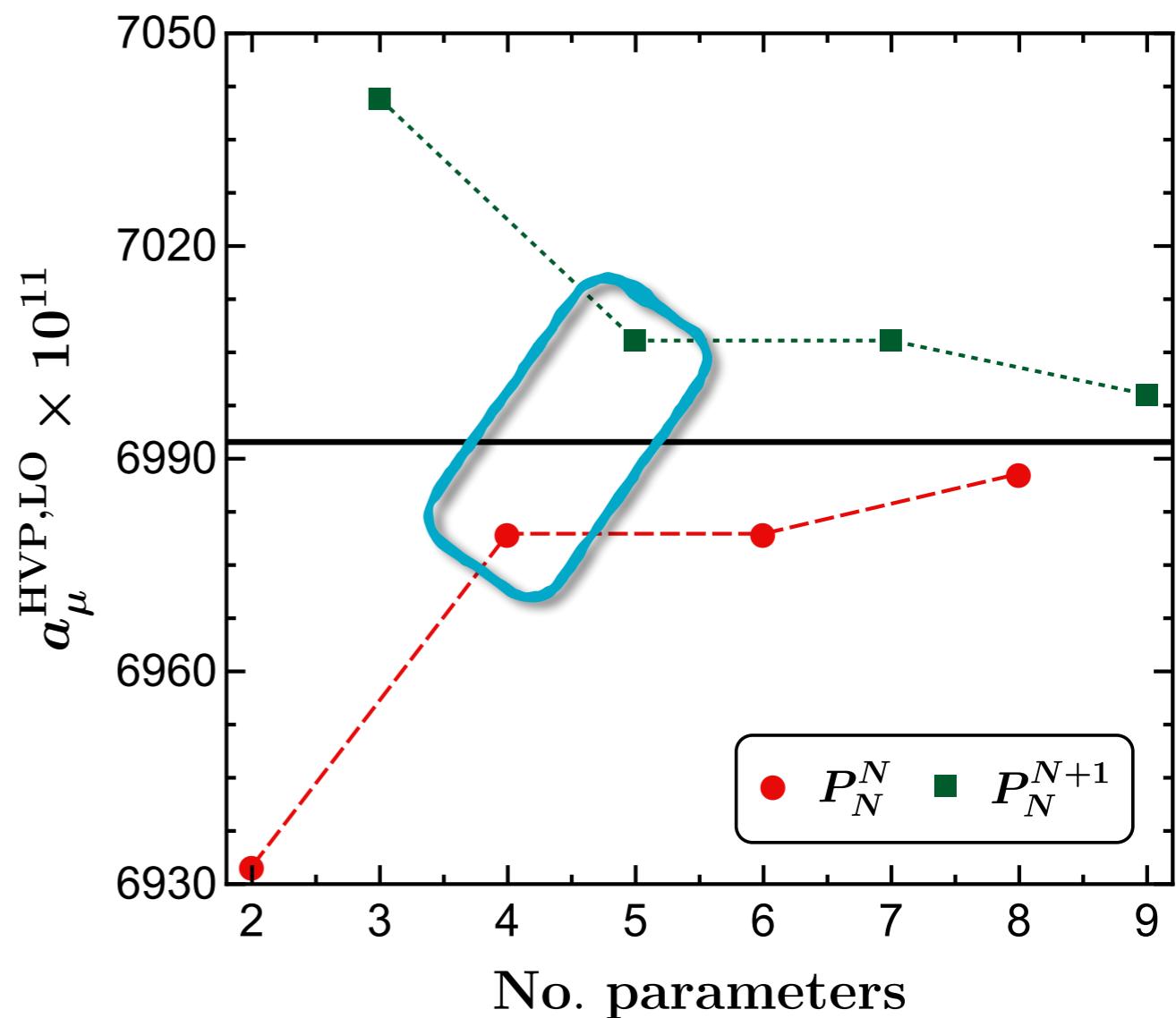
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- Error due to truncation

