

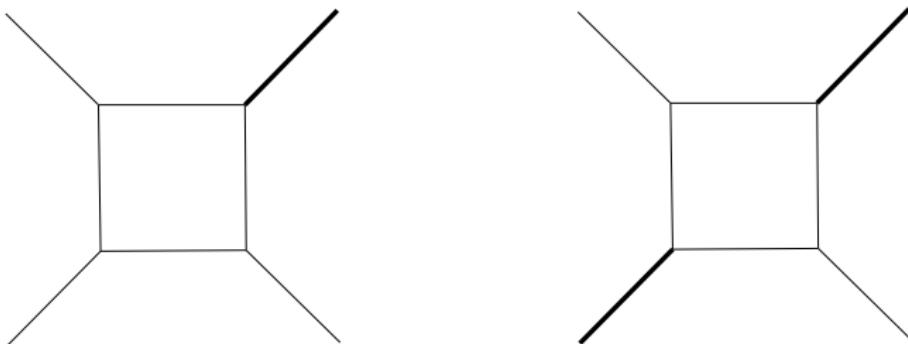
The massless single off-shell scalar box integral

Branch cut structure and all-order ε -expansion

Fabian Wunder

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QCD Masterclass in Saint-Jacut-de-la-Mer
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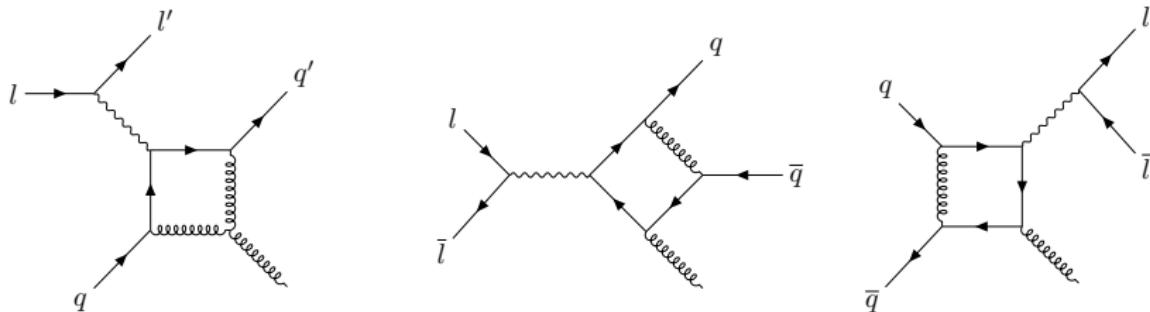
Juliane Haug, FW, arXiv:2211.14110, *JHEP* 02 (2023) 177
Juliane Haug, FW, arXiv:2302.01956, *JHEP* 05 (2023) 059

Outline

1. Motivation
2. Calculating the single off-shell scalar box integral in d dimensions
3. All-order ε -expansion of the scalar box integral
4. Conclusion and Outlook

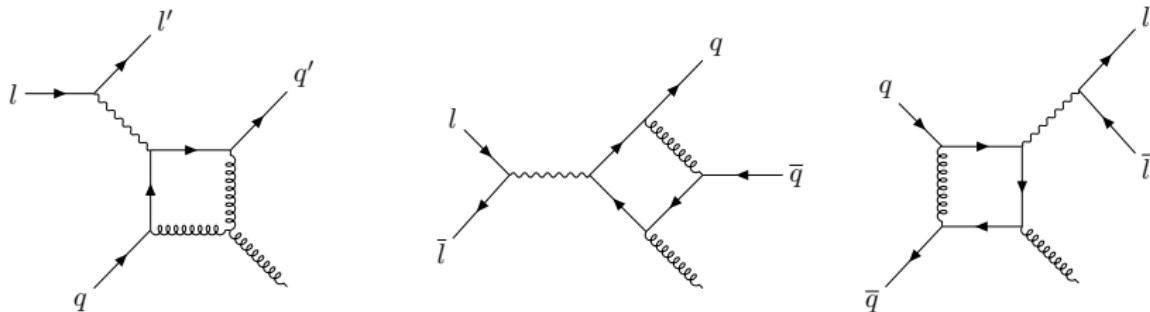
1. Motivation

Why do we care about the branch-cuts and all-order ε expansion of the massless single off-shell scalar box integral?



