What’s so bad about a negative coupling constant? A look at a QCD-like theory

Ryan Weller (Scott Lawrence, Christian Peterson, Paul Romatschke)

University of Colorado at Boulder

Outline

- The massless $\mathrm{O}(N)$ scalar model in $3+1\mathrm{D}$
  - Solvability
  - Landau pole
- Theories with negative coupling
  - $\mathcal{PT}$-symmetry
  - ABS formula
- Revisiting the $O(N)$ model: UV completion
  - QCD-like perks
- Takeaway and future work
  - Implications for QCD

The (massless) $O(N)$ scalar model in 3+1D

+ The Euclidean (thermal) partition function is

$$Z \propto \int \mathcal{D}^N \vec{\phi}\; e^{-\int \mathrm{d}^4 x \left(\frac{1}{2}\partial_i \vec{\phi}\cdot\partial^i \vec{\phi} + \frac{\lambda}{N}\left(\vec{\phi}\cdot\vec{\phi}\right)^2\right)}$$

+ There are N components of the field $\vec{\phi}$
+ Generalizes the familiar scalar field theory with a ϕ^4 interaction term
+ Consider the massless case (easy to generalize to massive case)

The (massless) $\mathrm{O}(N)$ scalar model in 3+1D

- Solvable in the large-N limit
- We can calculate the equation of state, $P(T)$
- With a mass term for $N = 4$ this is the Higgs

Solving the O(N) model

- Perform a Hubbard-Stratonovich transformation by first introducing auxiliary fields $\sigma = \vec{\phi} \cdot \vec{\phi}$ and ζ:

$$Z \propto \int \mathcal{D}^N \vec{\phi}\, \mathcal{D}\sigma\, \mathcal{D}\zeta\, e^{-\int \mathrm{d}^4 x \left(\frac{1}{2}\partial_i \vec{\phi} \cdot \partial^i \vec{\phi} + \frac{\lambda}{N}\sigma^2 + \frac{1}{2}\mathrm{i}\zeta\left(\vec{\phi} \cdot \vec{\phi} - \sigma\right)\right)}$$

- Action is quadratic in σ. Integrate out σ to get

$$Z \propto \int \mathcal{D}^N \vec{\phi}\, \mathcal{D}\zeta\, e^{-\int \mathrm{d}^4 x \left(\frac{1}{2}\partial_i \vec{\phi} \cdot \partial^i \vec{\phi} + \frac{1}{2}\mathrm{i}\zeta \vec{\phi} \cdot \vec{\phi} + \frac{N}{16\lambda}\zeta^2\right)}$$

Solving the O(N) model

+ Now the action is quadratic in $\vec{\phi}$. Integrate out $\vec{\phi}$ to get

$$Z \propto \int \mathcal{D}\zeta \; e^{-\frac{N}{2}\mathrm{tr}\,\ln\left(-\partial_i\partial^i + \mathrm{i}\zeta_0\right) - \int \mathrm{d}^4x \frac{N}{16\lambda}\zeta^2}$$

+ N in exponent guarantees that in large-N limit, only the saddle ζ^* contributes

Solving the O(N) model

+Split ζ into a zero mode and non-zero modes: $\zeta = \zeta_0 + \zeta'$

+The saddle ζ^* will be a constant: $\zeta' = 0, \zeta^* = \zeta_0^*$

+So at large N we can replace the integral with its value at the saddle:

$$Z \propto e^{-\frac{N}{2}\operatorname{tr}\ln\left(-\partial_i\partial^i + \mathrm{i}\zeta_0^*\right) - \int \mathrm{d}^4x \frac{N}{16\lambda}{\zeta_0^*}^2}$$

+Rewrite in terms of pressure per component $p(T, \zeta_0^*)$:

$$Z \propto e^{N\beta V p(T,\zeta_0^*)}$$

Solving the $O(N)$ model

+ The pressure per component is

$$p(T,\zeta_0^*) = -\frac{{\zeta_0^*}^2}{16\lambda} - \frac{T}{2}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln(\omega_n^2 + \mathbf{k}^2 + i\zeta_0^*)$$

