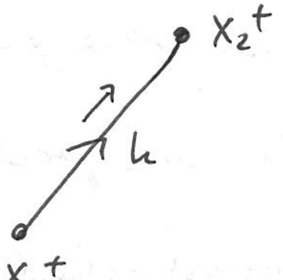
Further explanation of LCPT rules:

$\Rightarrow$ Imagine you have a quark propagator, which you are transforming into x^+-space:



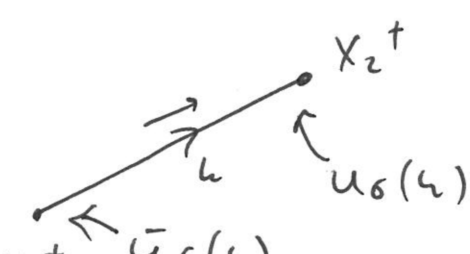
$$\int_{-\infty}^{\infty} \frac{dk^-}{2\pi} e^{-ik^- \frac{1}{2}(x_2^+ - x_1^+)} \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon}$$

After picking the pole at $k^2 - m^2 + i\varepsilon = 0$ we put the quark line on mass shell.

For an on-shell quark we have

$$\left[\not{k} + m\right]_{k^2=m^2} = \sum_\sigma u_\sigma(k)\, \bar{u}_\sigma(k) .$$

We can use this formula in the numerator of the propagator (now that $k^2 = m^2$), and assign $u_\sigma(k)$ and $\bar{u}_\sigma(k)$ to the adjoining vertices:



This way we treat the quark as outgoing for one vertex (x_1^+) and as incoming for the other (x_2^+).

$\Rightarrow$ For gluons (or photons) the story is similar:

$$\int_{-\infty}^{\infty} \frac{dk^-}{2\pi}\, e^{-i k^- \frac{1}{2}(x_2^+ - x_1^+)} \frac{i}{k^2 + i\varepsilon} \left[-g^{\mu\nu} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{\eta \cdot k} \right]$$

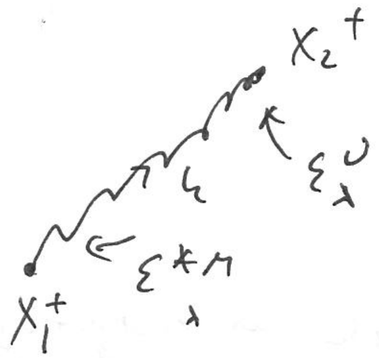
in the $A^+ = \eta \cdot A = 0$ light cone gauge.

Picking up the $k^2 = 0$ pole we put the gluon on mass shell. For a mass-shell gluon we have

$$\left[-g^{\mu\nu} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{\eta \cdot k} \right]_{k^2=0} = \sum_\lambda \varepsilon_\lambda^{\mu *}(k)\, \varepsilon_\lambda^\nu(k)$$

with the polarization vectors $\varepsilon_\lambda^\mu(k)$ defined in LCPT rule 2 above.

Again, we assign $\varepsilon_\lambda^{*\mu}$ and ε_λ^ν to the adjacent vertices:



$\Rightarrow$ For scalars there is no issue like this, since the scalar propagator $\frac{i}{k^2+i\varepsilon}$ has a trivial denominator.

$\Rightarrow$ Instantaneous terms: we have been a bit sloppy above. Consider the quark propagator again:

$$\int_{-\infty}^{\infty} \frac{dk^-}{2\pi}\, e^{-ik^- \frac{1}{2}(x_2^+ - x_1^+)} \frac{i\left[\frac{1}{2}k^-\gamma^+ + \frac{1}{2}k^+\gamma^- - \underline{\gamma}\cdot\underline{k}\right]}{k^2+i\varepsilon}$$

