Last time

# DIS

performed a decomposition of $W_{\mu\nu}$:

$$W_{\mu\nu} = -W_1(x, Q^2)\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \frac{W_2(x, Q^2)}{m_p^2} \cdot \left[p_\mu - q_\mu \frac{p\cdot q}{q^2}\right] \cdot \left[p_\nu - q_\nu \frac{p\cdot q}{q^2}\right]$$

$W_1, W_2$ = structure functions

**Def.** $F_1(x, Q^2) \equiv m_p W_1(x, Q^2)$

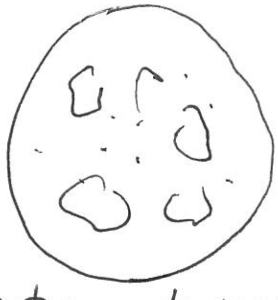
$$F_2(x, Q^2) \equiv \frac{Q^2}{2 m_p x} W_2(x, Q^2)$$

$F_1, F_2$ = dimensionless structure functions (more common)

$$\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4}\left[\frac{2}{m_p} F_1(x, Q^2) \sin^2\frac{\theta}{2} + \frac{2 m_p x}{Q^2} F_2(x, Q^2) \cos^2\frac{\theta}{2}\right]$$

DIS cross-section in proton's rest frame

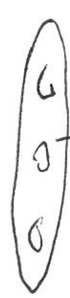
## The Parton Model (cont'd)



$\tau_{fluct} \approx \frac{1}{\Lambda_{QCD}}$

proton at rest

boost $\longrightarrow$



$\rightarrow P$

frozen in time

$t_{fluct} \approx \frac{P}{m_p} \frac{1}{\Lambda_{QCD}} \gg \tau_{fluct}$

infinite momentum frame (IMF

$$l_{\mu\nu} W^{\mu\nu} = 4\varepsilon\varepsilon' \left[ 2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2} \right]$$

$$\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4} \left[ 2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2} \right]$$

By varying the angle $\theta$ can separate $W_1$ & $W_2$ contributions in experiments. $\Rightarrow$ Rosenbluth separation

Usually one defines $F_1(x, Q^2) = m_p W_1(x, Q^2)$, $F_2(x, Q^2) = \nu\, W_2(x, Q^2)$

## The Parton Model.

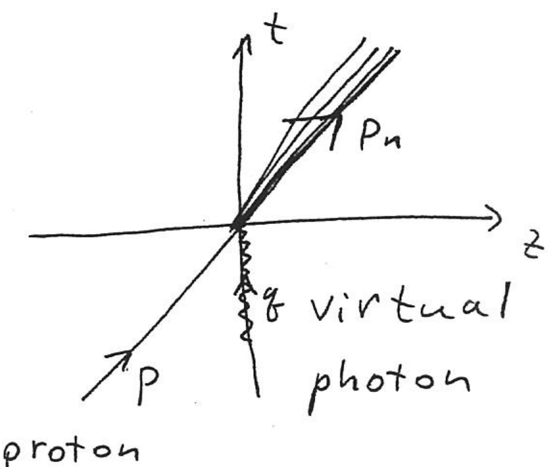
Sterman ch 14, Peskin 17.5
Y.K. & Levin, ch. 2.2

Go to Infinite Momentum Frame:

$$p_\mu \approx \Big( \underset{0}{p + \frac{m^2}{2p}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{p} \Big),$$

$\underline{q} = (q^1, q^2) \sim$ 2d vector in transverse plane

$$q_\mu = \Big( \underset{0}{q_0}, \underset{1,2}{\underline{q}}, \underset{3}{0} \Big),$$




$Q^2$ and $x$ are 2 invariants
$\uparrow$ large, $Q \gg \Lambda_{QCD}$

$$p \cdot q = m\nu = q_0 \cdot p$$

$$\Rightarrow q_0 = \frac{m\nu}{p} \sim \text{small as } p \text{ goes large}, \ p \gg Q$$

$\Rightarrow$ as $\nu = \dfrac{Q^2}{2m_p x} \Rightarrow q_0 = \dfrac{Q^2}{2xp}$

$$\Rightarrow Q^2 = -q^2 = \underline{q}^2$$

$\Rightarrow q_0 \ll Q$ since $xp \gg Q$.

$$F_1(x, Q^2) = m_p W_1(x, Q^2)$$

$$F_2(x, Q^2) = \nu W_2(x, Q^2) = \frac{Q^2}{2 m_p x} W_2(x, Q^2)$$

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4}\left[2\,\frac{1}{m_p} F_1(x, Q^2)\sin^2\frac{\theta}{2} + \frac{2 m_p x}{Q^2} F_2(x, Q^2)\cdot\cos^2(\theta/2)\right]}$$