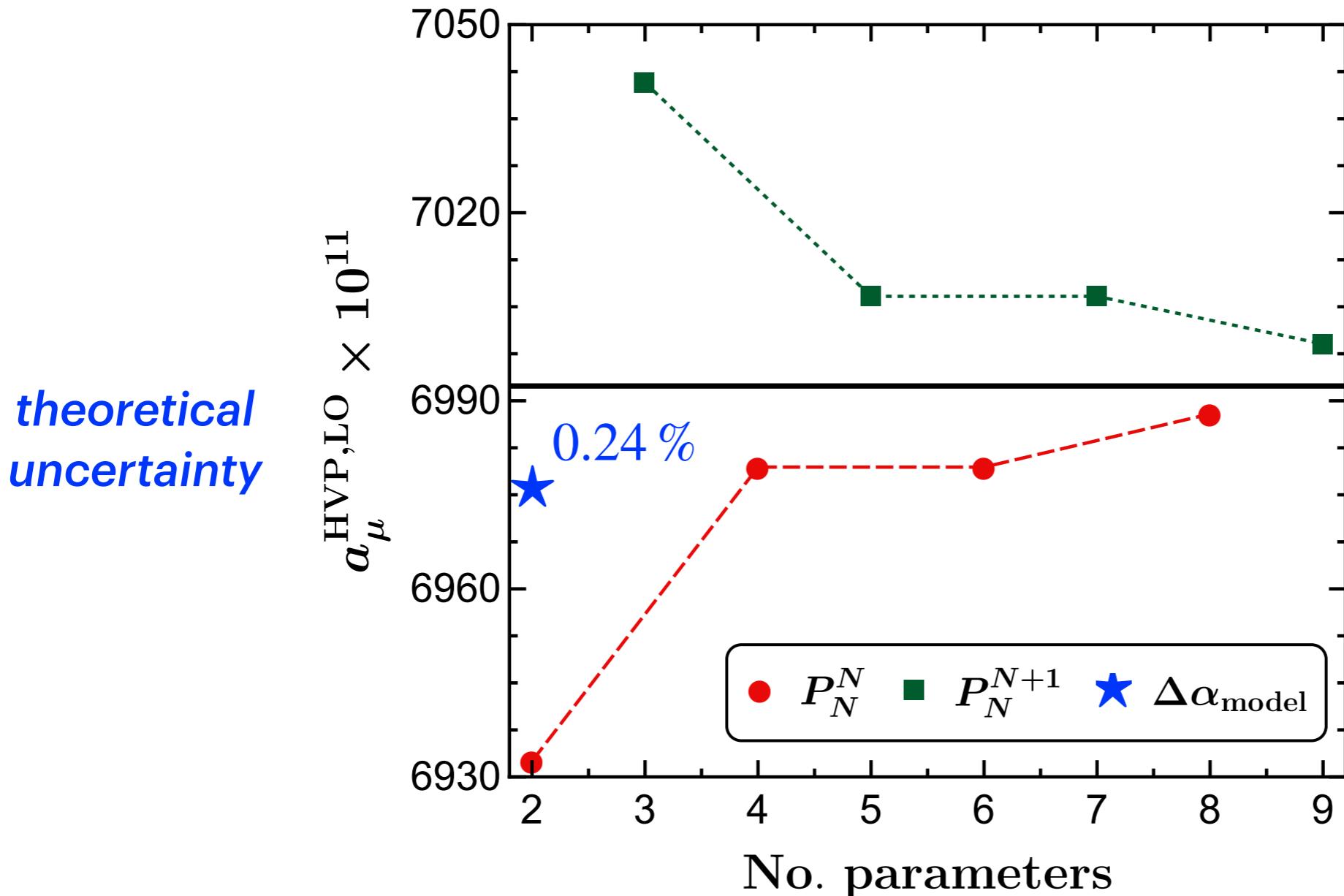
0.19%



Ideal Scenario

The MUonE Collaboration (2019)
Abbiendi (2022)