The $k^-\gamma^+$ term requires a special treatment. Indeed, in the integral

$$\int_{-\infty}^{\infty} \frac{dk^-}{2\pi}\, e^{-ik^- \frac{1}{2}(x_2^+ - x_1^+)} \frac{i\,\frac{1}{2}k^-\gamma^+}{k^+k^- - \underline{k}^2 + i\varepsilon}$$

we do not have enough convergence to close the contour in upper or lower half-plane. Instead we write $k^- = \left(k^- - \frac{\underline{k}^2+m^2}{k^+}\right) + \frac{\underline{k}^2+m^2}{k^+}$ in the numerator. We get

$$\int_{-\infty}^{\infty} \frac{dk^-}{2\pi} e^{-ik^-\frac{1}{2}(x_2^+ - x_1^+)} \frac{i\frac{1}{2}\gamma^+\Big[\overbrace{\Big(k^- - \frac{k^2+m^2}{k^+}\Big)}^{\frac{1}{k^+}k^2} + \frac{k^2+m^2}{k^+}\Big]}{k^2+i\varepsilon} =$$

$$= i\frac{\gamma^+}{2k^+}\frac{4\pi}{2\pi}\delta(x_2^+ - x_1^+) + \int_{-\infty}^{\infty}\frac{dk^-}{2\pi} e^{-ik^-\frac{1}{2}(x_2^+-x_1^+)} \frac{\frac{i}{2}\gamma^+\frac{k^2+m^2}{k^+}}{k^2+i\varepsilon}$$

~ the second term above can be placed back in the original expression for the Fourier transform of the quark propagator, giving

$$\frac{i(\tilde{\not{k}}+m)}{k^2-m^2+i\varepsilon}, \text{ where } \tilde{k}^\mu = \Big(k^+, \frac{k^2+m^2}{k^+}, k\Big). \Rightarrow$$

$\Rightarrow$ the standard steps described above apply to this term, $\tilde{\not{k}}+m = \sum_\sigma u_\sigma(k)\,\bar{u}_\sigma(k)$.

~ the first term, $i\frac{\gamma^+}{2k^+}4\pi\,\delta(x_2^+ - x_1^+)$, is instantaneous, due to $\delta(x_1^+ - x_2^+)$; it corresponds to Eq. (1.60) in the LCPT rules above.

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37

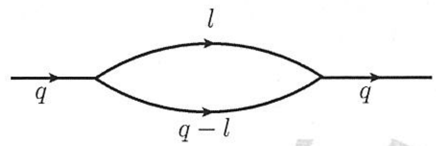

Fig. 1.1. A Feynman diagram in the ϕ^3-theory considered here. The arrows indicate the momentum flow.

We see that each light cone wave function $\Psi(n_G, n_q)$ is normalized to a number less than or equal to 1.

1.4 Sample LCPT calculations

While we expect that the reader has a fluent knowledge of Feynman rules, we realize that it is less likely that he or she is equally fluent with LCPT rules. Therefore, to help the reader become more familiar with LCPT, here we will perform two LCPT calculations. We will first "cross-check" LCPT by calculating a sample scattering amplitude using both the Feynman and LCPT rules and showing that we obtain the same result. We will then set up the rules for calculating light cone wave functions, by considering an example of a basic wave function containing $1 \to 2$ particle splitting.

1.4.1 LCPT "cross-check"

We begin by calculating a simple amplitude in a real scalar ϕ^3 field theory in two ways: using standard Feynman rules and using the rules of LCPT. We will show that the two ways give identical results. This demonstrates that LCPT is indeed equivalent to the standard Feynman diagram approach.

The process we consider is illustrated in Fig. 1.1. We consider a field theory for a real massive scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{3!}\,\phi^3. \tag{1.71}$$

The contribution of the diagram in Fig. 1.1 (henceforth labeled A) can be written down using the Feynman rules for the real scalar field theory having Lagrangian (1.71) (see e.g. Sterman (1993) on Peskin and Schroeder (1995)):

$$iA = \frac{(-i\lambda)^2}{2!}\int \frac{d^4l}{(2\pi)^4}\,\frac{i}{l^2 - m^2 + i\epsilon}\,\frac{i}{(q-l)^2 - m^2 + i\epsilon}. \tag{1.72}$$

Here $1/2!$ is a symmetry factor and m is the mass of the scalar particles.