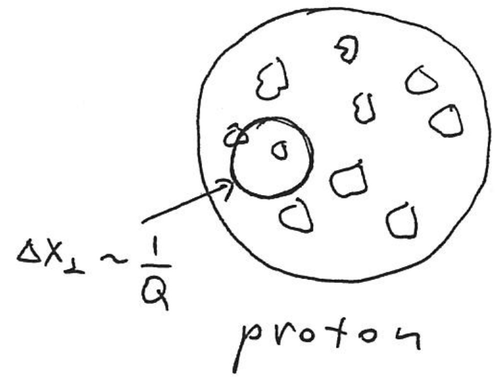
$F_1$, $F_2$ are dimensionless

$Q^2 = q^2 \Rightarrow$ photon acts like a <u>microscope</u>

in <u>transverse</u> plane:

$\Delta x_\perp \cdot q_\perp \sim 1 \qquad (\hbar = 1)$

$\Delta x_\perp \sim \frac{1}{q_\perp} \sim \frac{1}{Q}$

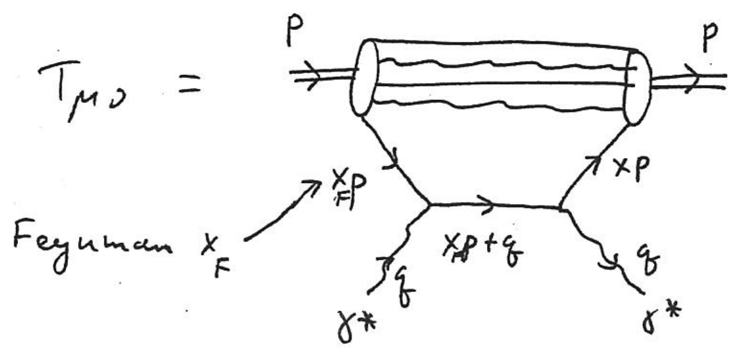



large $Q$ ~ resolve just 1 quark

Define $$\boxed{T_{\mu\nu} = \frac{1}{4\pi m_p} \int d^4x \, e^{iq\cdot x} \, \frac{1}{2}\sum_\sigma \langle p,\sigma | T j_\mu(x) j_\nu(0) | p,\sigma\rangle}$$

$W_{\mu\nu} = 2\,\mathrm{Im}\,(i\,T_{\mu\nu})$ (optical theorem)

$T_{\mu\nu} =$




"Forward Amplitude"

Def Feynman-x: the fraction of proton's longitudinal momentum carried by struck quarks

typical interaction time in proton's rest frame is $\frac{1}{\Lambda_{QCD}} \Rightarrow$ boost to get $\frac{p}{m}\frac{1}{\Lambda} \equiv \tau_\Lambda$

int. time of DIS is $\tau_{DIS} \simeq \frac{1}{q^0}$, where

$q^0 \simeq \frac{Q^2}{2x_B p}$ $\left(= \frac{m\nu}{p} \leftarrow \text{as } m\nu = \frac{Q^2}{2x}\right)$ is struck quark's velocity : $\tau_{DIS} \simeq \frac{2x_B p}{Q^2}$

time-ordered product: (denoted T)

$$\boxed{T\, j_\mu(x)\, j_\nu(y) \equiv \Theta(x^0-y^0)\, j_\mu(x)\, j_\nu(y) + \Theta(y^0-x^0)\, j_\nu(y)\, j_\mu(x)}$$

note: currents do not commute with each other in general $\Rightarrow$ not a trivial object.

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$$2\,\mathrm{Im}\left(i\,T_{\mu\nu}\right) = 2\,\mathrm{Im}\Big[i\cdot\frac{1}{4\pi m_p}\int d^4x\, e^{iq\cdot x}\langle p|\,\Theta(x^0)\,\underbrace{j_\mu(x)}_{e^{ip\cdot x} j_\mu(0) e^{-ip\cdot x}}\;\underset{\sum_n |n\rangle\langle n|}{\wedge}\; j_\nu(0) +$$

$$+\,\Theta(-x^0)\, j_\nu(0)\, j_\mu(x)\,|p\rangle\Big] = 2\cdot\frac{1}{4\pi m_p}\sum_n \mathrm{Re}\Big\{\int d^4x\, e^{iq\cdot x + ip\cdot x - ip_n\cdot x}\cdot$$