+ This thermal sum-integral can be done in dimensional regularization to get

$$p(T,\zeta_0^*) = -\frac{{\zeta_0^*}^2}{16\lambda} - \frac{{\zeta_0^*}^2}{64\pi^2}\left(\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}}{i\zeta_0^*}\right) + \frac{3}{2}\right) + \frac{i\zeta_0^* T^2}{2\pi^2}\sum_{n=1}^{\infty} \frac{K_2\left(n\beta\sqrt{i\zeta_0^*}\right)}{n^2}$$

Renormalization

+ Define the renormalized coupling as

$$\frac{1}{\lambda_{\mathrm{R}}} = \frac{1}{\lambda} + \frac{1}{4\pi^2\epsilon}$$

+ Then to make the pressure cutoff-independent, the renormalized coupling must depend on the $\overline{\mathrm{MS}}$ scale as

$$\lambda_{\mathrm{R}}(\bar{\mu}) = \frac{2\pi^2}{\ln\left(\frac{\Lambda_{\mathrm{LP}}}{\bar{\mu}}\right)}$$

Gap equation

- The saddle condition (or gap equation) is

$$\frac{\mathrm{d}p(T,\zeta_0^*)}{\mathrm{d}\zeta_0^*} = 0$$

- Expressed in terms of Λ_c, at zero temperature this gives two solutions

$$(\zeta_0^*)_0 = 0, \quad (\zeta_0^*)_1 = -\mathrm{i}\Lambda_{\mathrm{LP}}^2 e$$

- At higher temperatures it also gives two numerical solutions

Low-T and high-T behavior

- At low temperatures, in large-volume limit, the pressure $p_1(T)$ from the second saddle $(\zeta_0^*)_1$ dominates
- Above some $T_\mathrm{c} \approx 0.616\Lambda_{\mathrm{LP}}$ there are also two solutions to gap equation:

$$(\zeta_0^*)_+ = -\mathrm{i}m_+^2, \;\; (\zeta_0^*)_- = -\mathrm{i}m_-^2$$

  - with $m_+ = m_-^*$ complex conjugates
  - $p_+(T)$ and $p_-(T)$ are complex conjugate pairs
- Does this mean the theory breaks down at high T?

Landau pole

+ We non-perturbatively renormalized the theory
+ Has a Landau pole at some scale $\bar{\mu} = \Lambda_{\mathrm{LP}}$

$$\lambda_{\mathrm{R}}(\bar{\mu}) = \frac{2\pi^2}{\ln\left(\frac{\Lambda_{\mathrm{LP}}}{\bar{\mu}}\right)}$$

+ Oh no! This is a problem
+ But what if...

Negative coupling?

- $\bar{\mu}$ is not an observable
- Let's consider the continuum limit $\bar{\mu} \gg \Lambda_{\mathrm{LP}}$
- Coupling goes negative
- Isn't path integral unbounded?

Okay, so...

- Can theories with negative coupling make sense?

