UNIVERSITY OF HELSINKI

QCD MASTER CLASS
SAINT-JACUT-DE-LA-MER, FRANCE

Tackling the g^6 problem of the hot QCD pressure

QCD Master Class 2023

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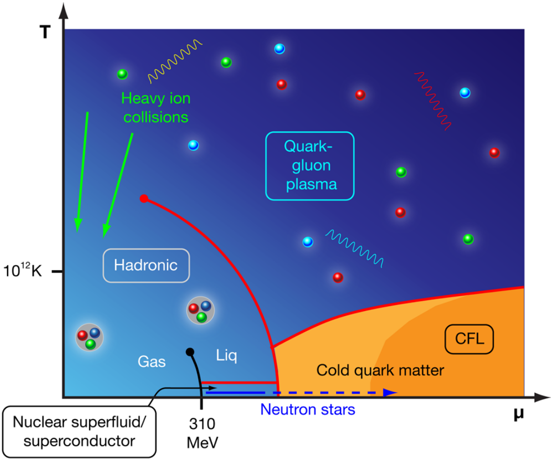
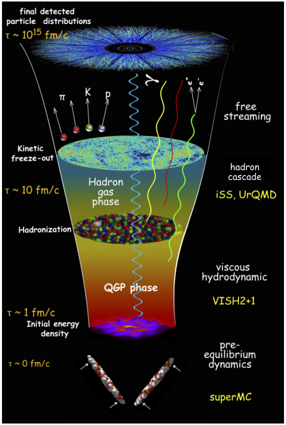
QCD Phase diagram



Motivation



Ongoing experiments:

- LHC: ALICE, ATLAS, CMS
- RHIC: Phenix, STAR

Interesting questions:

- Initial conditions
- Strongly interacting liquid-like QGP ($T_c \sim 170$ MeV)
- Thermal/Chemical freeze-out
- Relativistic hydrodynamic expansion

Hydrodynamics $\longrightarrow$ Pressure!

(among other things...)

Motivation

Matter in extreme physical conditions:

- Early Universe
- Quark-gluon plasma in heavy ion collisions
- Inner core of neutron stars ($n > n_s$, $n_s \approx 0.16\ \text{fm}^{-1}$)

$\longrightarrow$ Possibility of deconfined QCD matter at high enough T and/or μ.

$$\boxed{\text{Asymptotic freedom} \Longrightarrow \alpha_s \ll 1 \Longrightarrow \text{Perturbation Theory}}$$

Equilibrium thermodynamics

The fundamental object to consider is the (grand) partition function

$$\mathcal{Z}(\mu, T) = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{Q})].$$

$\longrightarrow$ all thermodynamic information derived from $\mathcal{Z}$.

For QFT, (Euclidean) path integral representation ($\tau = it$, $\mu = 0$):

$$\mathcal{Z}(T) = \int_{\mathrm{b.c}} \mathcal{D}[\text{fields}] \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, L_E\right\}.$$

- τ: compact dimension of length $\beta\hbar$
- Bosons: periodic in $\tau \Longrightarrow p_n = 2\pi n T$
- Fermions: anti-periodic in $\tau \Longrightarrow p_n = \pi T(2n+1)$

Equilibrium thermodynamics

Rest frame of heat bath $\Longrightarrow$ breaking of Lorentz invariance

$$i\int d^4p \longrightarrow T\sum_{n=-\infty}^{\infty}\int d^3\vec{p}, \quad i\int d^4x \longrightarrow \int_0^\beta d\tau \int d^3\vec{x}.$$

Fourier analysis is carried out in the Matsubara formalism
($d = 3 - 2\varepsilon$):

$$\phi^{b/\{f\}}(X) = \sum\!\!\!\!\!\!\!\int_{P/\{P\}} \tilde{\phi}^{b/\{f\}}(P)e^{iP\cdot X}, \quad \sum\!\!\!\!\!\!\!\int_{P/\{P\}} \equiv T\sum_{p_n}\int \frac{d^d\vec{p}}{(2\pi)^d},$$

with $P = (p_n, \vec{p})$.

Loop integrals $\longrightarrow$ Sum-integrals!