$$\cdot\,\Theta(x^0)\,\langle p|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle + \int d^4x\, e^{iq\cdot x + ip_n\cdot x - ip\cdot x}\,\Theta(-x^0)\cdot$$

$$\cdot\langle p|j_\nu(0)|n\rangle\langle n|j_\mu(0)|p\rangle\Big\} = \sum_n 2\,\frac{1}{4\pi m_p}\Big\{(2\pi)^3\,\delta(\vec{q}+\vec{p}-\vec{p}_n)\cdot\mathrm{Re}$$

$$\left(\frac{-1}{i(q^0+p^0-p_n^0+i\varepsilon)}\langle p|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle\right) + \underbrace{(2\pi)^3\,\delta(\vec{q}+\vec{p}_n-\vec{p})\cdot}_{\text{not physical}\ \Rightarrow\ \text{drop}}$$

$$\cdot\,\mathrm{Re}\left(\underbrace{\frac{1}{i(q^0+p_n^0-p^0-i\varepsilon)}}_{\propto\,\delta(q^0+p_n^0-p^0)}\langle p|j_\nu(0)|n\rangle\langle n|j_\mu(0)|p\rangle\right)\Big\}$$

$\overset{\shortparallel}{0}$ not physical $\Rightarrow$ drop (after including $\delta(q^0+p_n^0-p^0)$)

$$= 2\,\frac{1}{4\pi m_p}\sum_n (2\pi)^3\,\delta(\vec{q}+\vec{p}-\vec{p}_n)\,(-)\left(\mathrm{Im}\,\frac{1}{q^0+p^0-p_n^0+i\varepsilon}\right)\cdot\langle p|j_\mu(0)|n\rangle$$

$$\langle n|j_\nu(0)|p\rangle = \Big|\ \text{as } \mathrm{Im}\,\frac{1}{x+i\varepsilon} = -\pi\,\delta(x)\ = \frac{1}{4\pi m_p}\sum_n (2\pi)^4\,\delta^4(q+p-p_n)$$

$$\cdot\langle p|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle = W_{\mu\nu} \quad \text{as desired.}$$

(can prove that $\langle p|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle$ is real)

consider $\langle p | j_\mu(0) | n \rangle$. This is a 4-vector which depends on two 4-vectors, $p_\mu$ and $p_n^\mu$.
↖ simplifying assumption

We write
$$\boxed{\langle p | j_\mu(0) | n \rangle = \alpha e^{i\varphi_1} p_\mu + \beta e^{i\varphi_2} p_{n\mu}}, \quad (*)$$

where $\alpha, \beta, \varphi_1$ and $\varphi_2$ are all real. (and unknown)

Next consider $\langle p | j_\mu(x) | n \rangle$. Current conservation

$\partial_\mu j^\mu(x) = 0$, implies that $\partial^\mu \langle p | j_\mu(x) | n \rangle = 0$

$$\Rightarrow \partial^\mu \langle p | e^{i\hat{p}\cdot x} j_\mu(0) e^{-i\hat{p}\cdot x} | n \rangle = 0$$

$$\Rightarrow \left(\partial^\mu e^{i(p-p_n)\cdot x}\right) \langle p | j_\mu(0) | n \rangle = 0$$

$$\Rightarrow \boxed{(p-p_n)^\mu \langle p | j_\mu(0) | n \rangle = 0}$$

Now, plug in formula $(*)$:

$$(p-p_n)^\mu \left[\alpha e^{i\varphi_1} p_\mu + \beta e^{i\varphi_2} p_{n\mu}\right] = 0$$

$$\underbrace{\alpha\, p\cdot(p-p_n)}_{\text{real}} e^{i\varphi_1} = \underbrace{-\beta\, p_n\cdot(p-p_n)}_{\text{real}} e^{i\varphi_2}$$

$\Rightarrow$ for these two complex numbers to be equal, one has to have $\varphi_1 = \varphi_2 + \pi n$, $n \in \mathbb{Z}$

$$\Rightarrow e^{i\varphi_1} = \pm e^{i\varphi_2}$$

due to the $\pi n$ term possibility

back to (*)

$$\Rightarrow \langle p | j_\mu(0) | n \rangle = \left(\alpha p_\mu \pm \beta p_{n\mu}\right) e^{i\varphi_1}$$

$$\Rightarrow \langle p | j_\mu(0) | n \rangle \langle n | j_\nu(0) | p \rangle =$$

$$= \left(\alpha p_\mu \pm \beta p_{n\mu}\right)\left(\alpha p_\nu \pm \beta p_{n\nu}\right) = \text{real } \#.$$

if $x$ is small ($\lesssim 1$) and $Q$ is large

$\Rightarrow$ $\tau_{DIS} \ll \tau_\Lambda$ interaction is "instantaneous"

as $\frac{2xp}{Q^2} \ll \frac{p}{m}\frac{1}{\Lambda}$ $\Rightarrow$ as $2xm\Lambda < 2m\Lambda \ll Q^2$