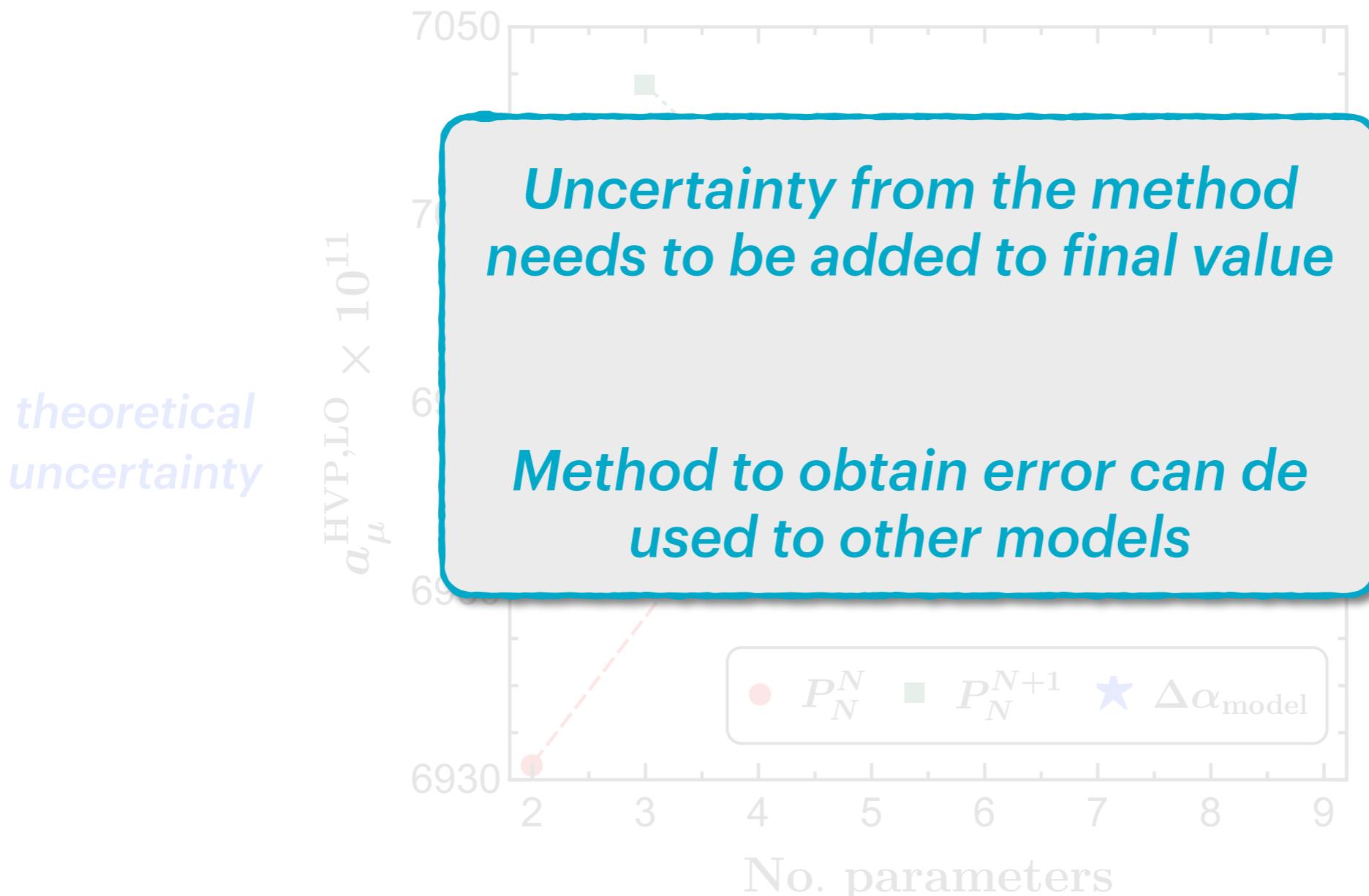
$$\Delta\alpha_{\text{model}}(t) = k \left\{ -\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6} \right) \frac{2}{\sqrt{1 - \frac{4M}{t}}} \log \left| \frac{1 - \sqrt{1 - \frac{4M}{t}}}{1 + \sqrt{1 - \frac{4M}{t}}} \right| \right\}$$



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More Realistic Data

- 1000 toy data sets generated
- $(x, \alpha \Delta \alpha_{\text{had}}(x) \times 10^5)$ – 30 data points equally spaced in $0.2 \leq x \leq 0.93$
- Central value randomly chosen from a gaussian distribution with error ranging from 0.7 % up to 6.7 % (small x)
- Analysis of the fits for each Padé approximant
- PAs used to calculate a_μ in the whole x region
- Value of $a_\mu^{\text{HVP,LO}}$ calculated for each data set