QCD pressure

We are interested in the QCD pressure:

$$p(T) = \lim_{V\to\infty} \frac{T}{V} \log \int_{\text{periodic}} \mathcal{D}A_\mu^a \int_{\text{periodic}} \mathcal{D}\bar{c}^a \mathcal{D}c^a \int_{\text{anti-periodic}} \mathcal{D}\bar{\psi}\mathcal{D}\psi$$

$$\times \exp\left\{ -\int_0^\beta \int_{\vec{x}} \left[\frac{1}{4}(F_{\mu\nu}^a)^2 + \sum_{i=1}^{N_f} \bar{\psi}_i \left(\not{D} + m \right) \psi_i + \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 + \partial_\mu \bar{c}^a \mathcal{D}_\mu^{ab} c^b \right] \right\}.$$

General strategy:

- Weak-coupling expansion in g
- $T \gg m$; N_f massless quarks and general $SU(N_c)$ gauge group
- Use general covariant R_ξ gauge; pressure is independent of ξ
- Feynman diagrams → Feynman rules → algebra → sum-ints

The Infrared Problem

- Electric screening for $A_0^a \longrightarrow m_{\text{eff}} \sim gT$
- Magnetic screening for $A_i^a \longrightarrow m_{\text{eff}} \sim g^2 T$

Fermionic/bosonic largest loop expansion parameters (Matsubara zero modes):

$$\epsilon_f \sim g^2, \quad \epsilon_b \sim \frac{g^2 T}{m_{\text{eff}}}.$$

Bosonic zero modes are static:

$$A_\mu^a(\tau, \vec{x}) = \sum\!\!\!\!\!\!\int_P \tilde{A}_\mu^a(P) e^{i(p_n\tau - \vec{p}\cdot\vec{x})}.$$

Conclusions:

- Electro<u>static</u> contributions: $\epsilon_b \sim g \longrightarrow$ barely perturbative ✓
- Magneto<u>static</u> contributions: $\epsilon_b \sim 1 \longrightarrow$ perturbative χ

The Infrared Problem

For diagrams containing N four-gluon vertices,

$$p_N(T)\Big|_{\text{dominant}} \sim (2\pi T)^{N+1} g^{2N} \int d^3p_1 \cdots \int d^3p_{N+1} \prod_{k=1}^{2N} \frac{1}{\vec{q}_k^{\,2} + m_{\text{eff}}^2},$$

$$\sim g^6 T^4 \left(\frac{g^2 T}{m_{\text{eff}}}\right)^{N-3}.$$

For magnetic modes $m_{\text{eff}} \sim g^2 T$, implying

- Severe infrared divergences for the pressure begin at $\mathcal{O}(g^6 T^4)$
- Perturbation theory breaks down [Linde '80; Gross *et. al* '80]

Dimensionally-reduced EFT

Hot QCD is a hierarchical multi-scale system $(T \gg gT \gg g^2 T)$:

- Hard scale T
- Soft scale gT
- Ultrasoft scale $g^2 T$

Massive hard modes decouple from the theory in the infrared:

$$G(\vec{x})\Big|_{\mathrm{IR}} \sim e^{-m|\vec{x}|}$$

Effective theory in three dimensions $(|\vec{x}| \gg 1/T)$:

$$S_{\mathrm{eff}} = \frac{1}{T}\int d^3x \left\{ -p^{hard} + \frac{1}{4}\bar{F}^a_{ij}\bar{F}^a_{ij} + \mathrm{Tr}\left([\bar{D}_i, \bar{A}_0][\bar{D}_i, \bar{A}_0]\right) + m_E^2 \mathrm{Tr}[\bar{A}_0^2] \right.$$
$$\left. + \lambda_E^{(1)}(\mathrm{Tr}[\bar{A}_0^2])^2 + \lambda_E^{(2)} Tr[\bar{A}_0^4] + (\text{higher order operators}) \right\}.$$

Electrostatic and Magnetostatic QCD

Further decoupling the massive $\bar{A}_0$ mode for $|\vec{x}| \gg 1/(gT)$, we define

Electrostatic QCD (EQCD) for soft scale:

$$S_E = \frac{1}{T}\int d^3x \left\{ \frac{1}{4}\bar{F}^a_{ij}\bar{F}^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_i \bar{A}^b_0)^2 + m_E^2 \mathrm{Tr}[\bar{A}_0^2] + \lambda_E^{(1)}(\mathrm{Tr}[\bar{A}_0^2])^2 \right.$$
$$\left. +\lambda_E^{(2)} Tr[\bar{A}_0^4] + \dots \right\}.$$