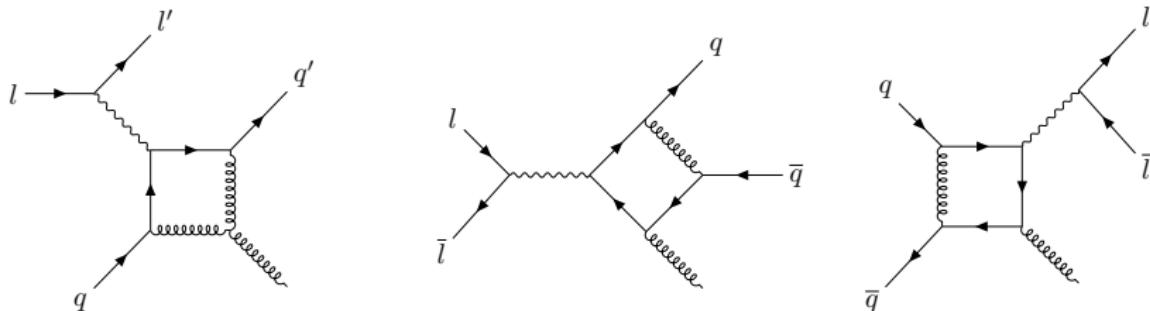
- ▶ **Why single off-shell box?** → important QCD processes (SI)DIS, SIA, DY feature single off-shell boson connected to box at NNLO
- ▶ **Why massless?** → Light quark masses negligible in high energy limit therefore massless propagators
- ▶ **Why scalar?** → Passarino-Veltman reduction of tensor one loop integrals to scalar integrals
- ▶ **Why branch-cut structure?** → Imaginary parts develop when certain momentum invariants change sign; e.g. difference between DY and DIS.
- ▶ **Why higher orders in ε ?** → box integral might get multiplied with other poles – e.g. from phase space, $(1-z)^{-1-\varepsilon} = -\frac{1}{\varepsilon}\delta(1-z) + \dots$

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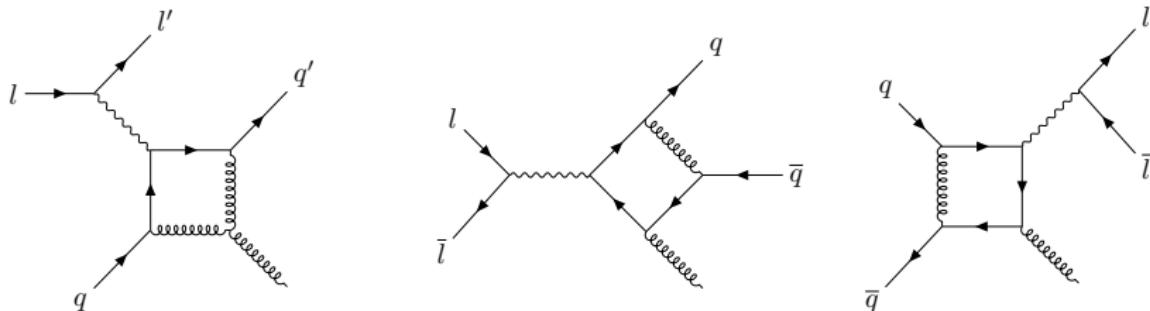
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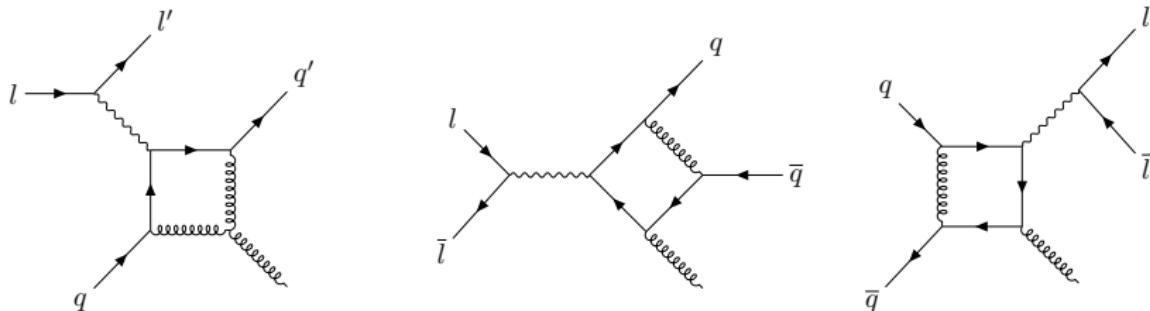
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The massless single-off shell scalar box integral in the literature

- ▶ K. Fabricius and I. Schmitt [1979]: Box integral with explicit imaginary part up to $\mathcal{O}(\varepsilon^0)$
- ▶ Matsuura et al. [1989]: Result in terms of 3 Gauss hypergeometric functions for general d
- ▶ Bern et al. [1994]: Rule for analytic continuation of result up to $\mathcal{O}(\varepsilon^0)$
- ▶ G. Duplančić and B. Nižić [2001]: Systematically keep causal $+i0$, result up to $\mathcal{O}(\varepsilon^0)$
- ▶ Lyubovitskij et al. [2021]: All-order ε -expansion, branch cuts not discussed

Literature summarized

Investigated for a long time, many aspects well understood but no all-order ε -expansion with explicit real and imaginary part valid in all kinematic regions.

