

The Shape of L

How a myriad of Lindblad operators behave.

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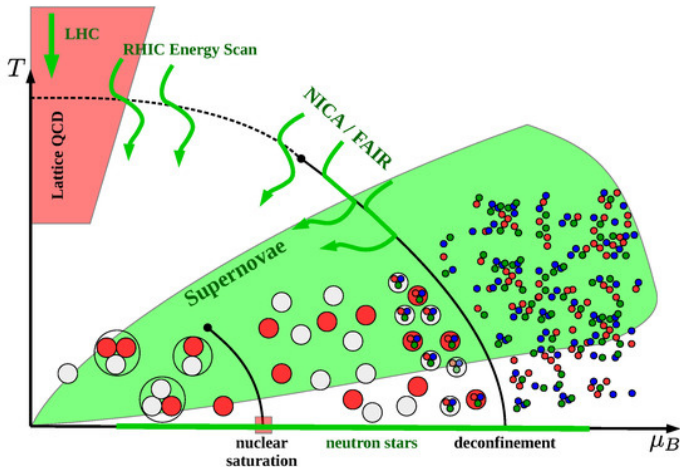
- ① Context
- ② Quantum Trajectories
- ③ New Implementation
- ④ Appendix



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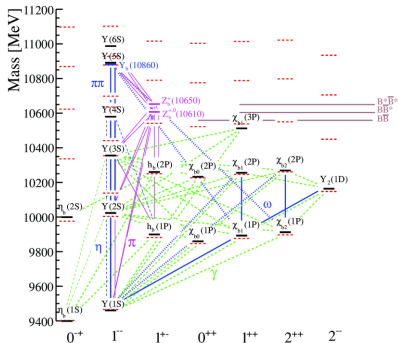
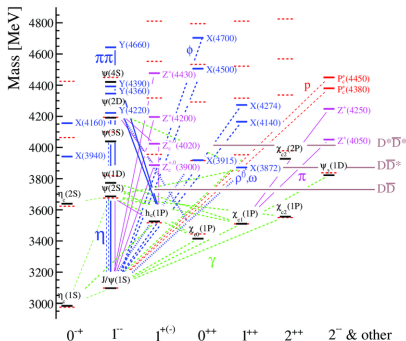
Quark-gluon plasma

It is a deconfined phase on the QCD phase diagram [1].



Quarkonia

They are heavy quark-antiquark ($q\bar{q}$) bound states [2].

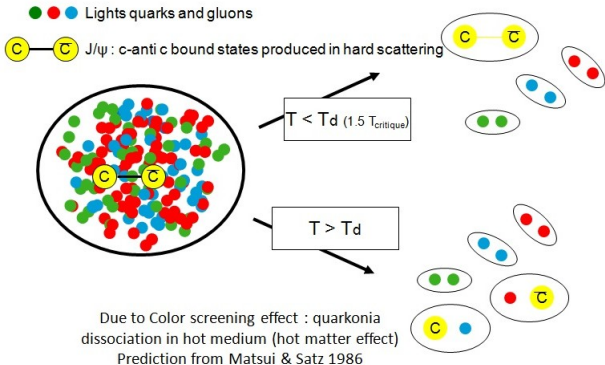


Why quarkonia as a probe

- 1 Hard scale: quarkonia mass.
- 2 Small radius: harder to dissociate from color screening than light quark matter.

$$\Delta E_{J/\psi} = 2M_D - M_{J/\psi} \approx 0.6 \text{ GeV} \gg \Lambda_{QCD} \approx 0.2 \text{ GeV}. \quad (1)$$

- 3 Well-known probe.



Notation disclaimer

Open Quantum System \longrightarrow we divide the full quantum system (T) into well-differentiated parts: the subsystem (S) and the environment (E) [3].

Main character \longrightarrow density matrix, ρ :

$$\rho = \rho_i \sum_i |\psi_i\rangle \langle \psi_i|. \quad (2)$$

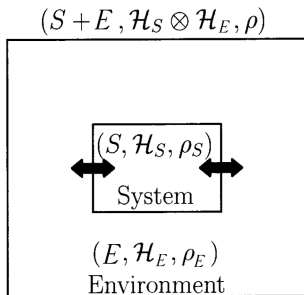
Observables \longrightarrow $\langle \mathcal{O} \rangle = \text{Tr}\{\rho \mathcal{O}\}$

Hamiltonian:

$$H_T = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I, \text{ where } H_I = V_S \otimes V_E.$$

Evolution: the Liouville - von Neumann equation.

$$\frac{d}{dt} \rho_T = -i[H_T, \rho_T]. \quad (3)$$



Steps to be taken

From Liouville-von Neumann's equation we can take the following steps:

- 1 Divide DoF into subsystem + background.
- 2 Trace out the environmental DoF \rightarrow loss of unitarity.
- 3 Make some further assumptions.