$$a_\mu^{\text{HVP,LO}} = (6991^{+22}_{-20}) \times 10^{-11}$$

More Realistic Data

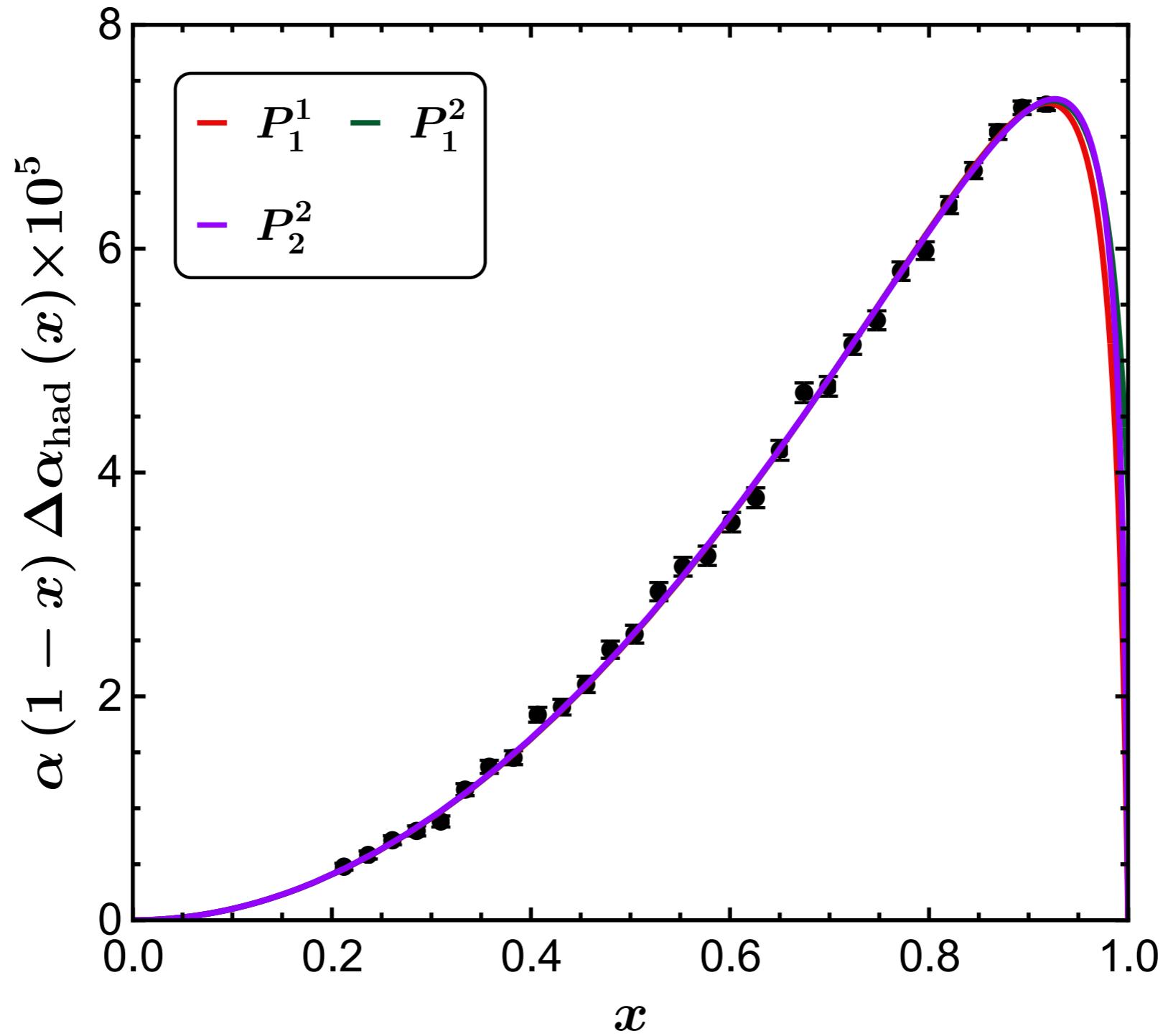
- 1000 toy data sets generated

Constraints

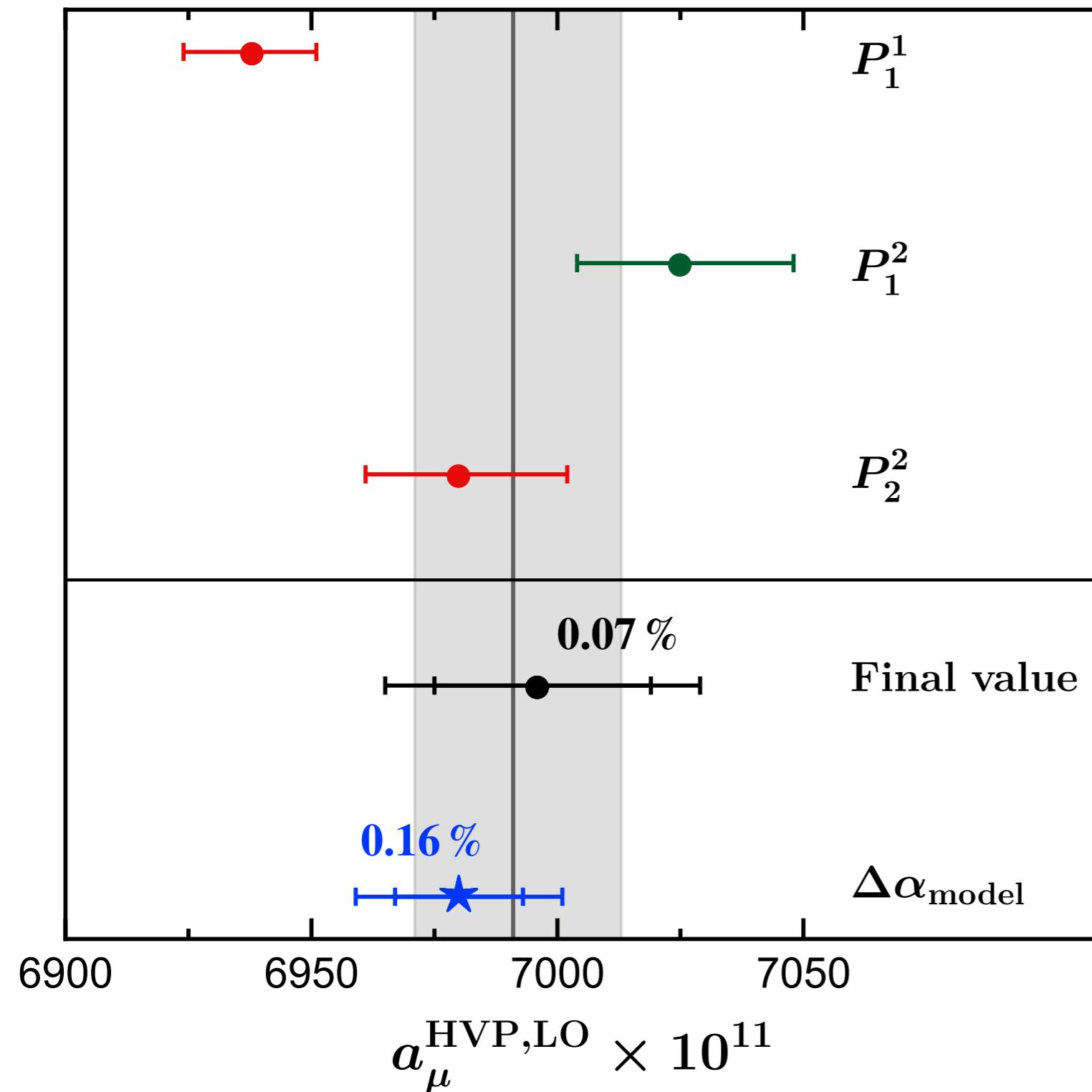
- $(x, \alpha \Delta \alpha)$ from 0.7 to 1.0, $\Delta \alpha \leq 0.93$
- Central values and error ranging from 0.7 to 1.0
- Analysis of quality of fit:
 - Quality of fit:
 - $0.5 < \chi^2/\text{dof} < 1.5$
 - $p - \text{value} > 10^{-4}$
- PAs used
- Value of a_μ :
 - Quality of Padés:
 - hierarchy of coefficients $a_i > a_{i+1}, i \in \mathbb{N}$
 - determinant condition $\begin{vmatrix} a_i & a_{i+1} \\ a_{i+1} & a_{i+2} \end{vmatrix} > 0$