$\mathcal{PT}$ symmetric quantum mechanics

- Bender and Boettcher considered Hamiltonians of the form

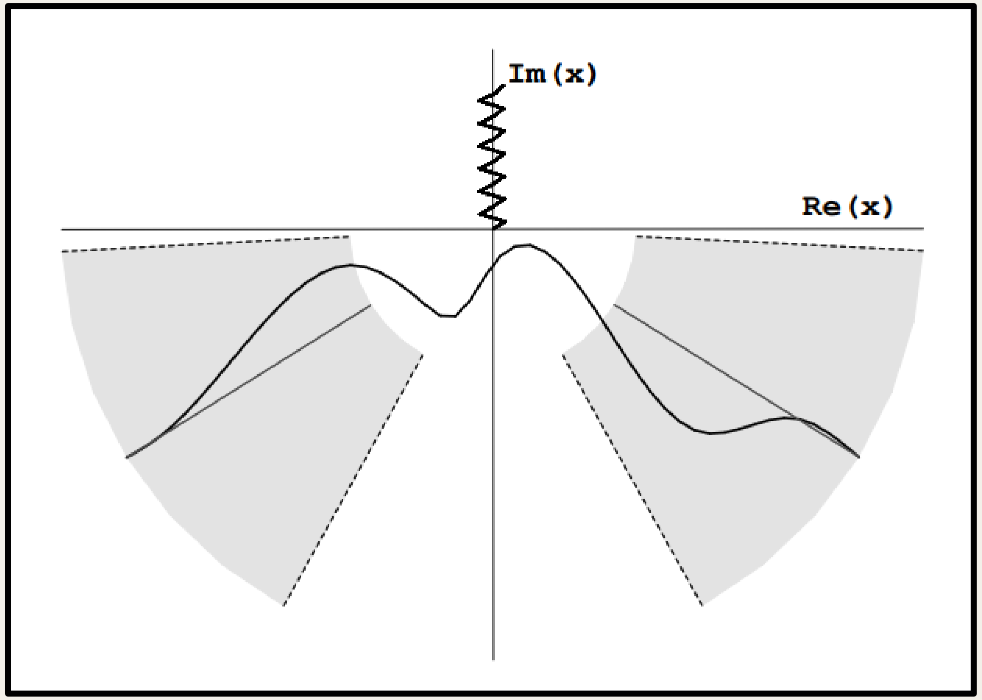
$$H = p^2 - g(\mathrm{i}x)^{2+\epsilon}, \qquad g > 0$$

- Unbounded or non-Hermitian, except for non-Hermitian x
- Have symmetry $x \to -x, \mathrm{i} \to -\mathrm{i}$ $(\mathcal{P}, \mathcal{T})$

$\mathcal{PT}$ symmetric quantum mechanics

- They solved Schrödinger equation for complexified fields x

C. Bender and S. Boettcher, Phys.Rev.Lett. 80 (1998)



$\mathcal{PT}$ symmetry: $\epsilon = 2$

+ Found real spectra for many values of ϵ
+ For $\epsilon = 2$ this is quartic oscillator with negative coupling

$$H = p^2 - gx^4, \qquad g > 0$$

+ Lower-dimensional massless ϕ^4 scalar field theory with $\lambda < 0$

ABS Conjecture

- Recently, Ai, Bender, Sarkar proposed relation for $\mathcal{PT}$-symmetric ϕ^4 scalar field theory: W. Ai, C. Bender, S. Sarkar, Phys. Rev. D 106, 125016 (2022)

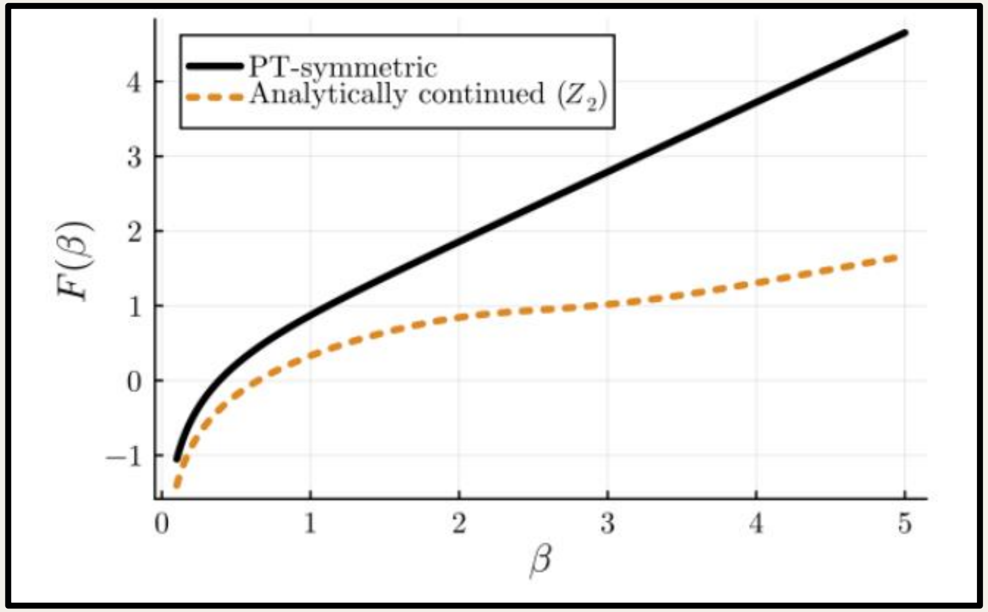
$$\ln Z_{\mathcal{PT}}(\beta, g) = \mathrm{Re} \ln Z_{\mathrm{Herm}}(\beta, \lambda \to -g + \mathrm{i}0^+)$$