General program



4-fold Feynman parameter integral



Calculating as (hypergeometric) function analytic in $d = 4 - 2\epsilon$



Transform to a series in ϵ

Goal

All-order ϵ -expansion valid in all kinematic regions (DIS $q^2 < 0$, SIA & DY $q^2 > 0$; DIS & DY $s > 0$ & $t, u < 0$, SIA $s, t, u > 0$) where real and imaginary parts are explicit.

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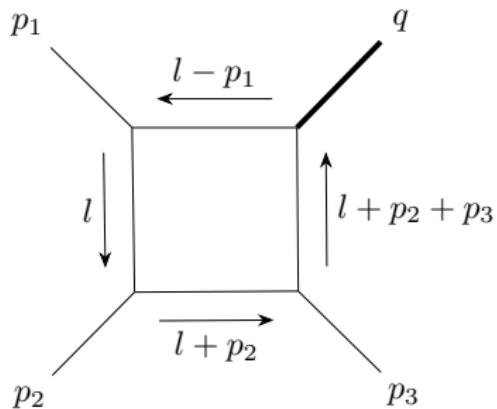
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2. Calculating the single off-shell scalar box integral in d dimensions

The scalar box integral in dimensional regularization



- ▶ External momenta taken to be incoming
- ▶ massless propagators
- ▶ Dimensional regularization with $d = 4 - 2\epsilon$
- ▶ Keep causal $+i0$ throughout
- ▶ $q^2 \neq 0, p_i^2 = 0$

$$D_0 \equiv \frac{\mu^{4-d}}{i\pi^{d/2}} \int d^d l \frac{1}{[l^2 + i0] [(l + p_2)^2 + i0] [(l + p_2 + p_3)^2 + i0] [(l - p_1)^2 + i0]}$$

Feynman parametrization

Feynman parametrization, evaluate loop integral →

$$D_0 = \mu^{2\varepsilon} \Gamma(2 + \varepsilon) \int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4)}{[-s_1 x_1(x_2 + x_3) - s_2 x_3(x_1 + x_4) - s_3 x_1 x_3 - i0]^{2+\varepsilon}},$$

with Mandelstam variables

$$s_1 = (p_1 + p_2)^2, \quad s_2 = (p_2 + p_3)^2, \quad s_3 = (p_1 + p_3)^2$$

Decouple Feynman parameter integrals through [Smirnov, 2012]

$$\begin{aligned} x_1 &\rightarrow \eta_1 \xi_1, & x_4 &\rightarrow \eta_1(1 - \xi_1), \\ x_3 &\rightarrow \eta_2 \xi_2, & x_2 &\rightarrow \eta_2(1 - \xi_2) \end{aligned}$$

Evaluate η -integrals in terms of Euler Beta function →

$$D_0 = \mu^{2\varepsilon} \frac{\Gamma(2 + \varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} \int_0^1 d\xi_1 \int_0^1 d\xi_2 [-s_1 \xi_1 - s_2 \xi_2 - s_3 \xi_1 \xi_2 - i0]^{-\varepsilon-2}$$

ξ -integrals symmetric under $s_1 \leftrightarrow s_2$, function of 3 variables

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Factoring out $s_1 s_2 / s_3$

Use

$$(a - i0)^\alpha = (b - i0)^\alpha \left(\frac{a}{b} - i0 \operatorname{sgn}(b) \right)^\alpha, \quad \text{where } a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}, \alpha \in \mathbb{C},$$

to factor out $s_1 s_2 / s_3 \rightarrow$

$$\begin{aligned} D_0(s_1, s_2, q^2) &= \mu^{2\varepsilon} \frac{\Gamma(2+\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} \left(\frac{s_1 s_2}{s_3} - i0 \right)^{-\varepsilon-2} \\ &\times \int_0^1 d\xi_1 \int_0^1 d\xi_2 \left[x_2 \xi_1 + x_1 \xi_2 - x_1 x_2 \xi_1 \xi_2 - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon-2}, \end{aligned}$$

depends on only 2 dimensionless variables

$$x_1 \equiv -\frac{s_3}{s_1}, \quad x_2 \equiv -\frac{s_3}{s_2}$$

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Several substitutions & evaluating 1 integral & splitting of integrals →

$$D_0(s_1, s_2, q^2) = -\frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^{\varepsilon} \times \{I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2))\}, \quad (1)$$

where

$$I(\chi) \equiv \int_0^\chi \frac{d\zeta}{1-\zeta} \left([\zeta - i0 \operatorname{sgn}_{123}]^{-\varepsilon-1} - 1 \right),$$

with abbreviation

$$\operatorname{sgn}_{123} \equiv \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right)$$

Remark

We essentially decomposed the *master integral* $D_0(s_1, s_2, q^2)$ into a sum of much simpler integrals $I(\chi)$ (→ also possible for other loop integrals?!)

