QCD AT SMALL x AND SATURATION

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LECTURE PLAN

- Light-Cone Perturbation Theory, light-cone wave functions
- Classical Small-x Physics:
  - DIS in the dipole picture, Glauber-Gribov-Mueller formula
  - Black disk limit, parton saturation, saturation scale
  - McLerran-Venugopalan model, saturation scale for a nucleus
- Nonlinear small-x evolution:
  - Non-linear BK and JIMWLK evolution equations
  - Solution of BK and JIMWLK equations, unitarity,
    energy dependence of the saturation scale, geometric scaling
  - Map of high-energy QCD

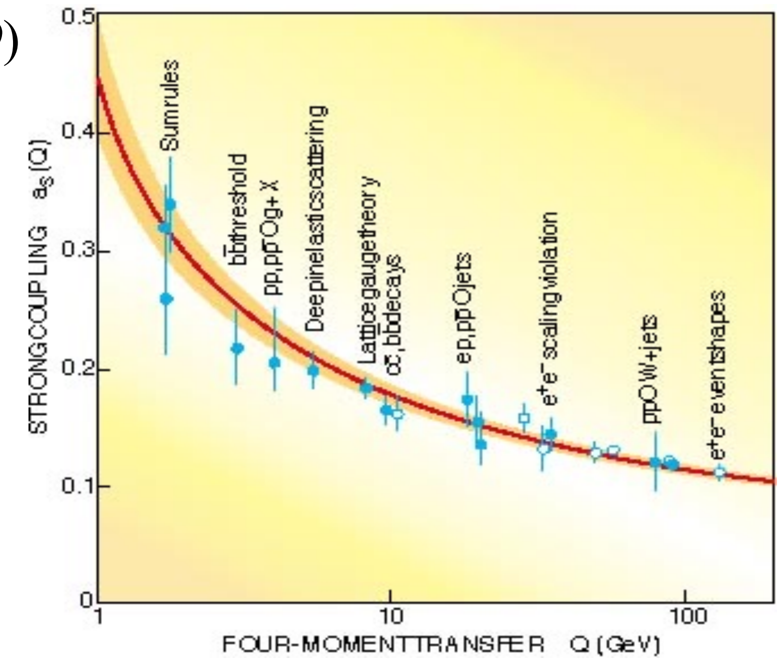
GENERAL CONCEPTS

Big goal: understand QCD at high energies.
What is the high-energy asymptotic behavior of QCD?

Running of QCD Coupling Constant

⇨ QCD coupling constant $\alpha_S = \frac{g^2}{4\pi}$ changes with the momentum scale involved in the interaction

$$\alpha_S = \alpha_S(Q)$$



Asymptotic Freedom!

Gross and Wilczek,
Politzer, ca ‘73

Physics Nobel Prize 2004!

For short distances x < 0.2 fm, or, equivalently, large momenta k > 1 GeV the QCD coupling is small $\alpha_S << 1$ and interactions are weak.

What sets the scale of running QCD coupling in high energy collisions?

- "Optimist": $$\alpha_S = \alpha_S\left(\sqrt{s}\right) << 1$$
- Pessimist: $\alpha_S = \alpha_S\left(\Lambda_{QCD}\right) \sim 1$ we simply can not tackle high energy scattering in QCD.
- pQCD: only study high-p_T particles such that
$$\alpha_S = \alpha_S\left(p_T\right) << 1$$
But: what about total cross section? bulk of particles?

What sets the scale of running QCD coupling in high energy collisions?

- Saturation physics is based on the existence of a large internal momentum scale Q_S which grows with both energy s and nuclear atomic number A

$$\boxed{Q_S^2 \sim A^{1/3}\ s^{\lambda}}$$

such that $$\boxed{\alpha_S = \alpha_S(Q_S)\ << 1}$$

and we can calculate total cross sections, particle spectra and multiplicities, etc, from <u>first principles</u>.