- Partition function in terms of Hermitian field theory partition function
- λ analytically continued to negative values $\lambda \to -g + \mathrm{i}0^+, \ g > 0$

(Dis)proving ABS conjecture

+ Solved for spectra of Hermitian and $\mathcal{PT}$-symmetric theory
+ Found that ABS conjecture does not hold in $D = 1$ massless case

S. Lawrence, C. Peterson, P. Romatschke, **R. W.**, arXiv:2303.01470



Proving related conjecture

- However, looked at path integral of ϕ^4 theory with negative and positive couplings at large N in $D = 0$ and $D = 1$
- Found that a related conjecture holds at large N in $D = 0{,}1$:

$$\ln Z_{\mathrm{wedge}}(\beta, g) = \operatorname{Re} \ln Z_{\mathrm{Herm}}(\beta, \lambda \to -g + \mathrm{i}0^+)$$

- “wedge” is some complexified path integration domain
- Expect (but haven’t proved) this generalizes to $D = 2{,}3{,}4, \ldots$

Back to $O(N)$ model

- There exist complexified path integration domains for which negative couplings $\lambda < 0$ give well-defined physical theories
- We can apply “ABS”-type formula to get pressure per component $p(T)$ at $T > T_\mathrm{c}$:

$$p(T > T_\mathrm{c}) = \mathrm{Re}\, p_+(T > T_\mathrm{c})$$

- Small imaginary part $\mathrm{i}0^+$ in coupling breaks degeneracy of real part between $p_+(T)$ and $p_-(T)$ → gives dominant phase

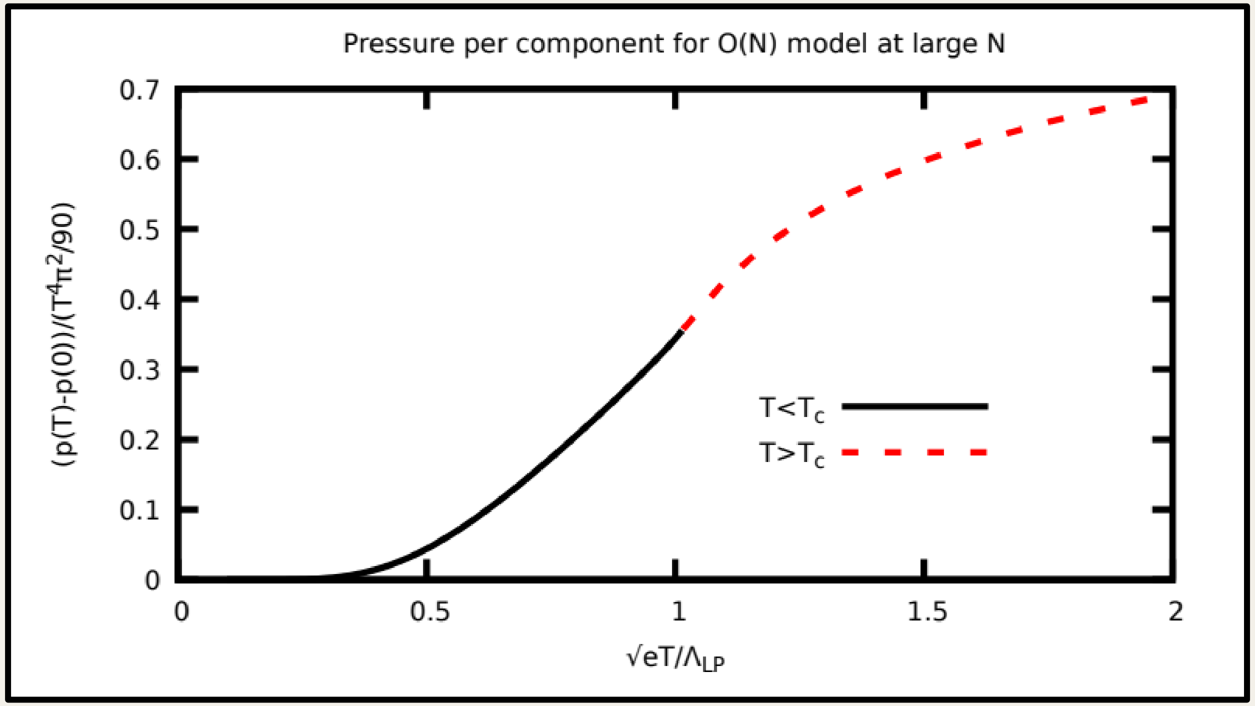
Pressure in massless $O(N)$ scalar model