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Denominator $1 - \chi\zeta$ diverges for $\chi > 1$

→ Introduce regulator $\chi \rightarrow \chi + i\tilde{\Omega}$ to split integral in two

$$\begin{aligned} I(\chi) &= [\chi - i0 \operatorname{sgn}_{123}]^{-\varepsilon} \int_0^1 d\zeta \zeta^{-\varepsilon-1} (1 - (\chi + i\tilde{\Omega})\zeta)^{-1} - \int_0^1 d\zeta \frac{\chi}{1 - (\chi + i\tilde{\Omega})\zeta} \\ &= -\frac{1}{\varepsilon} [\chi - i0 \operatorname{sgn}_{123}]^{-\varepsilon} {}_2F_1(1, -\varepsilon, 1 - \varepsilon; \chi + i\tilde{\Omega}) + \ln(1 - \chi - i\tilde{\Omega}) \end{aligned}$$

- ▶ Both ${}_2F_1$ and \ln evaluated on branch cut for $\chi > 1$
- ▶ Branch cuts are spurious and must cancel

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Cancellation of spurious branch cuts

Add integrals in eq. (1),

$$D_0(s_1, s_2, q^2) = -\frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^\varepsilon \\ \times \{ I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2)) \},$$

use different regulator $i\tilde{0}_i$ for each integral

Sum $I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2))$ contains following logarithms

$$\ln(1-x_1-i\tilde{0}_1) + \ln(1-x_2-i\tilde{0}_2) - \ln((1-x_1)(1-x_2)-i\tilde{0}_3)$$

- ▶ Real parts cancel
- ▶ Choose signs of $i\tilde{0}_i$ such that imaginary parts cancel as well

Cancellation of spurious branch cuts

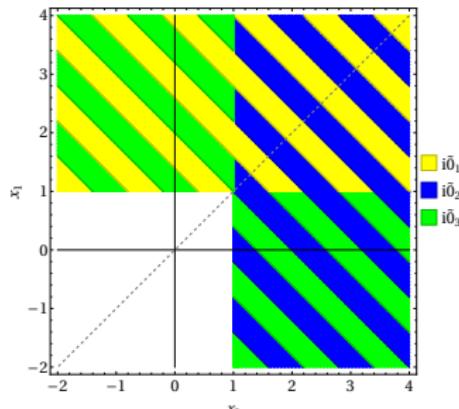
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Result in terms of hypergeometric functions

$$\begin{aligned}
 D_0(s_1, s_2, q^2) = & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^\varepsilon \\
 & \times \left\{ \left[-\frac{s_3}{s_1} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\
 & + \left[-\frac{s_3}{s_2} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
 & \left. - \left[-\frac{s_3 q^2}{s_1 s_2} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + i\tilde{0}\right) \right\} \quad (2)
 \end{aligned}$$

- ▶ Imaginary parts of hypergeometric functions will cancel by construction

Comparison to literature

Combine prefactors $(\dots)^\varepsilon$ and $[\dots]^{-\varepsilon} \rightarrow$

$$\begin{aligned}
 D_0(s_1, s_2, q^2) = & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \\
 & \times \left\{ \left[\frac{\mu^2}{-s_2 - i0} \right]^\varepsilon {}_2F_1 \left(1, -\varepsilon, 1 - \varepsilon; -\frac{s_3}{s_1} + i\tilde{0} \operatorname{sgn} \left(\frac{s_3}{s_2} - \frac{s_3}{s_1} \right) \right) \right. \\
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 \end{aligned}$$

- ▶ Agrees with [Matsuura et al., 1989], [Bern et al., 1994], and [Lyubovitskij et al., 2021] if all three hypergeometric functions are away from their branch cut

3. All-order ε -expansion of the scalar box integral

Starting point for ε -expansion of D_0

Write eq. (2) as

$$\begin{aligned} D_0(s_1, s_2, q^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left| \frac{s_3 \mu^2}{s_1 s_2} \right|^{\varepsilon} \\ &\times \exp \left[i\pi \varepsilon \Theta \left(-\frac{s_3}{s_1 s_2} \right) \right] \mathcal{D}_0 \left(\varepsilon; -\frac{s_3}{s_1}, -\frac{s_3}{s_2} \right), \end{aligned}$$

where \mathcal{D}_0 abbreviates the sum of hypergeometric functions (upper sign choice for $x_1 \geq x_2$, else lower sign),

$$\mathcal{D}_0(\varepsilon; x_1, x_2) = \mathcal{F}_{\pm}(\varepsilon; x_1) + \mathcal{F}_{\mp}(\varepsilon; x_2) - \mathcal{F}_{+}(\varepsilon; 1 - (1 - x_1)(1 - x_2))$$