Working in the light cone variables

$$q = (q^+, q^-, \vec{q}_\perp), \quad l = (l^+, l^-, \vec{l}_\perp), \tag{1.73}$$

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38

we write $l^2 = l^+ l^- - \vec{l}_\perp^2$ and $(q-l)^2 = (q^+ - l^+)(q^- - l^-) - (\vec{q}_\perp - \vec{l}_\perp)^2$. Equation (1.72) can now be rewritten as

$$i A = \frac{\lambda^2}{4} \int \frac{dl^+ \, dl^- \, d^2 l_\perp}{(2\pi)^4} \frac{1}{l^+ l^- - \vec{l}_\perp^2 - m^2 + i\epsilon} \times \frac{1}{(q^+ - l^+)(q^- - l^-) - (\vec{q}_\perp - \vec{l}_\perp)^2 - m^2 + i\epsilon}. \tag{1.74}$$

Now we need to integrate over l^-. In the complex l^--plane the integrand in Eq. (1.74) has two poles,

$$l_1^- = \frac{\vec{l}_\perp^2 + m^2 - i\epsilon}{l^+} \quad \text{and} \quad l_2^- = q^- - \frac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2 - i\epsilon}{q^+ - l^+}. \tag{1.75}$$

The l^--integral is nonzero only if these two poles lie in different half-planes. This happens for either (i) $l^+ > 0$, $q^+ - l^+ > 0$ or (ii) $l^+ < 0$, $q^+ - l^+ < 0$. As the incoming particle with momentum q is physical we have $q^+ > 0$, which makes case (ii) impossible to achieve, as there one has $q^+ < l^+ < 0$. We are left with case (i). Closing the l^--integration contour to be in the lower half-plane we pick up the pole at l_1^-, obtaining

$$\begin{aligned} A &= \frac{-\lambda^2}{2} \int \frac{dl^+ \, d^2 l_\perp}{2(2\pi)^3} \frac{\theta(l^+)\,\theta(q^+ - l^+)}{l^+ \, (q^+ - l^+)} \\ &\quad \times \frac{1}{q^- - \dfrac{\vec{l}_\perp^2 + m^2 - i\epsilon}{l^+} - \dfrac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2 - i\epsilon}{q^+ - l^+}} \\ &= \frac{-\lambda^2}{2!} \int \frac{dl^+ \, d^2 l_\perp}{2(2\pi)^3} \frac{\theta(l^+)\,\theta(q^+ - l^+)}{l^+ \, (q^+ - l^+)} \\ &\quad \times \frac{1}{q^- - \dfrac{\vec{l}_\perp^2 + m^2}{l^+} - \dfrac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2 + i\epsilon}{q^+ - l^+}}. \end{aligned} \tag{1.76}$$

We observe that Eq. (1.76) is identical to what one would obtain for the diagram in Fig. 1.1 if one calculated it using the rules of LCPT from Sec. 1.3 (modified for a scalar particle), as illustrated in Fig. 1.2. Indeed Eq. (1.76) can be obtained by assigning

$$\frac{\theta(l^+)}{l^+} \quad \text{and} \quad \frac{\theta(q^+ - l^+)}{q^+ - l^+} \tag{1.77}$$

for each internal line (LCPT rule 4), including an energy denominator

$$\frac{1}{\sum\limits_{inc} k^- - \sum\limits_{interm} k^- + i\,\epsilon} = \frac{1}{q^- - \dfrac{\vec{l}_\perp^2 + m^2}{l^+} - \dfrac{(\vec{q}_\perp - \vec{l}_\perp)^2 + m^2}{q^+ - l^+} + i\epsilon} \tag{1.78}$$

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39

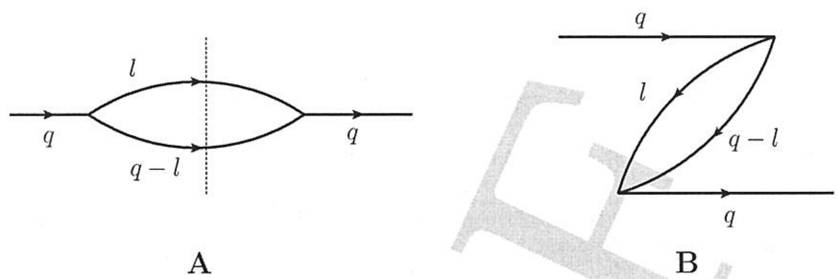

Fig. 1.2. Light cone perturbation theory diagrams in the ϕ^3-theory corresponding to the Feynman diagram in Fig. 1.1. Time flows to the right. The arrows indicate the momentum direction. The vertical dotted line indicates an intermediate state.