The main principle

- Saturation physics is based on the existence of a large internal transverse momentum scale Q_S which grows with both decreasing Bjorken *x* and with increasing nuclear atomic number A

$$Q_s^2 \sim A^{1/3} \left(\frac{1}{x}\right)^{\lambda}$$

such that

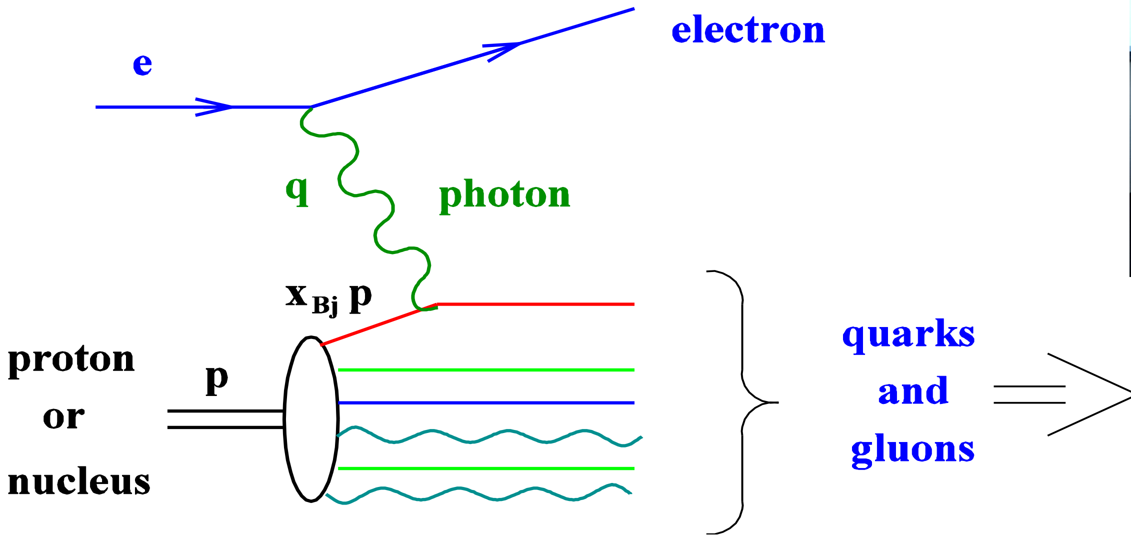
$$\boxed{\alpha_S = \alpha_S(Q_S) << 1}$$

and we can use perturbation theory to calculate total cross sections, particle spectra and multiplicities, correlations, etc, from first principles.

Quasi-classical approximation

A. Glauber-Mueller Rescatterings

Kinematics of DIS



- Photon carries 4-momentum q_μ , its virtuality is

$$Q^2 = -q_\mu \, q^\mu$$

- Photon hits a quark in the proton carrying momentum $x_{Bj} p$ with p being the proton's momentum. Parameter x_{Bj} is the Bjorken *x* variable.

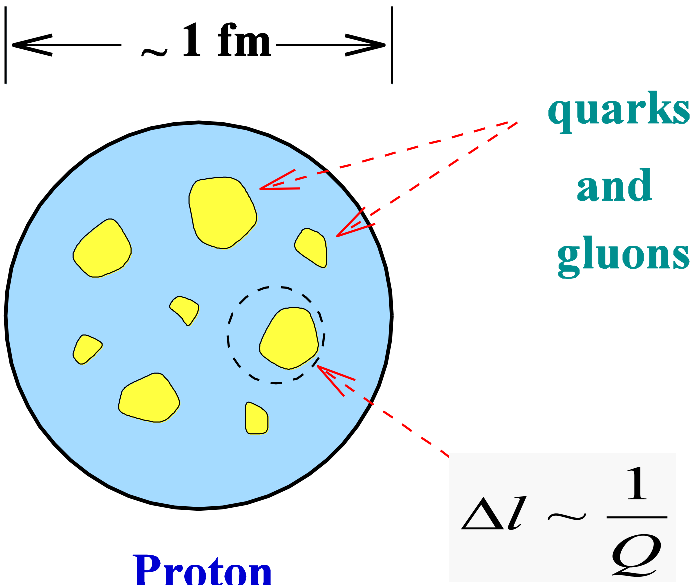
Physical Meaning of Q

Uncertainty principle teaches us that

$$\Delta p \Delta l \approx \hbar$$

which means that the photon probes the proton at the distances of the order (ħ=1)