Here,

$$\mathcal{F}_{\pm}(\varepsilon; x) \equiv (x - i0 \operatorname{sgn}_{123})^{-\varepsilon} {}_2F_1(1, -\varepsilon, 1 - \varepsilon; x \pm i\tilde{0}).$$

Epsilon expansion of $\mathcal{F}_\pm(\varepsilon; x)$

Function to be expanded

$$\mathcal{F}_\pm(\varepsilon; x) \equiv (x - i0 \operatorname{sgn}_{123})^{-\varepsilon} {}_2F_1(1, -\varepsilon, 1 - \varepsilon; x \pm i\tilde{0})$$

► Ingredients for expansion:

Epsilon expansion of ${}_2F_1$ (resp. Lerch Zeta function Φ)

$${}_2F_1(1, -\varepsilon, 1 - \varepsilon; z) = -\varepsilon \Phi(z, 1, -\varepsilon) = 1 - \sum_{n=1}^{\infty} \varepsilon^n \operatorname{Li}_n(z)$$

Inversion formula for ${}_2F_1$ (use for $x > 1$)

$${}_2F_1(1, -\varepsilon, 1 - \varepsilon; z \pm i\tilde{0}) + {}_2F_1\left(1, \varepsilon, 1 + \varepsilon; \frac{1}{z}\right) = 1 + (-z \mp i\tilde{0})^\varepsilon \frac{\pi \varepsilon}{\sin(\pi \varepsilon)}$$

Goal

Make cancellation of spurious branch cuts explicit

Epsilon expansion of $\mathcal{F}_\pm(\varepsilon; x)$

Function to be expanded

$$\mathcal{F}_\pm(\varepsilon; x) \equiv (x - i0 \operatorname{sgn}_{123})^{-\varepsilon} {}_2F_1(1, -\varepsilon, 1 - \varepsilon; x \pm i\tilde{0})$$

► Ingredients for expansion:

Epsilon expansion of ${}_2F_1$ (resp. Lerch Zeta function Φ)

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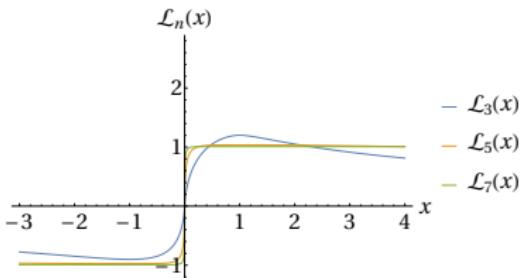
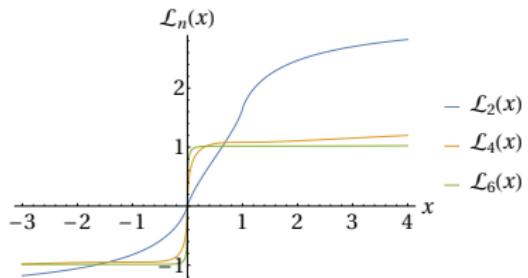
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Single-valued polylogarithms (inspired by [L. Lewin, 1991])

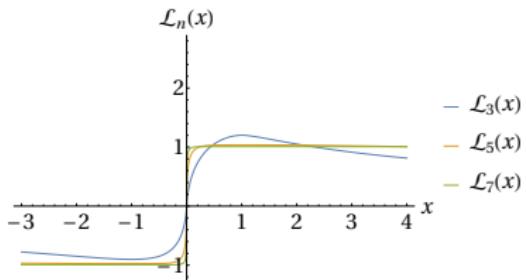
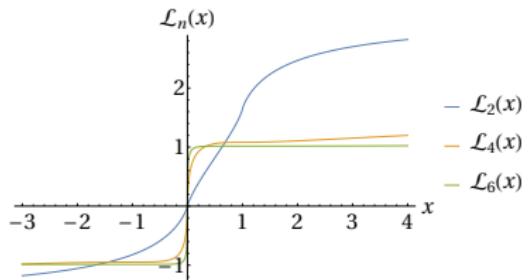
$$\mathcal{L}_n(x) \equiv \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \ln^k |x| \operatorname{Li}_{n-k}(x) + \frac{(-1)^{n-1}}{n!} \ln^{n-1} |x| \ln |1-x|$$