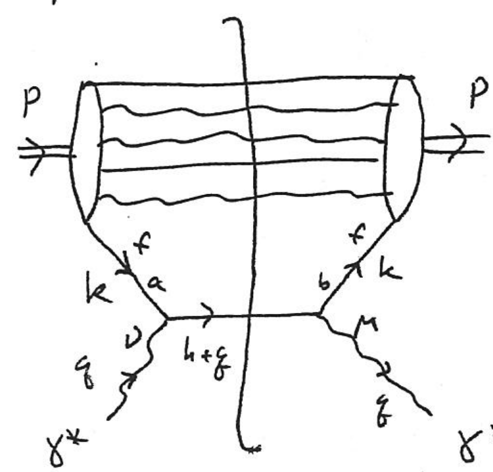
Define light cone variables:

for vector $V^\mu$ one has $V^+ = V^0 + V^3$

$\underline{V} = (V^1, V^2)$ $\qquad V^- = V^0 - V^3$

(2d transverse vector)

$$V_1 \cdot V_2 = V_{1\mu} V_2^\mu = \tfrac{1}{2} V_{1+} V_{2-} + \tfrac{1}{2} V_{1-} V_{2+} - \underline{V}_1 \cdot \underline{V}_2$$

$W_{\mu\nu} =$ 


$\Bigg\}$ $\underbrace{A^f_{ab}}_{\text{Dirac indices}}(p, k)$

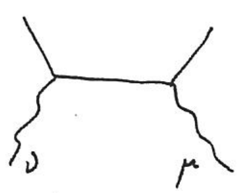
e.g.

$p^+ = p$

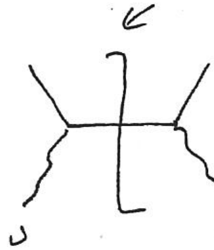
$p^- = \frac{m^2}{p}$

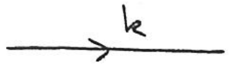
$p^+ \gg p^-$

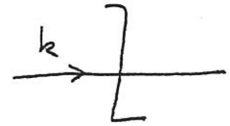
as $W_{\mu\nu} = 2\,\mathrm{Im}\,(i\,T_{\mu\nu})$

$T_{\mu\nu} =$ 


cut (Im part of a propagator)

$W_{\mu\nu} =$ 


$\frac{i}{k^2 - m^2 + i\varepsilon}$ $\Rightarrow$ 


$2\pi\,\delta^{(+)}(k^2 - m^2)$

as $2\,\mathrm{Im}\,\frac{-1}{k^2 - m^2 + i\varepsilon} = +2\pi\,\delta^{(+)}(k^2 - m^2)$

We write

$$W_{\mu\nu} = \frac{1}{\cancel{4\pi} 2 m_p} \sum_f e_f^2 \int \frac{d^4k}{(2\pi)^4} A^f_{ab}(p,k) \left[\gamma_\mu \gamma\cdot(k+q)\gamma_\nu\right]_{ba}$$

$$\cdot \cancel{(2\pi)}\, \delta((k+q)^2)$$ where $A^f_{ab}$ is the rest of the diagram (see p.89).

Start calculating assuming that

$$Q^2 \gg k^2, \ \underline{k}\cdot\underline{q}, \quad k^+ \gg k^- \ (\text{IMF})$$

$$\therefore (k+q)^2 = k^2 + 2k^+ q^- + 2k^- q^+ - 2\underline{k}\cdot\underline{q} - Q^2$$

$q_3 = 0 \Rightarrow q^+ = q^- \Rightarrow$ as $k^+ \gg k^- \Rightarrow$ drop $2k^- q^+$

dropping $k^2, \underline{k}\cdot\underline{q} \ll Q^2$ get

$$(k+q)^2 \simeq 2k^+ q^- - Q^2$$

$$\Rightarrow \delta((k+q)^2) \simeq \delta(2k^+ q^- - Q^2) = \delta\left(\frac{k^+}{p^+} 2p^+ q^- - Q^2\right)$$

as $p\cdot q \simeq p^+ q^- \Rightarrow$ and $x_{Bj} = \dfrac{Q^2}{2p\cdot q}$

$$\Rightarrow \delta((k+q)^2) \simeq \frac{x_{Bj}}{Q^2}\, \delta\left(x_{Bj} - \frac{k^+}{p^+}\right)$$

$$\Rightarrow \boxed{x_{Bj} = \frac{k^+}{p^+}}$$ Feynman $x$ = Bjorken $x$

physical meaning: light cone momentum fraction of the struck quark!