$$\begin{array}{ccc} \rho(0) = \rho_S(0) \otimes \rho_E & \xrightarrow{\text{unitary evolution}} & \rho(t) = U(t,0)[\rho_S(0) \otimes \rho_E]U^\dagger(t,0) \\ \text{tr}_E \downarrow & & \downarrow \text{tr}_E \\ \rho_S(0) & \xrightarrow{\text{dynamical map}} & \rho_S(t) = V(t)\rho_S(0) \end{array}$$

The assumptions

- ① Born approximation \rightarrow weakly interacting system.

$$\rho_T(t) \approx \rho_S(t) \otimes \rho_E(t) \approx \rho_S(t) \otimes \rho_E(0) \quad (4)$$

- ② Markov approximation \rightarrow no memory on the system (we could extend our integral up to minus infinity without affecting the result).

$$(0, t) \rightarrow (-\infty, t) \quad (5)$$

- ③ Born-Oppenheimer approximation \rightarrow the light degrees of freedom of the plasma accommodate very fast to changes produced by quarkonia (\sim atomic physics).

These approximations also refer to the characteristic timescales τ_i of the different parts of the system, namely:

$$\tau_S = 1/E, \quad \tau_E \sim 1/T, \quad \tau_R \sim M/T^2. \quad (6)$$

Here E is the binding energy of the state, T is the temperature, M the particle mass and g the coupling constant.

We look for the regime where:

$$\tau_E \ll \tau_R \longrightarrow \text{Born and Markov approximations}, \quad (7)$$

$$\tau_E \ll \tau_S \longrightarrow \text{Born-Oppenheimer approximation}. \quad (8)$$

As a result, after some rearranging, we get the Lindblad equation:

$$\frac{d\rho_S(t)}{dt} = -i[H_S(t), \rho_S(t)] + \sum_k \gamma_k \left(A_k \rho_S A_k^\dagger - \frac{1}{2} \{A_k^\dagger A_k, \rho_S(t)\} \right), \quad (9)$$

where, at least at first order, $A_k \approx V_S(t) + \frac{i}{4T} \frac{dV_S(t)}{dt}$ is called the Lindblad operator [4].

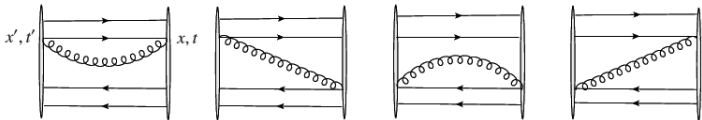
Conceptually, Lindblad operators are going to produce **jumps** between states (modifying the internal quantum numbers of the system). Thus, they are also called **jump operators**.

How to retrieve information: quantum trajectories

- 1 Projecting $\rho_S(t)$ into spherical harmonics.
- 2 Also, split into the singlet-octet colour basis.

$$\rho_S(t) = \text{diag}(\rho_S^{\text{sing},s}, \rho_S^{\text{oct},s}, \rho_S^{\text{sing},p}, \rho_S^{\text{oct},p}) \quad (10)$$

Great computational advantage: $3\text{D} \rightarrow 1\text{D} \cdot Y_m^\ell(\theta, \phi)$.



We compact the free subsystem hamiltonian with the 1-loop contributions, H_{1-loop} [4, 5, 6]. This is a **non-hermitian hamiltonian**.

$$H_{\text{eff}} = H_S + H_{1-loop} = H_S - \frac{i}{2} \sum_k \gamma_k A_k^\dagger A_k \quad (11)$$

Description of the algorithm

- 1 Non-hermitian hamiltonian evolution step is performed. Its non-unitarity makes the norm of the state decrease.

$$|\langle\psi(t_1)|\psi(t_1)\rangle| > |\langle\psi(t_2)|\psi(t_2)\rangle|, \text{ where } t_1 < t_2 \quad (12)$$

- 2 A random number decides if the jump is performed. The state will normally evolve until the norm goes below this random number.

$$\text{When } |\langle\psi(t)|\psi(t)\rangle| < \text{Random Number} \longrightarrow \text{jump}. \quad (13)$$

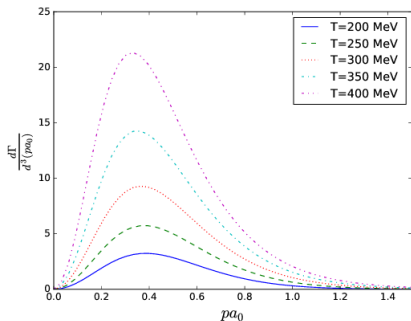
- 3 From the current state, another random number will decide the kind of jump done (next slide). We project using the corresponding jump operator:

$$|\psi_{new}\rangle = \hat{L}^x(\vec{q}) |\psi_{old}\rangle \quad (14)$$

- 4 Renormalize and back again.