$$\Delta l \sim \frac{1}{Q}$$

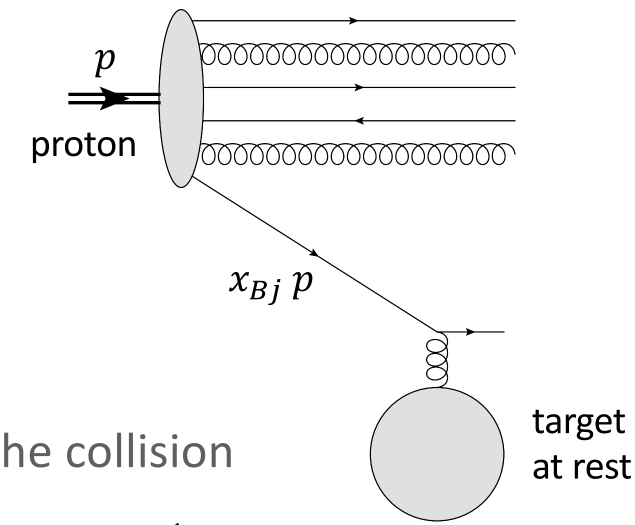


Large Momentum Q = Short Distances Probed

Physical Meaning of Bjorken x

The quarks and gluons that interact with the target have their typical momenta on the order of the typical momentum in the target,

$$x_{Bj}\, p \;\approx q \approx m.$$



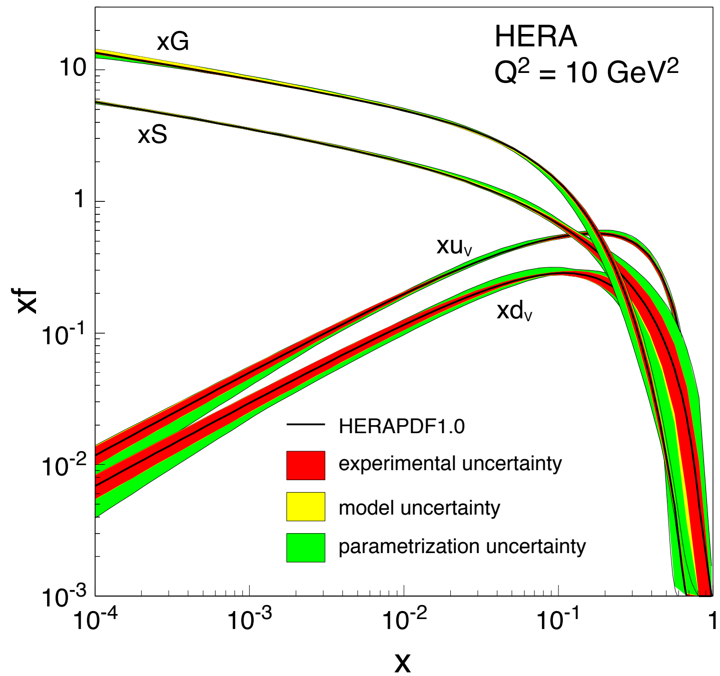
Then the energy of the collision

$$E \sim p \sim \frac{1}{x_{Bj}}$$

High Energy = Small x

Gluons at Small-x

- There is a large number of small-x gluons (and quarks) in a proton:

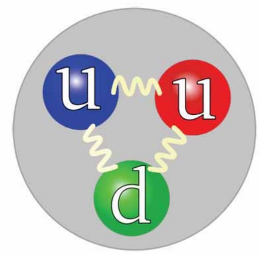


- $G(x, Q^2)$, $q(x, Q^2)$ = gluon and quark number densities (q=u,d, or S for sea).

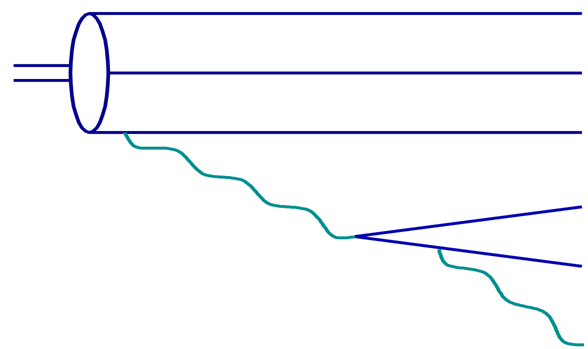
Gluons and Quarks in the Proton

⇨ There is a huge number of quarks, anti-quarks and gluons at small-x !