Magnetostatic QCD (MQCD) for ultrasoft scale:

$$S_M = \frac{1}{T}\int d^3x \left\{ \frac{1}{4}\bar{\bar{F}}^a_{ij}\bar{\bar{F}}^a_{ij} + \dots \right\}.$$

- Obtain effective parameters via perturbative matching of correlators, e.g.

m_E^2 : compare pole in static A_0 prop. for QCD and EQCD

The QCD pressure

We can split the full QCD pressure as [Braaten/Nieto '96]

$$p(T) = p^{hard}(T) + p^{soft}(T) + p^{ultrasoft}(T).$$

Non-trivial weak-coupling expansion:

$$p = p_0 + p_2 g^2 + p_3 g^3 + (p_4' \log g + p_4) g^4 + p_5 g^5 + (p_6' \log g + p_6) g^6 + \mathcal{O}(g^7).$$

The structure up to $\mathcal{O}(g^6)$ is the following:

$$p^h = T^4 \left[p_0^h + p_2^h g^2 + p_4^h g^4 + p_6^h g^6 + \dots \right]$$ 4d thermal QCD

$$p^s = m_E^3 T \left[p_3^s + \dots + p_6^s (g_E^2/m_E)^3 + \dots \right]$$ 3d YM + adjoint Higgs

$$p^u = g_M^6 T \left[p_6^u + \dots \right]$$ 3d pure YM

- p_6^u : non-pert. coeff. $\Longrightarrow$ lattice 3d YM + 4-loop Numerical Stochastic Perturbation Theory [Renzo/Schröder '04-'06]

The g^6 term of the QCD pressure

Each contributions starts at:

- p^{hard} at order g^0
- p^{soft} at order g^3
- $p^{ultrasoft}$ at order g^6

Order $\mathcal{O}(g^6)$ pressure: physical leading order.

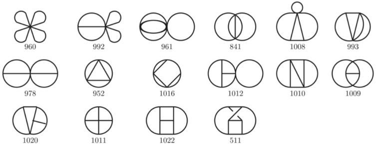
The contribution of order g^6 to p^{hard} is not known:

$$p^h\Big|_{\mathcal{O}(g^6)} = \sum \text{(connected 4-loop vacuum diagrams in 4d thermal QCD)}\,.$$

Strategy and reduction algorithm

1. Generate all 4-loop connected vacuum diagrams via `qgraf`
2. Assign Feynman rules in covariant R_ξ gauge
3. Solve Dirac, Lorentz and $SU(N)$ algebra via `FORM` algorithm
4. Map sum-integrals down to master sector, using momentum shifts, symmetries and integration-by-parts (IBP)

- Shifts: linear mapping to master sector
- Symmetries: linear mapping within the same master sector



Final step $\longrightarrow$ solve master sum-integrals

List notation

For L loops we have $L(L+1)/2$ independent scalar products between loop momenta $\{P_i\}_{i=1}^{L}$. We define the 4-loop momenta family

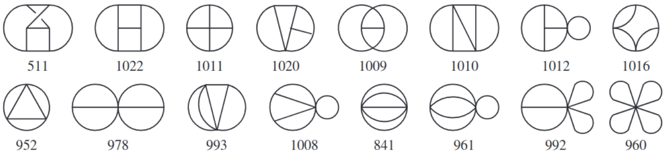
$$\{P_1, P_2, P_3, P_4, P_1 - P_4, P_2 - P_4, P_3 - P_4, P_1 - P_2, P_1 - P_3, P_1 - P_2 - P_3\}.$$

Scalar Feynman integrals can then be expressed as an ordered list of integer numbers, being the exponents of each corresponding propagator, e.g.,

$$\sum\!\!\!\!\!\!\!\int_{P_1} \cdots \sum\!\!\!\!\!\!\!\int_{P_4} \frac{(P_1 - P_2)^2}{[P_1^2]^2 P_2^2 P_3^2 P_4^2 (P_1 - P_2 - P_3)^2} \longleftrightarrow I(2, 1, 1, 1, 0, 0, 0, -1, 0, 1).$$