In arriving at Eq. (2.17) we have neglected the mass of the electron $m_e$, to write

$$d^3p' = p'^2\, dp'\, d\Omega \approx E'^2\, dE'\, d\Omega,$$

where $\Omega$ is the solid scattering angle. We have also used Eq. (2.3) to replace $Q^2$. Equation (2.17) demonstrates that the structure functions $W_1$ and $W_2$ can be measured experimentally by studying the angular dependence of the DIS cross section.

Note that the structure functions $W_1$ and $W_2$ have the dimension of inverse mass.[1] It is more convenient to define dimensionless structure functions $F_1$ and $F_2$, by

$$F_1(x_{Bj}, Q^2) \equiv m W_1(x_{Bj}, Q^2), \tag{2.18a}$$

$$F_2(x_{Bj}, Q^2) \equiv \nu W_2(x_{Bj}, Q^2) = \frac{Q^2}{2mx_{Bj}}\, W_2(x_{Bj}, Q^2). \tag{2.18b}$$

All the QCD physics in DIS is contained in $F_1$ and $F_2$. We will now attempt to calculate these structure functions.

## 2.2 Parton model and Bjorken scaling

To find the structure functions $F_1$ and $F_2$ it is easier to change the frame in which we are working. Instead of the proton's rest frame we will now use a frame in which the proton is ultrarelativistic. Such a frame is usually referred to as the *infinite momentum frame* (IMF) or Bjorken frame. The proton is taken to be moving along the $z$-axis, and its momentum in this frame is

$$P^\mu \approx \left(P + \frac{m^2}{2P}, 0, 0, P\right) \tag{2.19}$$

in the $(P^0, P^1, P^2, P^3)$ notation. We assume that the proton's momentum is much larger than its mass, $P \gg m$. The virtual photon in the IMF has $q^3 = 0$, so that

$$q^\mu = (q^0, q^1, q^2, 0). \tag{2.20}$$

The part of the DIS process relevant for the calculation of the structure functions, virtual photon–proton scattering, is depicted in Fig. 2.3. Note that, unlike Fig. 2.2, we now draw the proton at the top of the diagram. In fact, in our normal convention a proton at rest (or any other target) is drawn at the bottom of the diagram, while a proton (or any other projectile) moving at high energy is shown at the top of the diagram.

### *2.2.1 Warm-up: DIS on a single free quark*

As a warm-up calculation in preparation for the full *parton model*, let us simply assume that the proton consists of noninteracting quarks and gluons, which we will refer to as *partons*. As we will see below in Sec. 2.3, this is not such a bad approximation as in the IMF the

[1] Our single-particle states are normalized such that $\langle p|p'\rangle = (2\pi)^3\, 2E_p\, \delta^3(\vec p - \vec p\,')$, which allows one to see that the dimension of $W^{\mu\nu}$ in Eq. (2.14) is that of inverse mass.

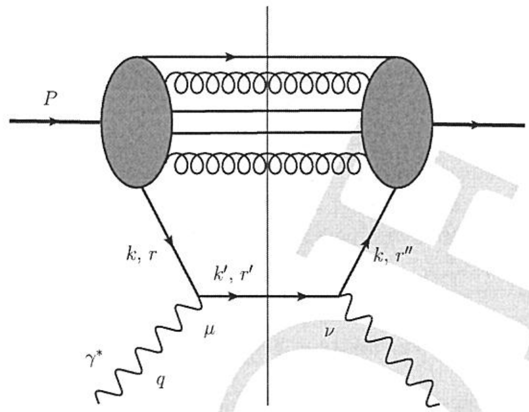



Fig. 2.3. Virtual photon–proton scattering in the IMF.

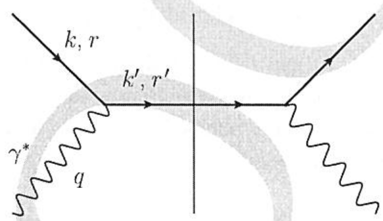



Fig. 2.4. Interaction of a virtual photon with one point-like particle (a parton), as the basic ingredient of the parton model. As usual, the vertical solid line denotes the final-state cut.

typical time scale of the quark and gluon interactions inside the proton is much longer than the time scale of DIS. Hence for the duration of the virtual photon–proton scattering we can assume that the quarks and gluons do not interact with each other. Thus the photon simply interacts with a quark in the proton. To better understand photon–quark scattering let us assume that we simply have one free quark instead of the proton. The diagram giving the cross section of the DIS process is shown in Fig. 2.4.