$$a_\mu = (0.991 \pm 20) \times 10^{-4}$$

More Realistic Data



More Realistic Data



Convergence pattern observed

PAs poles in t are all real and positive

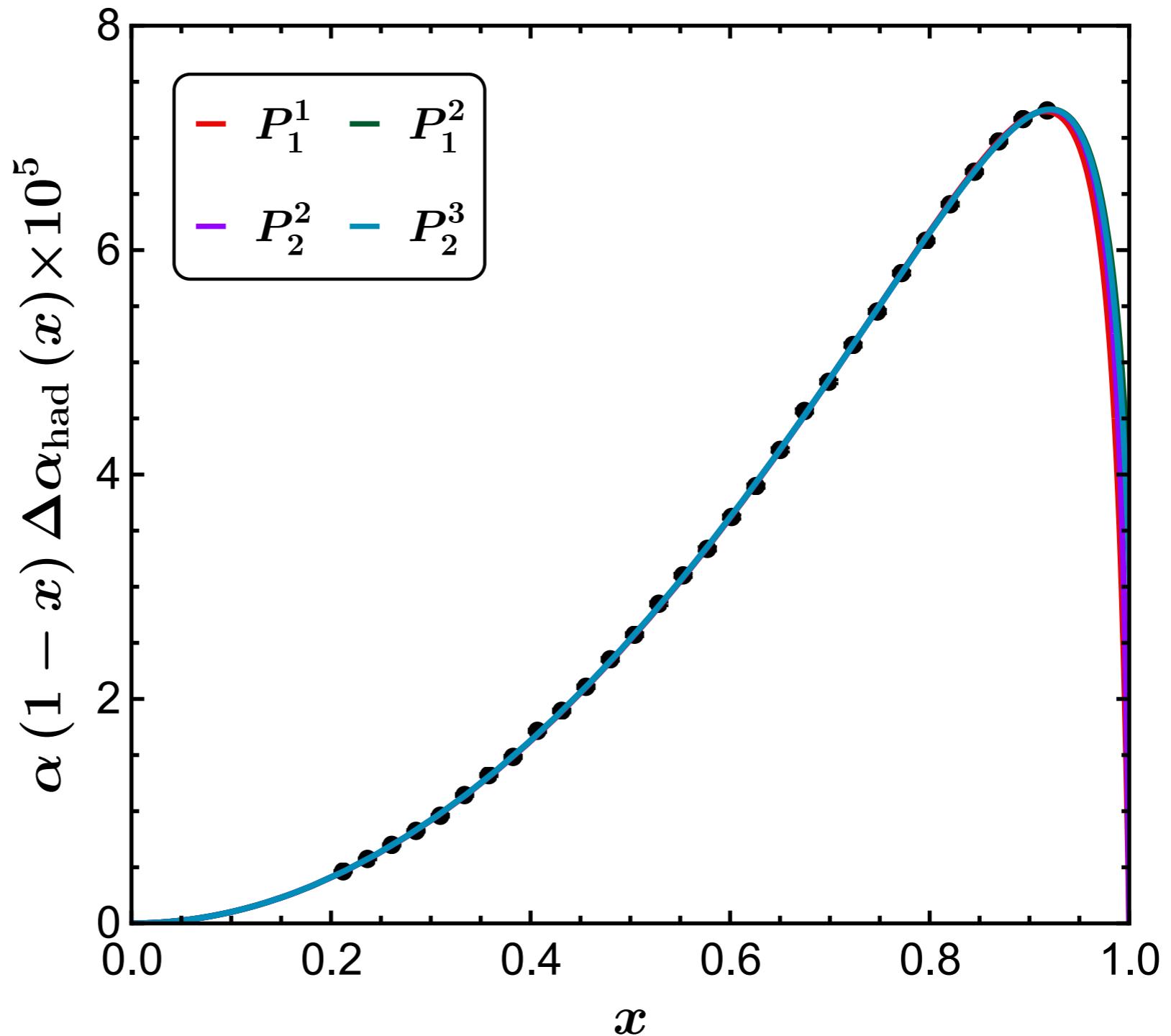