- Free to do momentum shifts

Master sectors



- Sectors are defined assigning a 0 for null and negative exponents, and 1 for positive exponents.
- For each list of 0s and 1s (binary), we can assign a decimal number
- Master sectors are defined by the sectors with the biggest decimal number
- Unique representatives with respect to linear momentum shifts

Thermal IBP

The idea is exploiting the identity

$$\SumInt_P \partial_{\vec{p}}(\cdots) = 0.$$

- Linear system of eqs. for sum-integrals of interest
- In general, coeffs. are rational functions of the dimension d
- Allows mapping to master sum-integrals
- Computationally expensive method for high orders

A trivial example for 1 loop gives the recursion relation

$$\SumInt_P \frac{(P_0)^{a+2}}{(P^2)^{b+1}} = \left(1 - \frac{d}{2b}\right) \SumInt_P \frac{(P_0)^a}{(P^2)^b}.$$

Thermal IBP

Non-trivial 2-loop IBP result

The most general massless bosonic 2-loop vacuum sum-integral is

$$L^{s_4 s_5 s_6}_{s_1 s_2 s_3} = \sum\!\!\!\!\!\!\!\int_{PQ} \frac{(P_0)^{s_4}(Q_0)^{s_5}(P_0 - Q_0)^{s_5}}{(P^2)^{s_1}(Q^2)^{s_2}[(P-Q)^2]^{s_3}}.$$

- Thermal IBP indicates factorization into a product of 1 loop ones [Nishimura/Schröder '12]
- Recently, proof of factorization [Davydychev/PN/Schröder in preparation]
- Trivial to solve in d dimensions in terms of 1-loop sum-integrals, $L = \sum (\text{1-loop}) \times (\text{1-loop})$

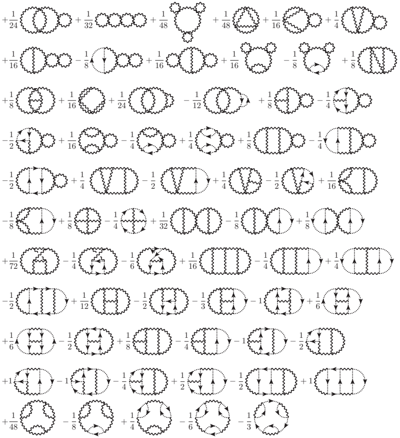
3-loop Sum-integrals

Main strategy:

1. Disentangling UV/IR div. from 1-loop selfE [Arnold/Zhai '94]
2. Divergences analytically + finite terms numerically
3. Many examples known in the literature

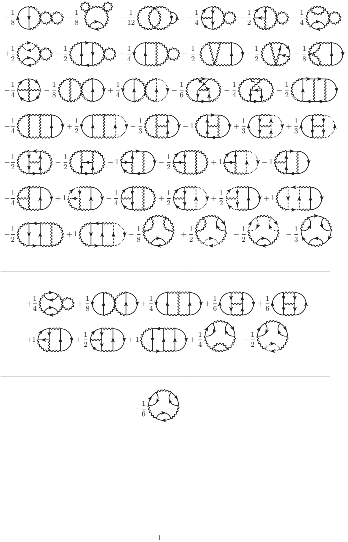
$$
\begin{aligned}
V_1 &= \sum\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\int_Q \sum\!\!\!\!\!\!\int_R \frac{1}{P^2(Q^2)^2(Q-P)^2R^2(R-P)^2}\,,\\
&= \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi T^2}\right)^{3\varepsilon}\frac{1}{6\varepsilon^3}\Big\{1+3\varepsilon+\Big(13-3\zeta_3+\frac{9}{2}\zeta_2-6\left(\gamma_E^2+2\gamma_1\right)\Big)\varepsilon^2\\
&+\Big\{51-42(\gamma_E^2+2\gamma_1)+24\zeta_2\left(\frac{19}{16}+\log 2\pi-12\log G\right)+2\log 2\left(12-12\gamma_E^2-24\gamma_1-\zeta_3\right)\\
&+6\gamma_E\left(3\zeta_3-4-4\gamma_1\right)-36\gamma_2+\frac{25}{2}\zeta_3-16\zeta_3'+6c_1+6c_2+6c_3\Big\}\varepsilon^3+\mathcal{O}(\varepsilon^4)\Big\},\\
c_2 &= \sum_{n=1}^{\infty}\int_0^\infty dx\,\frac{2e^{-x}}{n}\left[e^x\mathrm{Ei}(-x)+\gamma_E+\log\frac{x}{4n^2}\right]\times\\
&\times\left[\psi(n+1)+e^xB(e^{-x/n},n+1,0)+e^x\mathrm{Ei}(-x)-\log(1-e^{-x/n})+\log\frac{x}{n^2}-\frac{x}{12n^2}\right]\\
&\approx -3.2020672566(1).
\end{aligned}
$$