The hadronic tensor $W_{\mu\nu}$ for the interaction of the virtual photon with the point-like particle (a single quark) has a structure similar to $L_{\mu\nu}$ in Eq. (2.11), namely

$$\begin{aligned} W_{\mu\nu}^{quark} &= \frac{Z_f^2}{2} \sum_{r=\pm 1} \sum_{r'=\pm 1} \bar{u}_{r'}(k')\, \gamma_\mu\, u_r(k)\, \left[\bar{u}_{r'}(k')\, \gamma_\nu\, u_r(k)\right]^* \frac{1}{2m_q}\, \delta\big(k'^2 - m_q^2\big) \\ &= \frac{Z_f^2}{2} \operatorname{Tr}\left[\big(\not{k}' + m_q\big)\gamma_\mu \big(\not{k} + m_q\big)\gamma_\nu\right] \frac{1}{2m_q}\, \delta\big(k'^2 - m_q^2\big), \end{aligned} \tag{2.21}$$

where $k' = k + q$ while $r$ and $r'$ are the quark helicities (see Fig. 2.4) and $m_q$ is the quark mass. Equation (2.21) can be obtained from Eq. (2.13) by replacing $X$ in it by a single

Clarifications:

$$W_{\mu\nu} = \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \int d^4x \, e^{iq\cdot x} \langle p,\sigma | j_\mu(x) j_\nu(0) | p,\sigma \rangle$$

$$= \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \int \frac{d^4x \, d^4y}{\underset{= \int d^4z}{V_4}} e^{iq\cdot(x-y)} \underbrace{\langle p,\sigma | j_\mu(x) j_\nu(y) | p,\sigma \rangle}_{\text{function of } x-y \text{ only}}$$

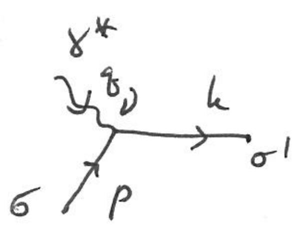
$$= \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \sum_n \int \frac{d^4x \, d^4y}{V_4} e^{iq\cdot(x-y)} \langle p,\sigma | j_\mu(x) | n \rangle \langle n | j_\nu(y) | p,\sigma \rangle$$

$$= \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \sum_n \frac{1}{V_4} \underbrace{\langle p,\sigma | \int d^4x \, e^{iq\cdot x} j_\mu(x) | n \rangle}_{-i M_\mu^* (2\pi)^4 \delta^4(p+q-p_n)} \underbrace{\langle n | \int d^4y \, e^{-iq\cdot y} j_\nu(y) | p,\sigma \rangle}_{i M_\nu (2\pi)^4 \delta^4(p+q-p_n)}$$

$$= \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \sum_n \cancel{\frac{1}{V_4}} M_\mu^* M_\nu (2\pi)^4 \delta^4(p+q-p_n) \underbrace{\cancel{(2\pi)^4 \delta^4(0)}}_{\substack{\| \\ \lim_{l\to 0} \left\{ \int d^4z \, e^{il\cdot z} \right\} = V_4}}$$

$$= \frac{1}{4\pi m_p} \frac{1}{2} \sum_\sigma \sum_n M_\mu^* M_\nu (2\pi)^4 \delta^4(p+q-p_n)$$

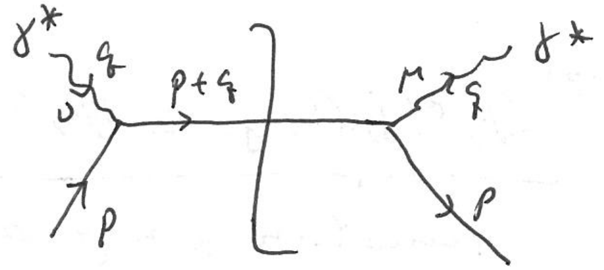
For a single quark target,

$M_\nu =$ 

, $p_n^\mu = k^\mu$, and $\sum_n = \sum_{\sigma'} \int \frac{d^3k}{(2\pi)^3 2E_k}$.