- At T_c, $\mathrm{Re}\, p_+(T_c) = p_1(T_c)$
- At $T > T_c$, we have $\mathrm{Re}\, p_+(T_c)$
- At $T < T_c$, we have $p_1(T)$
- Now we have $p(T)$ calculated for all temperatures T

Equation of state

+ The vacuum-subtracted pressure normalized by the Stefan-Boltzmann (free field theory) limit:

P. Romatschke, arXiv:2212.03254



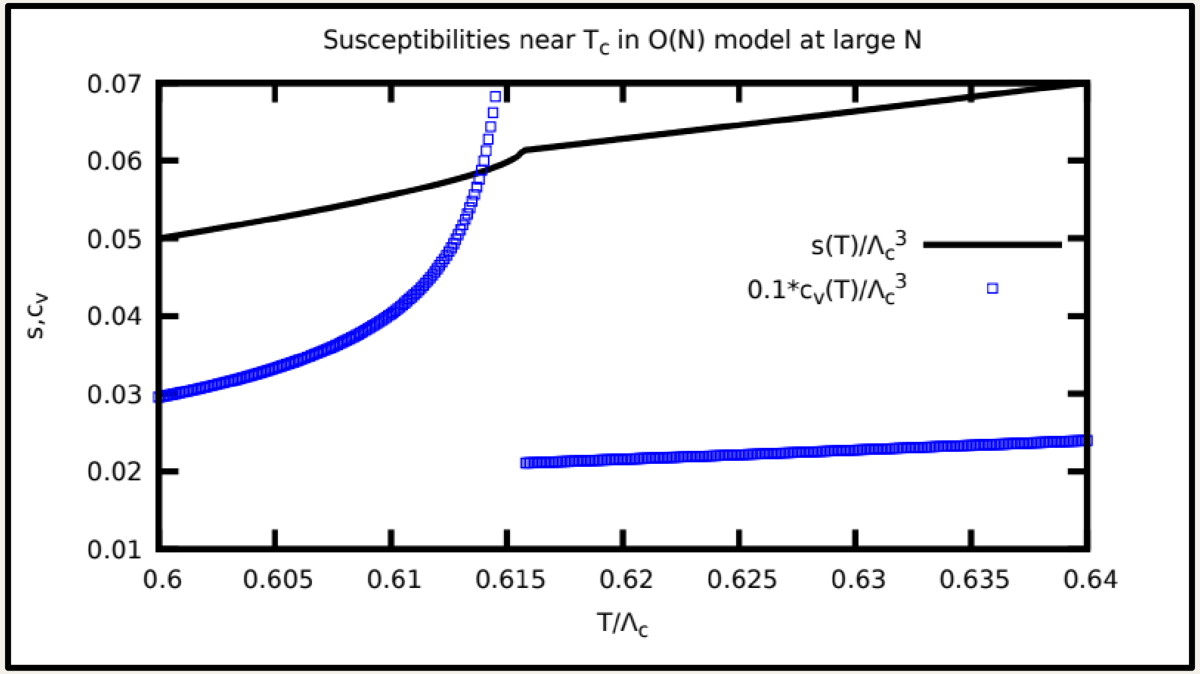
Remarks

- Second-order phase transition at T_c when $\mathrm{Re}\, p_+(T)$ takes over from $p_1(T)$
- Just like QCD! (Albeit at finite baryon density)
- Asymptotic freedom! (Approaches free field theory limit at high T)