Results: Bosonic sector



- 65 bosonic Feynman diagrams
- Hardest: $2^9\,6^6 \approx 24M$ terms
- Polynomial up to ξ^6
- Sum-ints to solve: 176119
- After shifts $\to$ 25047
- After symmetries $\to$ 945
- Sum all diagrams $\to$ 21
- Explicit gauge invariance in d dimensions is obtained [PN/Schröder in preparation]
- Finally: solve 21 sum-ints (IBP improvement?)

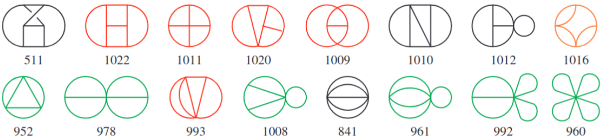
Results: Fermionic sector



- 53 fermionic Feynman integrals:
  - N_f^1: 42; N_f^2: 10; N_f^3: 1
- Poly. in gauge parameter up to ξ^5
- Nr. of sum-ints to solve: 106212
- Apply shifts $\rightarrow$ 22479
- Apply symmetries $\rightarrow$ 1009
- Sum all diags. $\rightarrow$ 134
- Explicit gauge invariance in d dimensions is obtained
- Use IBP $\rightarrow$ 117 $\rightarrow$ Y (in progress)
- Finally: solve Y sum-ints (IBP improvement?)

Status of the four-loop QCD pressure

- 2 factorized as (1-loop)^4: all known
- 3 factorized as (1-loop)$^2\times$(2-loop): all known
- 6 factorized as (1-loop)$\times$(3-loop): all known [PN/Schröder '22]
- 10 genuine 4-loops: 9 unknown



$$\sum\!\!\!\!\!\!\int_P [\Pi(P)]^3 = \frac{T^4}{(4\pi)^4}\frac{1}{16\epsilon^2}\left[1+\epsilon t_{11}+\epsilon^2 t_{12}+\mathcal{O}(\epsilon^3)\right] \text{ [Gynther } et.\ al \text{ '07]},$$

$$\Pi(P) = \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P-Q)^2}.$$ only known 4-loop sum-int !

Status of the Bosonic sector

For the contribution coming from topology 1016, need to evaluate the beasts

$$\sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q \sum\!\!\!\!\!\!\!\int_R \sum\!\!\!\!\!\!\!\int_S \frac{[(P-Q-R)^2]^2}{P^2Q^2R^2(S^2)^2(P-S)^2(Q-S)^2(R-S)^2},$$

$$\sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q \sum\!\!\!\!\!\!\!\int_R \sum\!\!\!\!\!\!\!\int_S \frac{[(P-Q-R)^2]^3}{P^2Q^2R^2(S^2)^3(P-S)^2(Q-S)^2(R-S)^2}.$$

After tensor decomposition, need to evaluate [PN/Schröder '22]

$$\sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^2}\Pi\Pi^{\mu\nu}\Pi^{\mu\nu}, \quad \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^3}\Pi^{\mu\nu}\Pi^{\nu\sigma}\Pi^{\sigma\mu}; \quad \Pi^{\mu\nu} \equiv \sum\!\!\!\!\!\!\!\int_Q \frac{Q^\mu Q^\nu}{Q^2(P-Q)^2}.$$

Outlook

Possible generalizations and future directions...