- ▶ $\mathcal{L}_n(x)$ is single-valued, in contrast to $\operatorname{Li}_n(x)$
- ▶ $\mathcal{L}_n(x)$ is continuous for all $x \in \mathbb{R}$
- ▶ $\mathcal{L}_n(x)$ is bounded on \mathbb{R} , in contrast to $\operatorname{Li}_n(x)$
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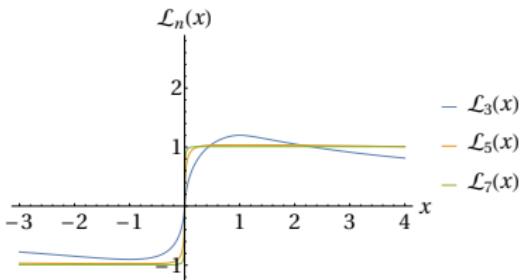
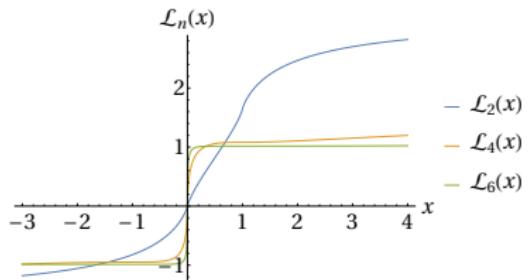
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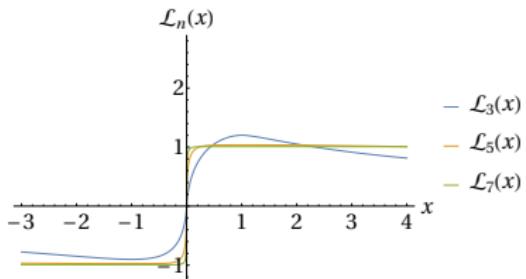
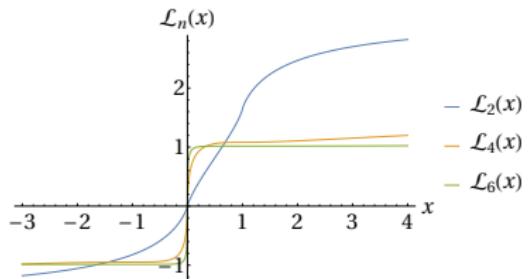
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Expansion of $\mathcal{F}_\pm(\varepsilon; x)$ on entire real axis

$$\mathcal{F}_\pm(\varepsilon; x) = e^{i\pi\varepsilon \operatorname{sgn}_{123} \Theta(-x)} \mathfrak{F}(\varepsilon; x) \mp i\pi\varepsilon \Theta(x - 1),$$

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- ▶ spurious branch cut explicit
- ▶ all functions of real variable x manifestly real
- ▶ \mathfrak{F} finite for $x \rightarrow \pm\infty$ in every order in ε

Resum $\ln |x|$ terms to obtain

$$\mathfrak{F}(\varepsilon; x) = |x|^{-\varepsilon} + \varepsilon \ln |1-x| - \sum_{n=2}^{\infty} \varepsilon^n \left[\frac{(-1)^n \ln |1-x| \ln^{n-1} |x|}{n!} + \mathcal{L}_n(x) \right]$$

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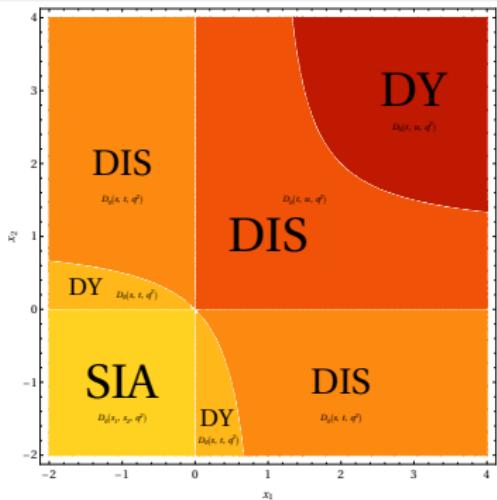
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\mathcal{D}_0 suffering from spurious branch-cuts

$$\mathcal{D}_0(\varepsilon; x_1, x_2) = \mathcal{F}_{\pm}(\varepsilon; x_1) + \mathcal{F}_{\mp}(\varepsilon; x_2) - \mathcal{F}_{+}(\varepsilon; 1 - (1 - x_1)(1 - x_2))$$



- ▶ logarithmic divergence for $x_1, x_2 = 1$ in DIS region cancels between 3 \mathcal{F} -functions
- ▶ spurious branch cuts $\mp i\pi\varepsilon\Theta(x-1)$ cancel in all kinematic regions