for the intermediate state (denoted by the dotted line in Fig. 1.2A), according to LCPT rule 3, and integrating over the internal momentum l with the integration measure

$$\int \frac{dl^+ \, d^2 l_\perp}{2(2\pi)^3}, \tag{1.79}$$

as prescribed by LCPT rule 6. In LCPT each vertex gives a factor λ (a modification of rule 5 for ϕ^3-theory) and one has to include the symmetry factor $1/2!$ as well. (Scalar particles obviously have no polarization. Neither do they have instantaneous terms.)

We have demonstrated that starting from the Feynman diagram amplitude expression (1.72) we can reduce it to the result that one would obtain by the rules of LCPT. Hence the two approaches in the end give identical expressions for the amplitudes, as expected.

A few words of caution are in order here. In principle the Feynman diagram in Fig. 1.1 corresponds to the two LCPT diagrams A and B shown in Fig. 1.2, which correspond to two different orderings of the vertices (see LCPT rule 1). The two graphs A and B in fact correspond to cases (i) and (ii) considered after Eq. (1.75). Our argument above was simplified by the fact that diagram B in Fig. 1.2 is zero as, according to the LCPT rules, it comes with a factor $\theta(-l^+)\,\theta(l^+ - q^+)$, which is zero for $q^+ > 0$. The physical meaning of this is quite clear: one cannot generate three particles with positive plus momenta out of nothing (see the lower vertex in Fig. 1.2B). Conversely, three particles with positive plus momenta cannot combine to give nothing (see the upper vertex in Fig. 1.2B). Because of this simplification, we have a one-to-one correspondence between the Feynman diagram in Fig. 1.1 and the LCPT diagram in Fig. 1.2A. In general, each Feynman diagram corresponds to a sum of all the LCPT diagrams with the same topology, including all possible time-orderings and instantaneous terms. A general derivation of an LCPT diagram starting from a Feynman diagram does not simply involve integration over the minus components of the internal momenta; one has to assign each vertex an x^+-coordinate and Fourier transform the diagram (by integrating over the minus momenta) into x^+ coordinate space. One then

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40

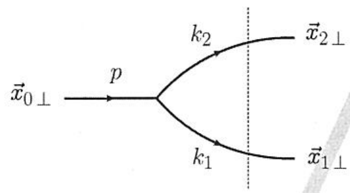

Fig. 1.3. Light cone wave function for a scalar particle splitting into two. The vertical dotted line denotes an intermediate state.

has to integrate over all the x^+-coordinates of the vertices, imposing different orderings: each ordering will lead to a different LCPT diagram.

1.4.2 A sample light cone wave function

Let us calculate, using the rules of LCPT, a sample light cone wave function. The calculation will be instructive, as the wave function we will calculate is similar to certain light cone wave functions that we will use throughout the book. In this calculation we will also illustrate in more detail what is actually meant by the light cone wave function definition (1.69) and will set up the rules for wave function calculations.

The sample wave function is depicted in Fig. 1.3. Again we are working in ϕ^3 real scalar field theory, with the Lagrangian (1.71). The wave function describes a single incoming particle splitting into two. For the scalar field theory only rules 1, 3, 4, and 6 from Sec. 1.3 apply. On top of these rules there is a factor equal to the coupling λ coming from the vertex. In calculating light cone wave functions one has to treat the "outgoing" state on the right of the diagram (the state denoted by the dotted line in Fig. 1.3) as an intermediate state. The reason is that, in describing a scattering process, the light cone wave function is thought of as a part of a larger diagram in which this "outgoing" state in fact undergoes subsequent interactions with other particles and therefore is truly an intermediate state. Our definition of the boost-invariant integration measure (1.67) dictates a slight modification of LCPT rule 4 as well, when calculating light cone wave functions: we treat the incoming lines (the external lines on the left, e.g. line p in Fig. 1.3) as "internal" and include a factor $1/p^+$ for them, while the outgoing lines (the lines on the right, e.g. lines k_1 and k_2 in Fig. 1.3) will be treated as "external" and so will not bring in such factors.