- Mean value of χ^2/dof close to 1 and high p -value, close to 0.5
- Statistical and theoretical error of the same order

Inner error — statistical one

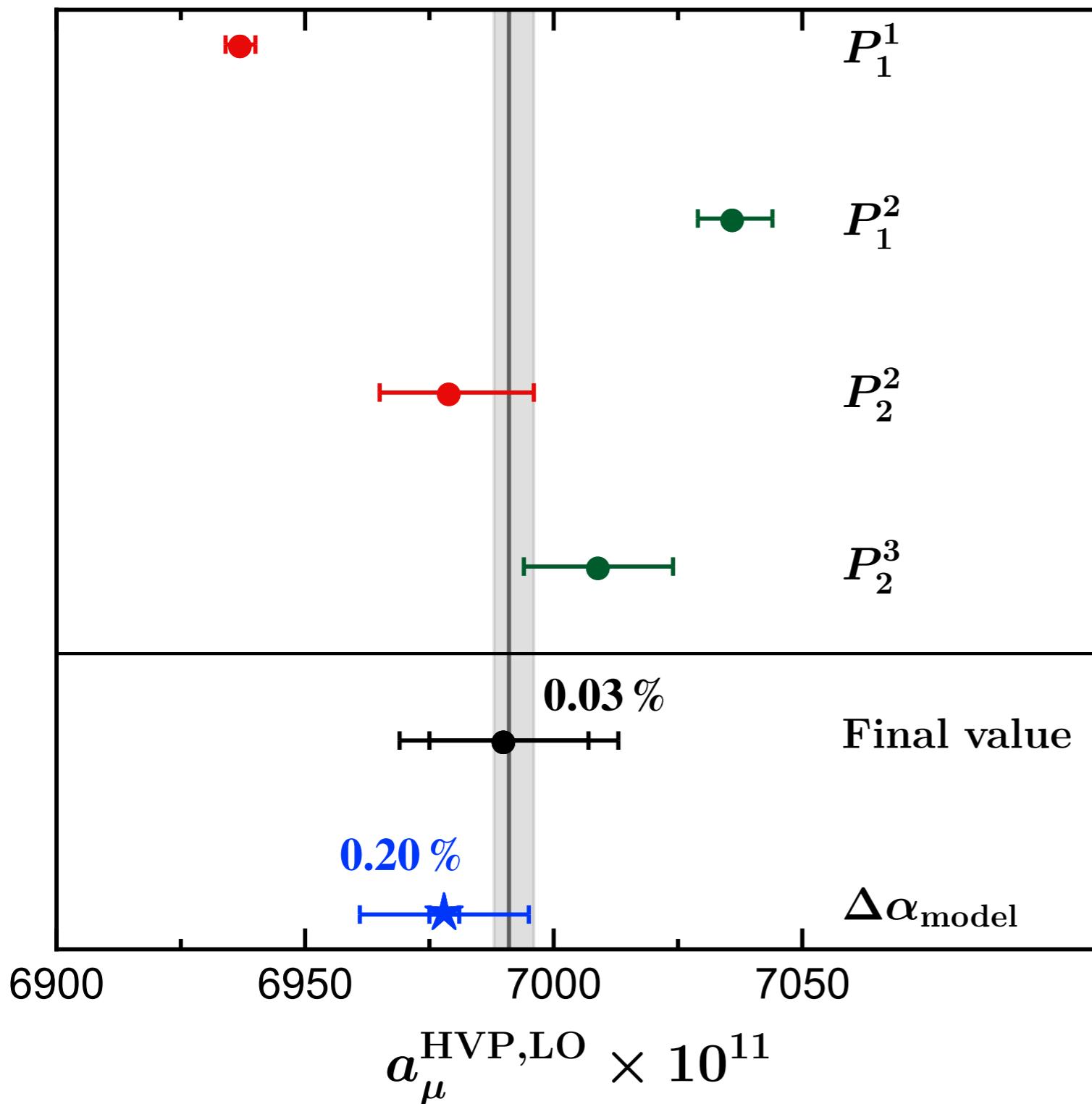
Exterior error — statistical and systematic errors summed in quadrature