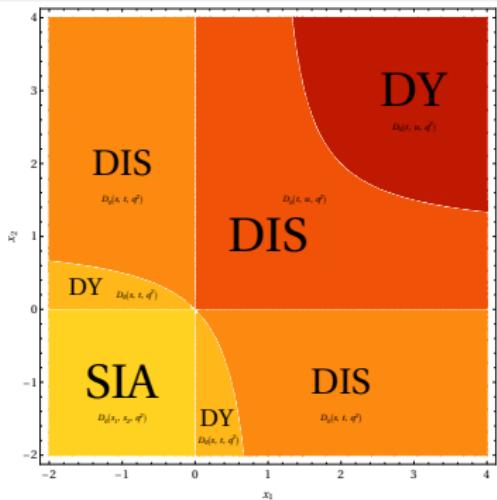
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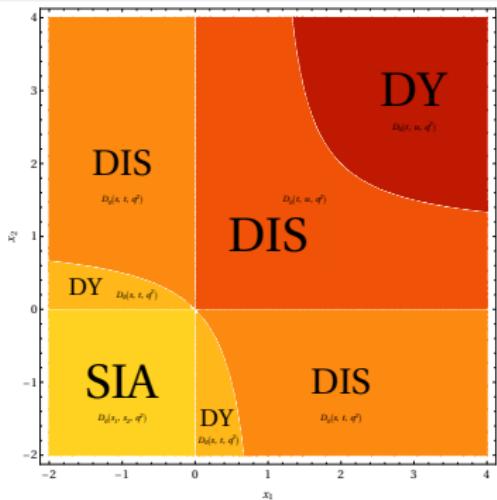
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Epsilon expansion of the scalar box integral

$$D_0(s_1, s_2, q^2) = \frac{1}{\varepsilon^2} \frac{\Gamma(1 + \varepsilon) \Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \frac{2}{s_1 s_2} \left| \frac{s_3 \mu^2}{s_1 s_2} \right|^{\varepsilon} \\ \times \left[(\Theta(-s_2) + \Theta(s_2) e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1) + (\Theta(-s_1) + \Theta(s_1) e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_2) \right. \\ \left. - (\Theta(-q^2) + \Theta(q^2) e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1 + x_2 - x_1 x_2) \right]$$

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Alternative representation of ε -expansion

Bring factor $|s_3\mu^2/s_1s_2|^\varepsilon$ into square brackets

$$\begin{aligned} D_0(s_1, s_2, q^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1s_2} \\ &\times \left[\left(\frac{\mu^2}{-s_2 - i0} \right)^\varepsilon \left| \frac{s_3}{s_1} \right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3}{s_1}\right) + \left(\frac{\mu^2}{-s_1 - i0} \right)^\varepsilon \left| \frac{s_3}{s_2} \right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3}{s_2}\right) \right. \\ &\quad \left. - \left(\frac{\mu^2}{-q^2 - i0} \right)^\varepsilon \left| \frac{s_3 q^2}{s_1 s_2} \right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3 q^2}{s_1 s_2}\right) \right] \end{aligned}$$

- Compared to result in terms of hypergeometric functions, ${}_2F_1$ was replaced by a single-valued version,

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4. Conclusion and Outlook

Conclusion and Outlook

Summary of results:

- ▶ hypergeometric representation in d -dimensional has spurious branch-cuts
- ▶ Real and imaginary parts made explicit to all orders in ε , results free of spurious branch cuts
- ▶ All-order ε -expansion in terms of special class of single-valued polylogarithms
- ▶ Any kinematic divergences contained in logarithms, as single-valued polylogarithms are bounded

Outlook on possible generalizations:

- ▶ Same calculation goes through for the non-adjacent double off-shell case; more sophisticated methods needed for further generalization of results to more off-shell particles
- ▶ Differential equations [Gehrmann and Remiddi, 2000], Mellin-Barnes integrals [Smirnov, 1999], negative dimensions [Anastasiou et al., 2000], recurrence relations w.r.t. d [Fleischer et al., 2003]
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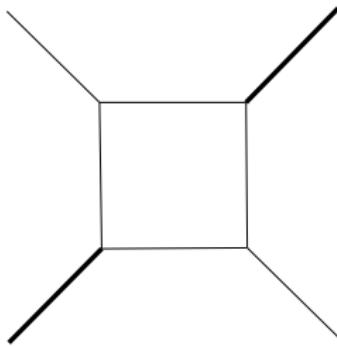
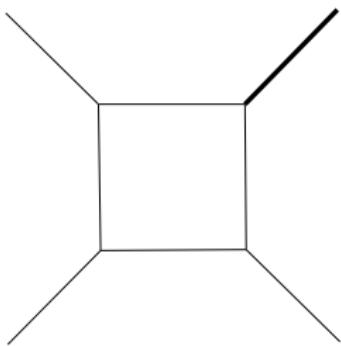
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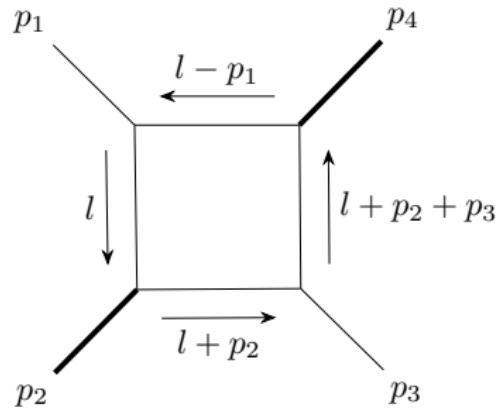
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Thank you for your attention!