To summarize, when calculating the light cone wave function using LCPT one should follow the rules stated in Sec. 1.3, with the following modifications.

(i) The outgoing state on the right of a diagram is treated as an internal state and brings in an energy denominator according to LCPT rule 3.
(ii) At the same time the outgoing external lines on the right of the diagram bring in only factors $\theta(k^+)$, in modification of LCPT rule 4. (As usual, light cone time flows to the right.)

Light Cone Wave Function

Wave functions are present in quantum field theory, even if scarcely mentioned in a typical QFT course. Here's how they come in.

Consider a scalar field theory for simplicity. The states of this theory are the vacuum, single-particle state, two-particle state, etc.

$|0\rangle$ ~ vacuum (ground) state

$\hat{a}^+_{\vec{k}}|0\rangle = |\vec{k}\rangle$ ~ single-particle state

$\hat{a}^+_{\vec{k}_1}\hat{a}^+_{\vec{k}_2}|0\rangle = |\vec{k}_1, \vec{k}_2\rangle$ ~ two-particle state

$\vdots \qquad\qquad \vdots$

$\hat{a}^+_{\vec{k}_1}\hat{a}^+_{\vec{k}_2}\cdots\hat{a}^+_{\vec{k}_n}|0\rangle = |\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle$ ~ n-particle state

These are known as Fock states.

A general state in QFT can be expanded over Fock states:

$$\boxed{|\psi\rangle = c_0|0\rangle + \int\frac{d^3k}{(2\pi)^3\,2E_k}\,c_{\vec{k}}\,|\vec{k}\rangle + \frac{1}{2!}\int\frac{d^3k_1}{(2\pi)^3\,2E_{k_1}}\frac{d^3k_2}{(2\pi)^3\,2E_{k_2}}\cdot c_{\vec{k}_1,\vec{k}_2}\,|\vec{k}_1,\vec{k}_2\rangle + \ldots}$$

Note that $[\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = (2\pi)^3\, 2E_k\, \delta^3(\vec{k}-\vec{k}')$

with $E_k = \sqrt{\vec{k}^2+m^2}$. (Our convention.)

Hence $\langle \vec{k} | \vec{k}' \rangle \underset{}{\overset{\langle 0|\hat{a}_{\vec{k}}\hat{a}^{+}_{\vec{k}'}|0\rangle}{=}} (2\pi)^3\, 2E_k\, \delta^3(\vec{k}-\vec{k}')$.

Def. Define momentum-space single-particle wave function by $\boxed{\Psi_{\vec{k}} \equiv \langle \vec{k} | \Psi \rangle}$. (Just like in quantum mechanics.)

Using the definition of the state $|\Psi\rangle$ above, one can show that $\boxed{\Psi_{\vec{k}} = C_{\vec{k}}}$ $\Rightarrow$ coefficients in the decomposition of $|\Psi\rangle$ over Fock states are the wave functions!

Def. n-particle wave function is

$$\boxed{\Psi_{\vec{k}_1,\ldots,\vec{k}_n} = \langle \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n | \Psi \rangle}$$

$$\Rightarrow \quad \boxed{\Psi_{\vec{k}_1,\ldots,\vec{k}_n} = C_{\vec{k}_1,\ldots,\vec{k}_n}}.$$

(Note that $\langle \vec{k}_1, \ldots, \vec{k}_m | \vec{k}'_1, \ldots, \vec{k}'_n \rangle = 0$ unless $m=n$.)

Normalization:

$$1 = \langle \Psi | \Psi \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}} \cdot$$

$$\cdot \frac{d^3k_1'}{(2\pi)^3 2E_{k_1'}} \cdots \frac{d^3k_n'}{(2\pi)^3 2E_{k_n'}} \; C_{\vec{k}_1,\dots,\vec{k}_n} C^*_{\vec{k}_1',\dots,\vec{k}_n'} \langle \vec{k}_1,\dots,\vec{k}_n | \vec{k}_1',\dots,\vec{k}_n' \rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3 2E_{k_1}} \cdots \frac{d^3k_n}{(2\pi)^3 2E_{k_n}} \left| C_{\vec{k}_1,\dots,\vec{k}_n} \right|^2$$

$\Rightarrow$ each n-particle wave function is normalized to a number ≤ 1.