More Realistic Data

Dividing the
error of the
data points
by 5



More Realistic Data



Dividing the error of the data points by 5

Convergence pattern observed

PAs poles in t are all real and positive

- Good quality fits
- Statistical uncertainty 26 % smaller and theoretical one 37 %
- Without theoretical error, a_μ^{model} outside expected region

Conclusions

Conclusions

- Model-independent method to fit and extrapolate the data from the MUonE experiment
- Uses knowledge of Stieltjes functions
- Statistical and theoretical uncertainties can be lowered – D-log Padés
- Technique to estimate theoretical uncertainty for any model
- Padés can be a good method to fit the MUonE data

*work in
progress*

*Thank you
for your
attention!*

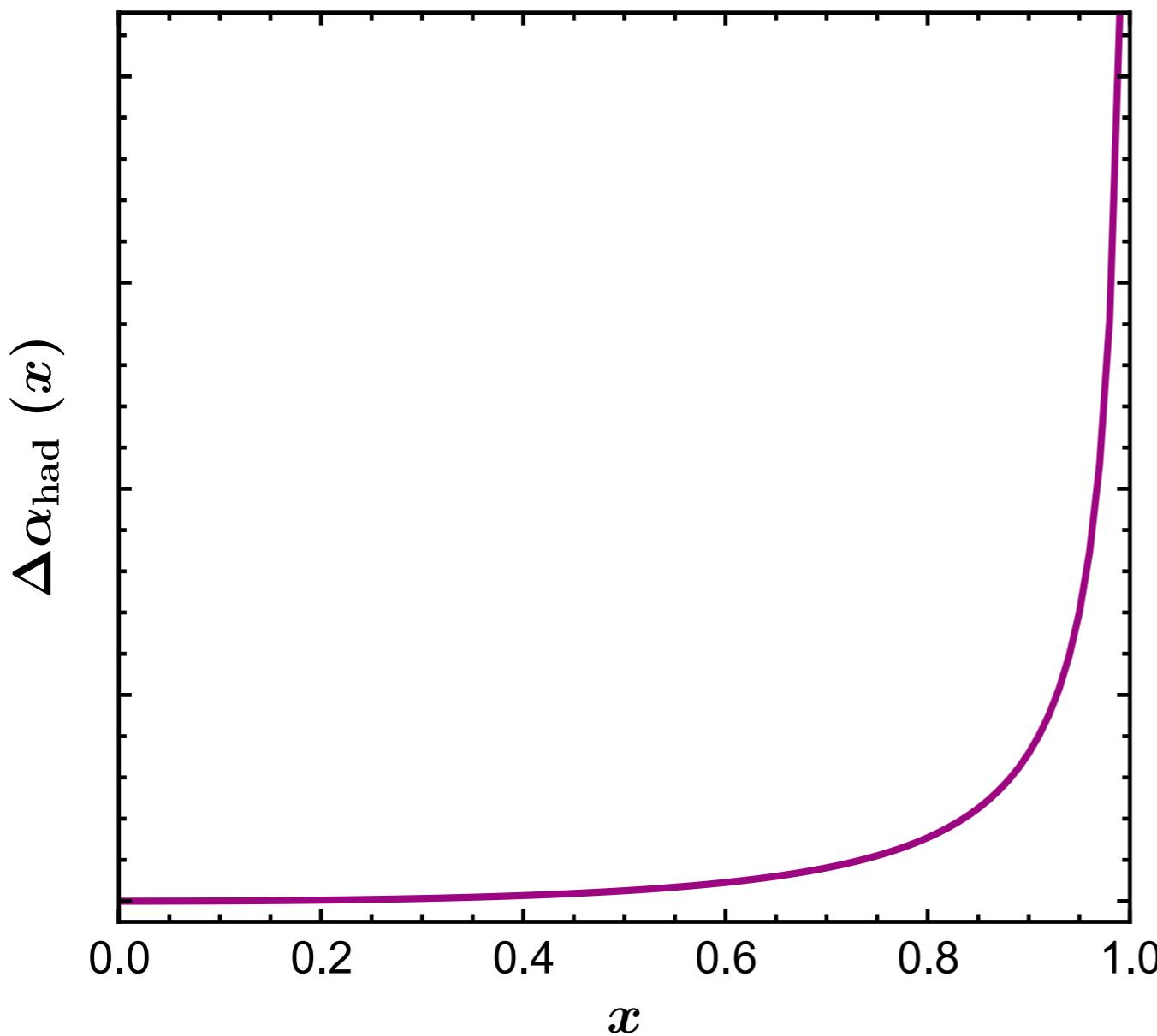
Acknowledgements



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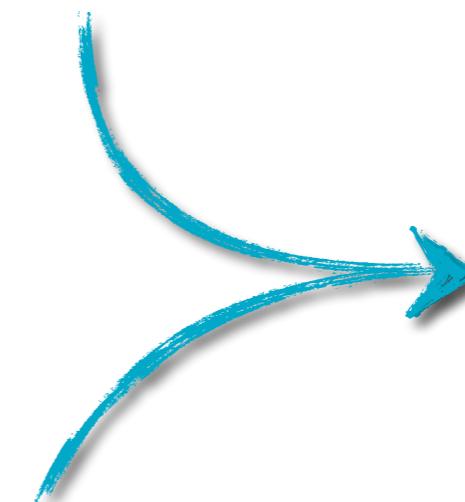
Extras

Data Generation



$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{\pi} \int_0^1 dx (1-x) \Delta\alpha_{\text{had}}(t)$$

$$\Delta\alpha_{\text{had}}(t) = -\text{Re}[\bar{\Pi}_{\text{had}}(t)]$$



**Stieltjes
functions**

$$\bar{\Pi}_{\text{had}}(q^2) \equiv \Pi_{\text{had}}(q^2) - \Pi_{\text{had}}(0)$$

$$= q^2 \int_{4m_\pi^2}^\infty \frac{\text{Im } \Pi_{\text{had}}(s)}{s(s - q^2 + i\epsilon)} ds$$

Model of Greynat and de Rafael

$$\text{Im } \Pi_{\text{had}}(s) = \left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \left(\frac{|F(s)|^2}{12} + \sum_i q_i^2 \Theta(s) \right) \theta(s - 4m_\pi^2)$$

model used to generate toy data

Greynat, de Rafael (2022)

$$|F(s)|^2 = \frac{m_\rho^4}{(m_\rho^2 - s)^2 + m_\rho^2 \Gamma(s)^2}$$

$$\begin{aligned} \Gamma(s) = & \frac{m_\rho s}{96\pi f_\pi^2} \left[\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \theta(s - 4m_\pi^2) \right. \\ & \left. + \frac{1}{2} \left(1 - \frac{4m_k^2}{s}\right)^{3/2} \theta(s - 4m_k^2) \right] \end{aligned}$$

$$\Theta(s) = \frac{2}{\pi} \left[\frac{\arctan\left(\frac{s-s_c}{\Delta}\right) - \arctan\left(\frac{4m_\pi^2-s_c}{\Delta}\right)}{\frac{\pi}{2} - \arctan\left(\frac{4m_\pi^2-s_c}{\Delta}\right)} \right]$$