Description of the algorithm

Jump operators must be obtained in a matrix fashion matching this way of expressing the density matrix. [5]. We would call the Lindblad operators $\hat{L}^x(\vec{q})$ from now on.



Decay rate is defined as:

$$\Gamma(p)_k = \sum_k L_k(p)L_k^\dagger(p). \quad (15)$$

$$\Gamma(p) = \sum_k \Gamma(p)_k \quad (16)$$

The curve depends on the physics of the system \rightarrow the explicit expression of the Lindblad operators.

QTRAJ (1.0 + ϵ)

QTRAJ 1.0 [7]: C-based code which simulates through the quantum trajectories algorithm and shows the relative population of colour and wave states for quarkonia.

The current potential available compatible with the Lindblad formalism is the Munich potential. This approach is adequate for a regime where $rT \ll 1$ and is performed with a finite number of Lindblad operators.

Goal of $+\epsilon$: New potentials \longrightarrow Infinite number of Lindblad operators \longrightarrow reach regime where $rT \approx 1$.

How?:

- ➊ Adding definitions of new potentials to QTRAJ.
- ➋ Modifying the selection rules \longleftrightarrow Defining new Lindblad operators.

Current efforts

Implementing new potentials, less restrictive, to try to perform up to $rT \approx 1$ [5].

$$\operatorname{Re}\{V_S(r)\}_{\text{singlet}} = -C_F\alpha_s(1/a_0)\frac{e^{-m_D r}}{r}, \quad (17)$$

$$\operatorname{Im}\{V_S(r)\} \frac{g^2 T}{2\pi} \int_0^\infty dx \frac{x}{(x^2 + 1)^2} \left[1 - \frac{\sin(xr m_D)}{xr m_D} \right] \quad (18)$$

where m_D is the Debye mass:

$$m_D = \sqrt{\frac{2N_c + N_f}{6}} gT \quad (19)$$

This will give a contribution to an infinite number of Lindblad operators.

New Lindblad operators:

Lindblad operators are in this framework:


$$\hat{L}^x(\vec{q}) = K_x \sqrt{\Delta(\vec{q})} cs\left(\frac{\vec{q} \cdot \hat{\vec{r}}}{2}\right), \quad (20)$$

where cs stands for $\sin\left(\frac{\vec{q} \cdot \hat{\vec{r}}}{2}\right)$ if $x \in \{s \rightarrow o, o \rightarrow s, o \rightarrow o (1)\}$ and $\cos\left(\frac{\vec{q} \cdot \hat{\vec{r}}}{2}\right)$ is $x \in \{o \rightarrow o (2)\}$. Using:

$$e^{-i\vec{k}\vec{r}} = \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kr) Y_{\ell m}(\vec{k}_u) Y_{\ell, m}^*(\vec{r}_u), \quad (21)$$

we get:

$$\hat{L}^x(\vec{q}) = K_x \sqrt{\Delta(\vec{q})} \sum_t^{\infty} \sum_{m=-\ell}^{\ell} j_\ell(qr) Y_\ell^m(\Omega_r) = \sum_t^{\infty} \hat{L}_\alpha^x(\vec{q}), \quad (22)$$

where for the case of the cosine $\alpha = 2t$ and for the sine $\alpha = 2t + 1$. 

Slight modification of the algorithm:

The change is in the third step, how selection rules are implemented:

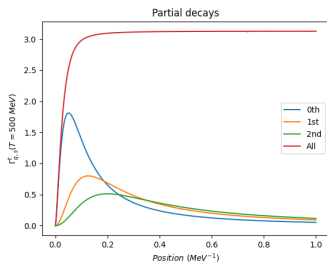
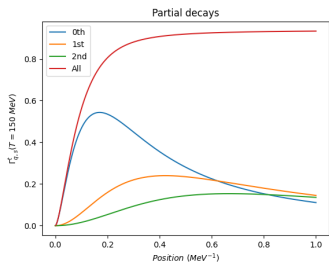
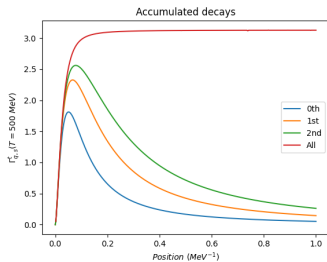
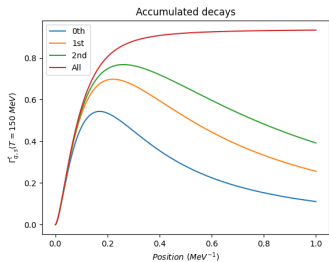
- 1 We choose, according to the current state, the kind of transition that quarkonia will undergo to apply its proper Lindblad operator:

$$\hat{L}^{s \rightarrow o}(\vec{q}), \quad \hat{L}^{o \rightarrow s}(\vec{q}), \quad \hat{L}^{o \rightarrow o(1)}(\vec{q}), \quad \hat{L}^{o \rightarrow o(2)}(\vec{q}). \quad (23)$$