⇨ How do we reconcile this result with the picture of the proton made up of three valence quarks?



⇨ Qualitatively we understand that these extra quarks and gluons are emitted by the original three valence quarks in the proton.



Dipole picture of DIS

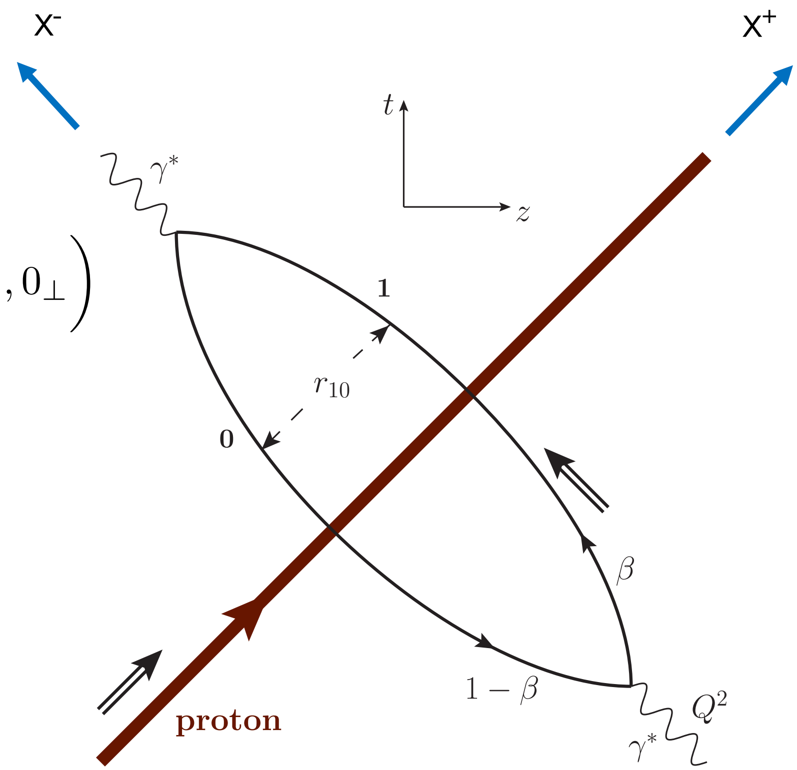
$$W^{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x \, e^{iq\cdot x} \, \langle P | j^\mu(x) \, j^\nu(0) | P \rangle$$

$$q^\mu = \left(\frac{Q^2}{2q^-}, q^-, 0_\perp \right)$$

Large q^- → large x^- separation

$$e^{iq\cdot x} = e^{i\frac{Q^2}{2q^-}x^- + iq^- x^+}$$

$$x^\pm = \frac{t \pm z}{\sqrt{2}}$$

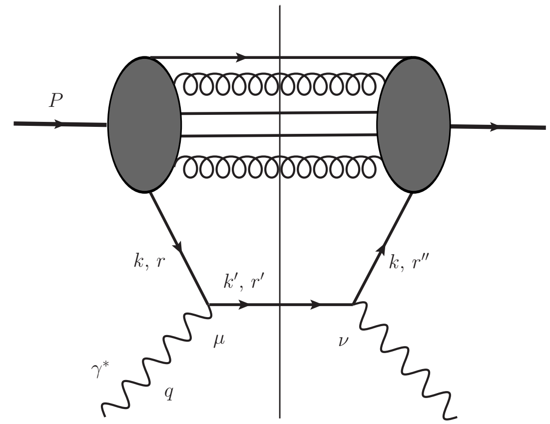


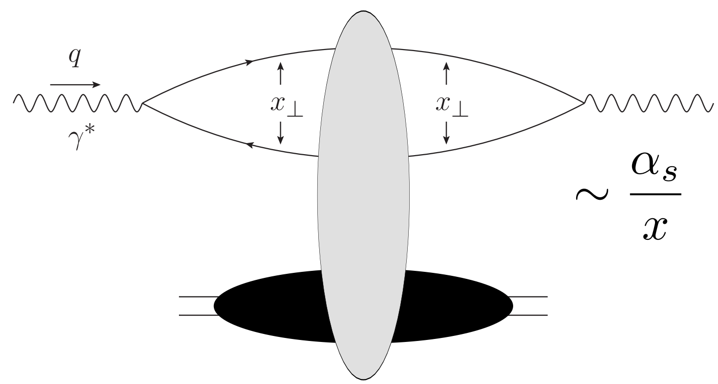
aka the "shock wave"

Dipole picture of DIS

- At small x, the dominant contribution to DIS structure functions does not come from the handbag diagram.