2^{nd}-order phase transition

+ Second-order phase transition is apparent in the divergence of the specific heat $c_V(T)$ at $T = T_c$:

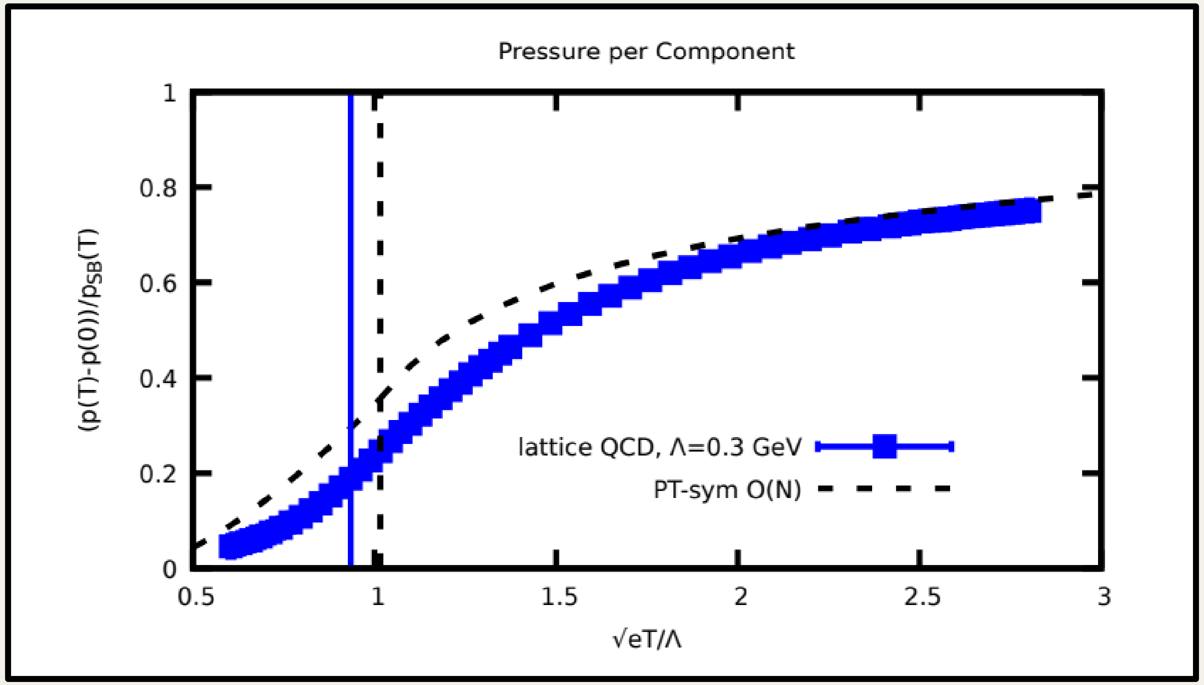
P. Romatschke, arXiv:2211.15683



Comparison to QCD

+ Comparison to Lattice QCD equation of state:

P. Romatschke, arXiv:2212.03254



Bound state

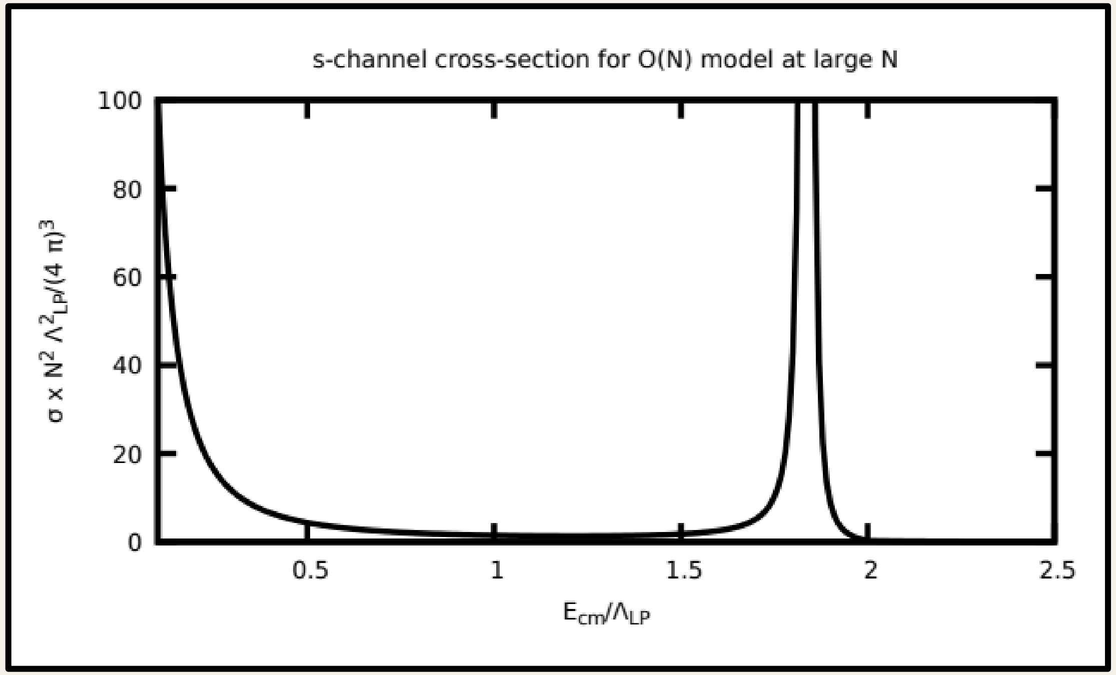
- Can calculate pole mass for Minkowski propagator of ζ'
- Need to reinstate non-zero modes of auxiliary field ζ' and calculate self-energy correction in propagator $D(k)$ for ζ'

$$D^{-1}(k) = \frac{N}{32\pi^2}\left(2 + 2\ln\left(\frac{\Lambda_{\mathrm{LP}}}{\sqrt{\mathrm{i}\zeta_0}}\right) - 2\sqrt{\frac{k^2 + 4\mathrm{i}\zeta_0}{k^2}}\operatorname{arctanh}\sqrt{\frac{4\mathrm{i}\zeta_0}{k^2 + 4\mathrm{i}\zeta_0}}\right)$$

- $D(k_0 \to \mathrm{i}\omega - 0^+, \mathbf{k}) = 0 \to$ gives mass of bound state

Bound state

+ Pole of Minkowski propagator with mass $m_\zeta \approx 1.84\sqrt{\mathrm{i}(\zeta_0^*)_1} \approx 3\Lambda_{\mathrm{LP}}$
+ $\vec{\phi}$ pair: Bound states just like QCD! P. Romatschke, arXiv:2305.05678



The $O(N)$ model: takeaway

- A playground for QCD-like physics!
- Can test assumptions in heavy ion physics
  - Thermalization in collisions of bound states
  - Bound-state multiplicities compared to Cooper-Frye formula

More takeaway

- Negative (or even complex) potentials in Lagrangians are meaningful if path integral is integrated over certain complex domains
- Proposed continuum limit for theories with Landau poles in the UV: coupling becomes negative
- But the fields become complexified such that the path integral is defined and physical
- Theories normally with a Landau pole in UV become asymptotically free

Future outlook: QCD?

- We might expect QCD to have Landau pole at Λ_{QCD}
- Perturbative $\alpha_s(\bar{\mu})$ has divergence just like QED and ϕ^4 theory
- Large-N-like techniques might be applicable to QCD calculations
- Can “get around” the Landau pole by taking $\alpha_s < 0$ in the IR