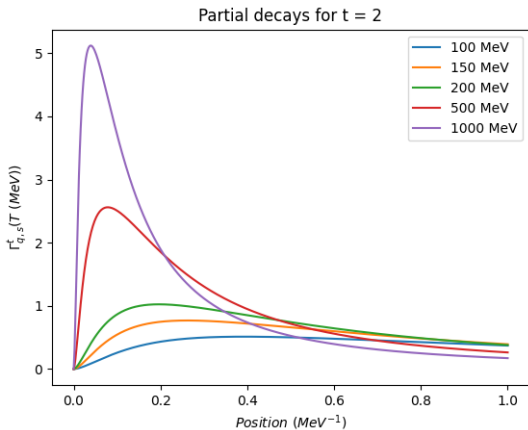
- 2 We choose, once the jump channel is known, the value of t of $\hat{L}_t^x(\vec{q})$, which we understand it as the virtual angular momentum of the one gluon exchange.
- 3 We choose q from its momentum distribution.
- 4 We apply the Lindblad operator so:

$$\hat{L}_t^x(\vec{q}) |\psi_{old}\rangle = |\psi_{new}\rangle. \quad (24)$$

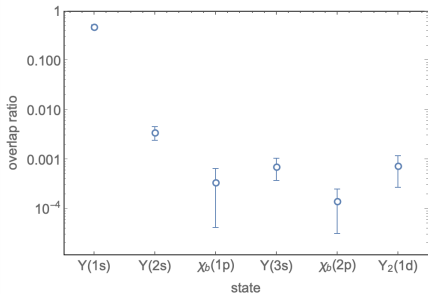
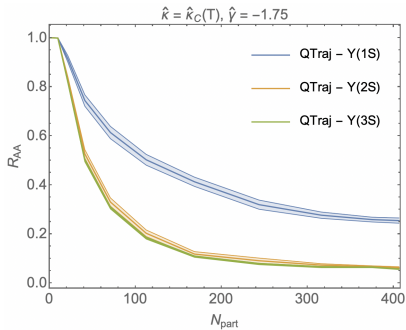
Behaviour of the jump operators



Behaviour of the jump operators



Plots that can be retrieved.



These results are from Strickland's original code [7].

Conclusions

- 1 The inclusion of less restrictive potentials allows the expansion the regime of validity of the simulations.
- 2 This means two things: either temperature does not have to be as high as before for applying this formalism or the small dipole approximation implicit in the Boltzmann equation is no longer applied. The latter case is of our greater interest.
- 3 The new shape of the Lindblad operators depend on the momentum exchanged with the medium particles. In the region of interest, $\Delta J = 1$ dominates.

Thank you!

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Approximations: Born approximation

It is a weak coupling between the subsystem and the environment,
 $H_I \ll 1$.

$$\rho_T(t) = \rho_S(t) \otimes \rho_E(t) + \rho_{corr}(t) \approx \rho_S(t) \otimes \rho_E(t), \quad (25)$$

where ρ_{corr} is the correlation component between the environment and the subsystem.

$$\frac{d\rho_{T,I}(t)}{dt} \approx - \int_0^t d\tau [H_I(t), [H_I(\tau), \rho_{S,I}(\tau) \otimes \rho_{E,I}(0)]] \quad (26)$$

Approximations: Markov approximation

Taking into account only the current step in order to obtain the next one $\rho_{S,I}(\tau) \longrightarrow \rho_{S,I}(t)$. We will perform the change of variable $\tau \longrightarrow \tau' = t - \tau$ so:

- $\tau = 0 \longrightarrow \tau' = t - \tau = t$
- $\tau = t \longrightarrow \tau' = t - \tau = 0$
- Since the correlation time of the environment is much less than the average relaxation time of the system we can take $t \longrightarrow \infty$.

If we also trace over the environment, we get:

$$\frac{d\rho_{S,I}(t)}{dt} \approx - \int_0^\infty d\tau \operatorname{tr}_E \{ [H_I(t), [H_I(t - \tau), \rho_{S,I}(t) \otimes \rho_{E,I}(0)]] \}. \quad (27)$$

Redfield equation.

Approximations: Born-Oppenheimer approximation

The environmental degrees of freedom move much faster than the quarkonium so effectively they instantly change to any changes that the quarkonium may induce.

$$V_S(t-s) \approx V_S(t) - s \frac{dV_S(t)}{dt} + \dots = V_S(t) - is[H_S, V_S(t)] + \dots \quad (28)$$

Gradient expansion for Brownian motion.