Using the above we arrive at

$$W_{\mu\nu}^{quark} = \frac{Z_f^2}{2} \mathrm{Tr}\left[ (\not p + \not q + m_q)\gamma_\mu(\not p + m_q)\gamma_\nu \right] \frac{1}{2m_q} \delta((p+q)^2 - m_q^2)$$




$$\delta((p+q)^2 - m_q^2) = \frac{1}{2p\cdot q}\delta(1-x)$$

$$W_{\mu\nu}^{quark} = \frac{Z_f^2}{4m_q}\frac{1}{2p\cdot q}\delta(1-x)\cdot 4\left[(p+q)_\mu p_\nu + (p+q)_\nu p_\mu\right.$$

$$\left. - g_{\mu\nu}\, p\cdot(\cancel{p}+q) + \cancel{g_{\mu\nu} m_q^2}\right] = \frac{Z_f^2}{m_q}\frac{1}{2p\cdot q}\delta(1-x)\cdot$$

$$\cdot\left[-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)p\cdot q \underbrace{- q_\mu q_\nu \frac{p\cdot q}{q^2} + 2p_\mu p_\nu + p_\mu q_\nu + p_\nu q_\mu}_{\displaystyle 2\left[p_\mu - q_\mu \frac{p\cdot q}{q^2}\right]\left[p_\nu - q_\nu\frac{p\cdot q}{q^2}\right]}\right]$$

as $\delta(1-x) = \delta\left(1 - \frac{Q^2}{2p\cdot q}\right)$ ensures that

$$\frac{p\cdot q}{q^2} = -\frac{1}{2}.$$

Thus,
$$W_{\mu\nu}^{quark} = \frac{Z_f^2}{m_q}\cdot\frac{x}{Q^2}\delta(1-x)\left[-p\cdot q\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\right.$$

$$\left. + 2\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)\right].$$

We read off the structure functions

$$\boxed{\begin{aligned} W_1^{quark}(x, Q^2) &= \frac{Z_f^2}{2m_q}\,\delta(1-x) \\ W_2^{quark}(x, Q^2) &= 2\,Z_f^2\,\frac{m_q x}{Q^2}\,\delta(1-x) \end{aligned}}$$

The dimensionless structure functions are

$$\boxed{\begin{aligned} F_1^{quark}(x, Q^2) &= m_q\, W_1^{quark} = \frac{Z_f^2}{2}\,\delta(1-x) \\ F_2^{quark}(x, Q^2) &= \frac{Q^2}{2m_q x}\, W_2^{quark} = Z_f^2\,\delta(1-x) \end{aligned}}$$

$F_1$ & $F_2$ are functions of $x$ only (not $Q^2$)

$\Rightarrow$ Bjorken scaling.

63



particle (a quark), so that

$$\sum_{X=\text{one particle}} = \int \frac{d^3k'}{2k'^0\,(2\pi)^3} \sum_{r'=\pm 1}$$

along with $p_X \to k'$ and $P \to k$. It is then easy to show that

$$\frac{1}{4\pi m_q} \int \frac{d^3k'}{2k'^0\,(2\pi)^3} (2\pi)^4\,\delta^4(k+q-k') = \frac{1}{2m_q}\,\delta\left((k+q)^2 - m_q^2\right), \tag{2.22}$$

justifying the delta function factor in Eq. (2.21).

We can rewrite $\delta((k+q)^2 - m_q^2)$ as follows:

$$\delta\left((k+q)^2 - m_q^2\right) = \delta\left(2k\cdot q - Q^2\right) = \frac{1}{2k\cdot q}\,\delta\left(1 - \frac{Q^2}{2k\cdot q}\right), \tag{2.23}$$

where we have used the fact that the incoming quark is on mass shell.

Calculating the trace in Eq. (2.21), comparing the result with Eq. (2.16), and using Eqs. (2.18a) and (2.18b) with $P$ replaced by $k$ we obtain for DIS on a point-like particle (a quark)

$$F_1^{quark}\left(x_{Bj}, Q^2\right) = m_q W_1^{quark}\left(x_{Bj}, Q^2\right) = \frac{Z_f^2}{2}\,\delta\left(1 - x_{Bj}\right) \tag{2.24}$$

$$F_2^{quark}\left(x_{Bj}, Q^2\right) = \frac{Q^2}{2m_q x_{Bj}}\,W_2^{quark}\left(x_{Bj}, Q^2\right) = Z_f^2\,\delta\left(1 - x_{Bj}\right). \tag{2.25}$$

We have used the fact that, for DIS on a single quark, $x_{Bj} = Q^2/(2k\cdot q)$. We see that for DIS on a point-like particle the structure functions $F_1$ and $F_2$ turn out to depend only on one variable, $x_{Bj}$. This behavior is known as *Bjorken scaling* (Bjorken 1969).