Now open for questions.

5. Generalization to two non-adjacent off-shell external particles



Juliane Haug, FW, arXiv:2302.01956, *JHEP* 05 (2023) 059

Generalization to two non-adjacent off-shell external particles

General box integral with massless propagators after Feynman parametrization:

$$D_0 = \mu^{2\varepsilon} \Gamma(2 + \varepsilon) \times \int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4)}{[-x_1 x_2 s_1 - x_1 x_3 p_4^2 - x_1 x_4 p_1^2 - x_2 x_3 p_3^2 - x_2 x_4 p_2^2 - x_3 x_4 s_2 - i0]^{2+\varepsilon}}$$

With the same substitution as before,

$$x_1 = \eta_1 \xi_1, \quad x_2 = \eta_2 (1 - \xi_2), \quad x_3 = \eta_2 \xi_2, \quad x_4 = \eta_1 (1 - \xi_1),$$

term in denominator becomes

$$\begin{aligned} & -\eta_1 \eta_2 (1 - \xi_1) (1 - \xi_2) s_1 - \eta_1 \eta_2 \xi_1 \xi_2 s_2 - \eta_1 \eta_2 \xi_1 (1 - \xi_2) p_2^2 - \eta_1 \eta_2 \xi_2 (1 - \xi_1) p_4^2 \\ & - \eta_1^2 \xi_1 (1 - \xi_1) p_1^2 - \eta_2^2 \xi_2 (1 - \xi_2) p_3^2 - i0 \end{aligned}$$

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Result in terms of 4 hypergeometric functions

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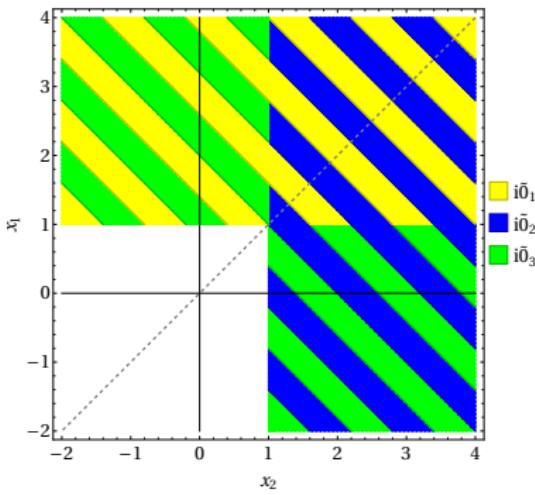
Epsilon expansion in the non-adjacent double off-shell case

$$\begin{aligned}
 D_0(s_1, s_2, p_2^2, p_4^2) = & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \left| \frac{(p_2^2 + p_4^2 - s_1 - s_2)\mu^2}{s_1 s_2 - p_2^2 p_4^2} \right|^{\varepsilon} \\
 & \times \left\{ \left(\Theta(-s_1) + \Theta(s_1)e^{i\pi\varepsilon} \right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2}\right) \right. \\
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 \end{aligned}$$

- ▶ All-order ε -expansion (not previously known)
- ▶ Real and imaginary parts can be easily read off
- ▶ Valid in all kinematic regions ($s_1, s_2, p_2^2, p_4^2 \in \mathbb{R}$)
- ▶ No spurious branch cuts

Cancellation of imaginary parts of logarithms

$$\begin{aligned} i\pi \left[-\operatorname{sgn}(\tilde{0}_1) \Theta(x_1 - 1) - \operatorname{sgn}(\tilde{0}_2) \Theta(x_2 - 1) \right. \\ \left. + \operatorname{sgn}(\tilde{0}_3) \{\Theta(x_1 - 1) \Theta(1 - x_2) + \Theta(1 - x_1) \Theta(x_2 - 1)\} \right] \stackrel{!}{=} 0 \quad (3) \end{aligned}$$



► Conditions for eq. (3) to hold:

$$\operatorname{sgn}(\tilde{0}_1) \stackrel{!}{=} \operatorname{sgn}(\tilde{0}_3) \text{ (yellow-green region)}$$

$$\operatorname{sgn}(\tilde{0}_1) \stackrel{!}{=} -\operatorname{sgn}(\tilde{0}_2) \text{ (yellow-blue region)}$$

$$\operatorname{sgn}(\tilde{0}_2) \stackrel{!}{=} \operatorname{sgn}(\tilde{0}_3) \text{ (blue-green region)}$$

► Choose (only relative signs matter)

$$\tilde{i0}_1 \equiv i\tilde{0} \operatorname{sgn}(x_1 - x_2)$$

$$\tilde{i0}_2 \equiv i\tilde{0} \operatorname{sgn}(x_2 - x_1)$$

$$\tilde{i0}_3 \equiv i\tilde{0}$$

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