- Instead, the dominant terms comes from the dipole picture of DIS, where the virtual photon splits into a quark-antiquark pair, which then interacts with the target.



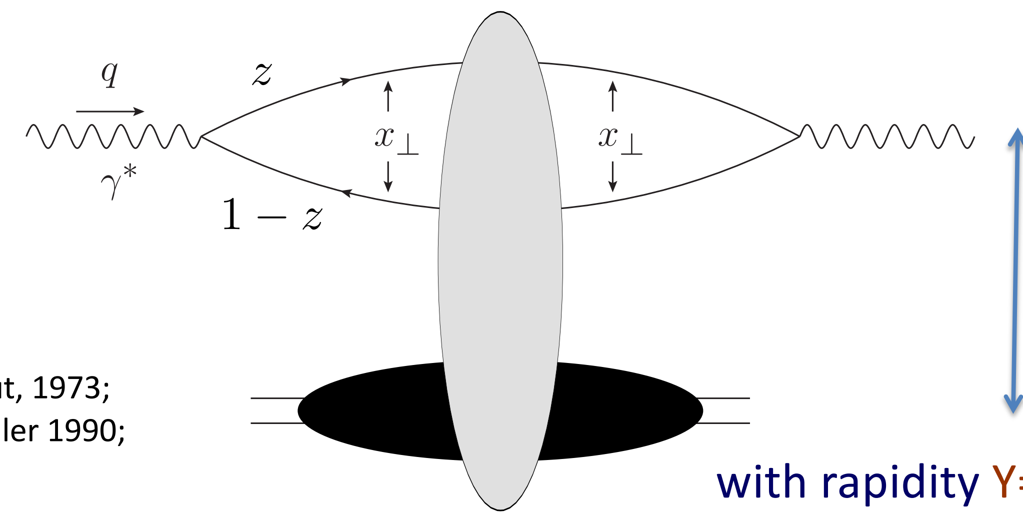


$$\sim \frac{\alpha_s}{x}$$

Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)}\; |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2 \; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



b is the Fourier conjugate to q with t = - q^2, making the dipole amplitude N similar to the GPDs at zero skewedness.

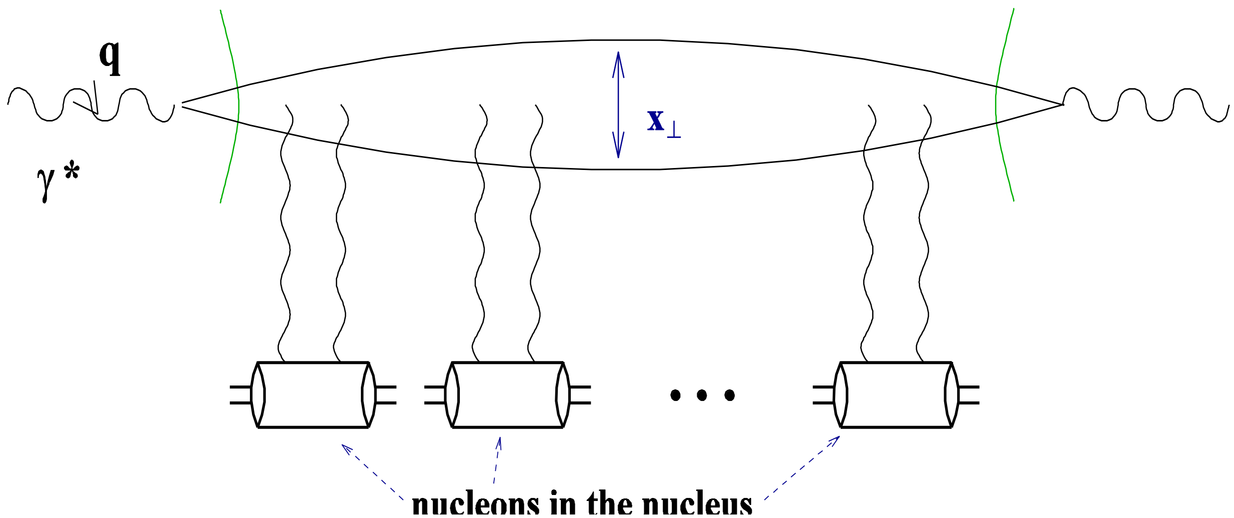
$b_\perp,\ Y$

Gribov, 1970; Bjorken and Kogut, 1973; Frankfurt, Strikman 1988; Mueller 1990; Nikolaev and Zakharov 1991

with rapidity Y=ln(1/x)