- Finite chemical potentials [A. Vuorinen *et. al*]:

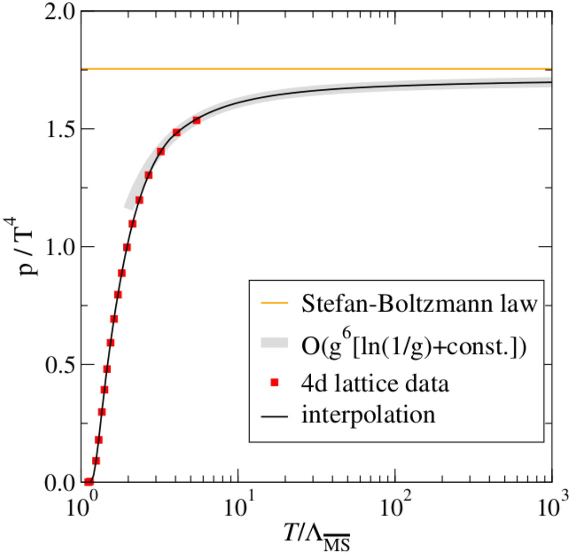
$$P_0 \longrightarrow P_0 + i\mu$$

$p(T \to 0, \mu)$ at large μ relevant for neutron star EoS [Tyler Gorda *et. al*]

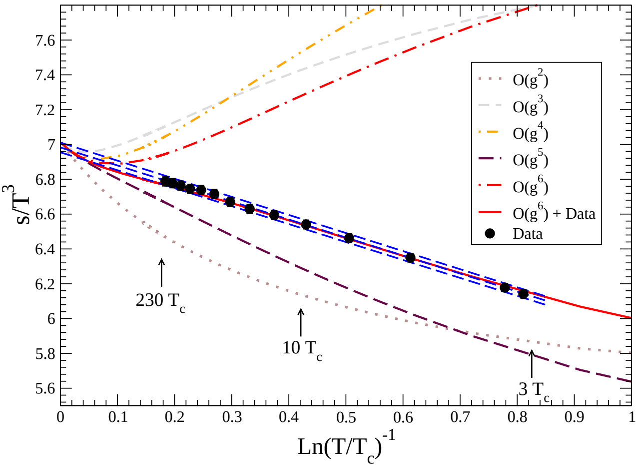
- Quark masses no analytic solution even at 1 loop [Laine/Schröder '06]
- A lot of room for improvement on sum-integral technology
- full physical leading order QCD pressure
- Possibility of directly comparing with lattice results [Borsanyi *et. al* '12; Weber *et. al* '21]
- $\mathcal{N} = 4$ Super Yang-Mills [Strickland *et. al* '22]

thank you

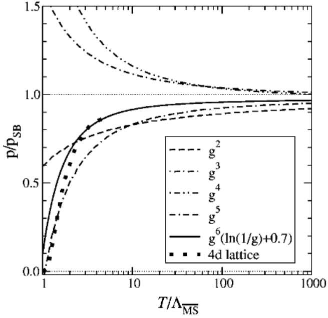
Backup: QCD pressure

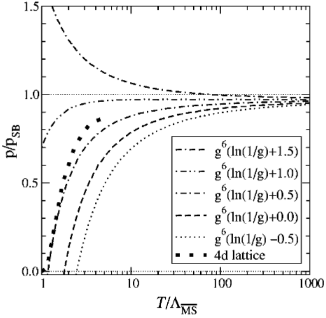


Backup: Entropy density



Backup: Pressure up to order $g^6 \log g$





Status of the four-loop QCD pressure

- Tensor structures difficult to handle
- Trade tensors for higher-dimensional scalar integrals [Ghişoiu/Schröder '12]

Problem is reduced to compute the structures

$$U^{s_{8\ldots11}}_{s_{1\ldots7}} = \sum\!\!\!\!\!\!\int_P \frac{(P_0)^{s_8}}{(P^2)^{s_1}} \Pi^{s_9}_{s_2 s_5} \Pi^{s_{10}}_{s_3 s_6} \Pi^{s_{11}}_{s_4 s_7}; \quad \Pi^c_{ab} \equiv \sum\!\!\!\!\!\!\int_Q \frac{(Q_0)^c}{(Q^2)^a[(P-Q)^2]^b}$$

- Approach: Subtract divergences from $\Pi(P)$ [Arnold/Zhai '94]
- Otherwise, need completely new approaches!