### 2.2.2 Full calculation: DIS on a proton

The idea that the actual interaction in DIS occurs with the point-like constituents of a hadron (the partons) can be illustrated by studying the full DIS process. Let us consider DIS on the whole proton, as shown in Fig. 2.3. We want to calculate the diagram in Fig. 2.3 using the rules of light cone perturbation theory (LCPT) outlined in Sec. 1.3 (see also Sec. 1.4). We first rewrite all four-momenta in the light cone $(+, -, \perp)$ notation. In the IMF/Bjorken frame the proton has a very large momentum. The proton's momentum in Eq. (2.19) becomes, in light cone notation,

$$P^\mu \approx (P^+, 0, 0_\perp) \tag{2.26}$$

with very large $P^+ \approx 2P$. Quarks and gluons in such an ultrarelativistic proton also have very large light cone plus momenta. The quark in Fig. 2.3 has four-momentum $k^\mu = (k^+, (\vec{k}_\perp^2 + m_q^2)/k^+, \vec{k}_\perp)$; we assume that it has a large $k^+$ component. We define the Feynman-$x$ variable as the fraction of the light cone momentum of the proton carried by

64



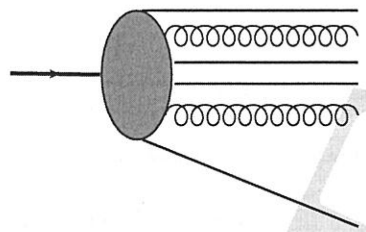



Fig. 2.5. Light cone wave function of the proton.

this quark[2]

$$x \equiv \frac{k^+}{P^+}, \tag{2.27}$$

writing $k^\mu = (x P^+, (\vec{k}_\perp^2 + m_q^2)/(x P^+), \vec{k}_\perp)$.

In LCPT every particle is on mass shell. However, we want to calculate the virtual photon–proton scattering cross section for the process shown in Fig. 2.3. By the definition of the problem the incoming photon is virtual, $q^2 = -Q^2$. Hence in LCPT we can treat this virtual photon as having an imaginary mass $iQ$. The virtual photon momentum (2.20) becomes, in light cone notation,

$$q^\mu = \left( q^+, \frac{\vec{q}_\perp^{\,2} - Q^2}{q^+}, \vec{q}_\perp \right) \tag{2.28}$$

with $(q^+)^2 = \vec{q}_\perp^{\,2} - Q^2$ in the IMF.

In the calculations below we will assume that $Q^2$ is very large. First, for QCD perturbation theory to be applicable $Q^2$ has to be much larger than the confinement scale $\Lambda_{QCD}$: $Q^2 \gg \Lambda_{QCD}^2$. Second, for the parton model (which we are about to present) to be valid, $Q$ has to be much larger than the transverse momentum of any other particle in the problem. This applies to the quark line carrying momentum $k$ in Fig. 2.3, for which we have $Q^2 \gg \vec{k}_\perp^2, m_q^2$. If, for a particular wave function configuration the upper boxed part of Fig. 2.3 contains $n$ partons with transverse momenta $\vec{k}_{i\perp}$ for $i = 1, \ldots, n$, then we will assume that $Q^2 \gg \vec{k}_{i\perp}^2$ for any $i$. Note that $\vec{q}_\perp^{\,2} = Q^2 + (q^+)^2 > Q^2$ is also very large.

Now let us assume that these $n$ partons carry light cone momentum components $k_i^+$ or, equivalently, have Feynman-$x$ values given by $x_i$ for $i = 1, \ldots, n$. We can then define the light cone wave function of the $(n+1)$-parton Fock state of the proton and denote it by $\Psi_n^f(\{x_i, k_{i\perp}\}; x, k_\perp; r)$. The proton has $n$ "spectator" partons (both quarks and gluons) and one quark carrying momentum $k$ in Fig. 2.3 that interacts with the photon. This quark has helicity $r$ and flavor $f$. The light cone wave function $\Psi_n^f(\{x_i, k_{i\perp}\}; x, k_\perp; r)$ is illustrated in Fig. 2.5. In our discussion and notation we will suppress the polarization indices of the

[2] The Feynman-$x$ variable was originally defined as $x = 2k^3/\sqrt{s}$ in the center-of-mass frame with $k^\mu$ the momentum of the produced outgoing particle (Feynman 1969). Our definition here is different, but is also widely used in the community: it maps back onto the original definition at large $x$.