DIS in the Classical Approximation

The DIS process in the rest frame of the target nucleus is shown below.



$$\sigma_{tot}^{\gamma * A}(x_{Bj}, Q^2) = |\Psi^{\gamma * \rightarrow q\bar{q}}|^2 \otimes N(x_\perp, Y = \ln 1/x_{Bj})$$

with rapidity Y=ln(1/x)

Dipole Amplitude

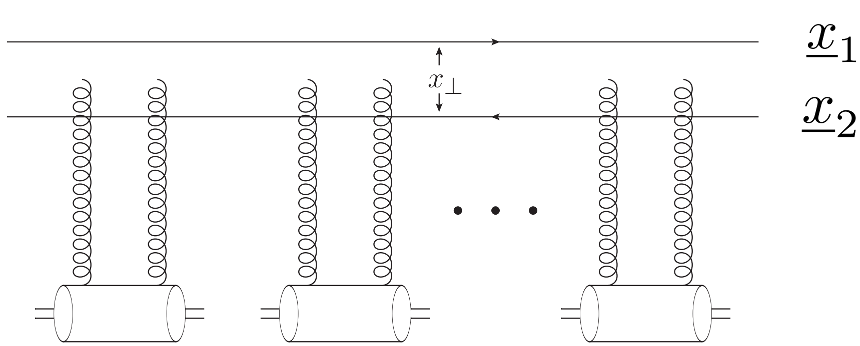
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr} \left[V(\underline{x}_1) \, V^\dagger(\underline{x}_2) \right] \right\rangle$$

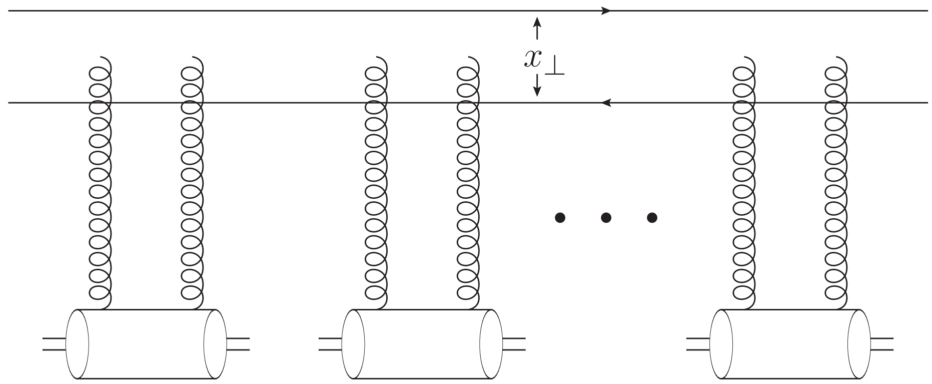
- Here we use the Wilson lines along the light-cone direction

$$V_{\underline{x}} = \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^- A^+(0^+, x^-, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:



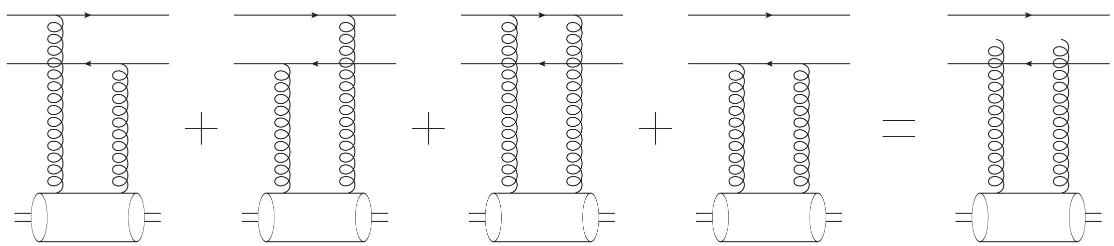
Quasi-classical dipole amplitude



A.H. Mueller, '90

Lowest-order interaction with each nucleon – two gluon exchange – lead to the following resummation parameter:

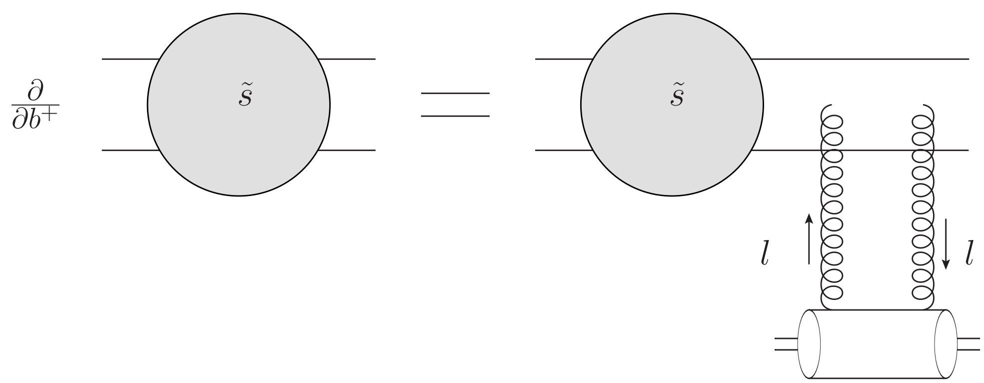
$$\alpha_s^2 \, A^{1/3}$$



Quasi-classical dipole amplitude

- To resum multiple rescatterings, note that the nucleons are independent of each other and rescatterings on the nucleons are also independent.

- One then writes an equation (Mueller '90)



Each scattering!

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\, \ln \frac{1}{x_\perp\, \Lambda}\right]$$

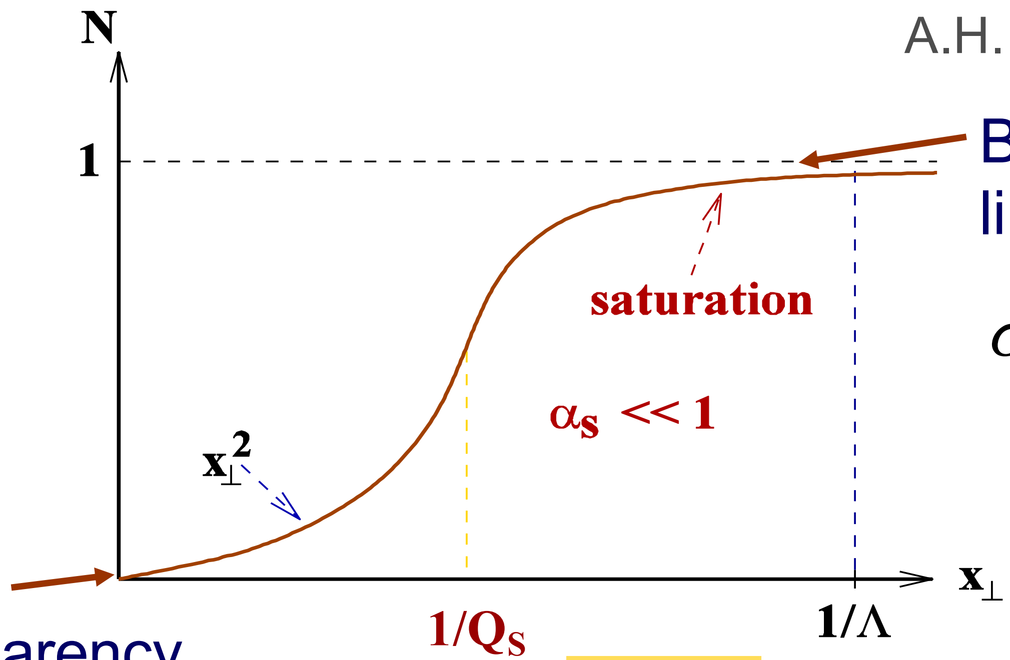
DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$\sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)$$

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2 \, Q_s^2}{4} \, \ln \frac{1}{x_\perp \, \Lambda}\right]$$

A.H. Mueller, ‘90



Black disk limit,

$$\sigma_{tot} < 2\pi R^2$$

Color transparency

$Q_S^2 \sim A^{1/3}$

Black Disk Limit

- Start with basic scattering theory: the final and initial states are related by the S-matrix operator,

$$|\psi_f\rangle = \hat{S}\,|\psi_i\rangle$$

- Write it as $$|\psi_f\rangle = |\psi_i\rangle + \left[\hat{S} - 1\right]\,|\psi_i\rangle$$

- The total cross section is

$$\sigma_{tot} \propto \left|\left[\hat{S} - 1\right]\,|\psi_i\rangle\right|^2 = 2 - S - S^*$$

where the forward matrix element of the S-matrix operator is

$$S = \langle\psi_i|\,\hat{S}\,|\psi_i\rangle$$

and we have used unitarity of the S-matrix

$$\hat{S}\,\hat{S}^\dagger = 1$$

Black Disk Limit

- Now, since $$|\psi_f\rangle = |\psi_i\rangle + \left[\hat{S} - 1\right] |\psi_i\rangle$$

  the elastic cross section is

$$\sigma_{el} \propto \left| \langle\psi_i| \left[\hat{S} - 1\right] |\psi_i\rangle \right|^2 = |1 - S|^2$$

- The inelastic cross section can be found via

$$\sigma_{tot} = \sigma_{inel} + \sigma_{el}$$

- In the end, for scattering with impact parameter b we write

$$\sigma_{tot} = 2 \int d^2b \, [1 - \mathrm{Re}\, S(b)]$$

$$\sigma_{el} = \int d^2b \, |1 - S(b)|^2$$

$$\sigma_{inel} = \int d^2b \left[1 - |S(b)|^2\right]$$

Unitarity Limit

- Unitarity implies that

$$1 = \langle \psi_i | \, \hat{S} \, \hat{S}^\dagger \, | \psi_i \rangle = \sum_X \langle \psi_i | \, \hat{S} | X \rangle \, \langle X | \hat{S}^\dagger \, | \psi_i \rangle \geq |S|^2$$

- Therefore

$$|S| \leq 1$$

leading to the unitarity bound on the total cross section

$$\sigma_{tot} = 2 \int d^2 b \; [1 - \mathrm{Re}\, S(b)] \leq 4 \int d^2 b = 4\pi R^2$$

- Notice that when S=-1 the inelastic cross section is zero and

$$\sigma_{tot} = 2 \int d^2 b \, [1 - \mathrm{Re}\, S(b)]$$

$$\sigma_{tot} = 4\pi R^2 = \sigma_{el}$$

$$\sigma_{el} = \int d^2 b \, |1 - S(b)|^2$$

This limit is realized in low-energy scattering!

$$\sigma_{inel} = \int d^2 b \left[1 - |S(b)|^2 \right]$$

Black Disk Limit

- At high energy inelastic processes dominate over elastic. Imposing

$$\sigma_{inel} \geq \sigma_{el}$$

we get

$$\mathrm{Re}\, S \geq 0$$

- The bound on the total cross section is (aka the **black disk limit**)

$$\sigma_{tot} = 2 \int d^2b \, [1 - \mathrm{Re}\, S] \leq 2 \int d^2b = 2\pi R^2$$

- The inelastic and elastic cross sections at the black disk limit are

$$\sigma_{inel} = \sigma_{el} = \pi R^2$$

$$\sigma_{tot} = 2 \int d^2b \, [1 - \mathrm{Re}\, S(b)]$$

$$\sigma_{el} = \int d^2b \, |1 - S(b)|^2$$

$$\sigma_{inel} = \int d^2b \left[1 - |S(b)|^2\right]$$

Notation

- At high energies $$\mathrm{Im}\, S \approx 0$$

  while the dipole amplitude N is the imaginary part of the T-matrix (S=1+iT), such that
  $$\mathrm{Re}\, S = 1 - N$$

- The cross sections are
$$\sigma_{tot} = 2 \int d^2 b\, N(x_\perp, b_\perp)$$
$$\sigma_{el} = \int d^2 b\, N^2(x_\perp, b_\perp)$$
$$\sigma_{inel} = \int d^2 b \left[2\, N(x_\perp, b_\perp) - N^2(x_\perp, b_\perp)\right]$$

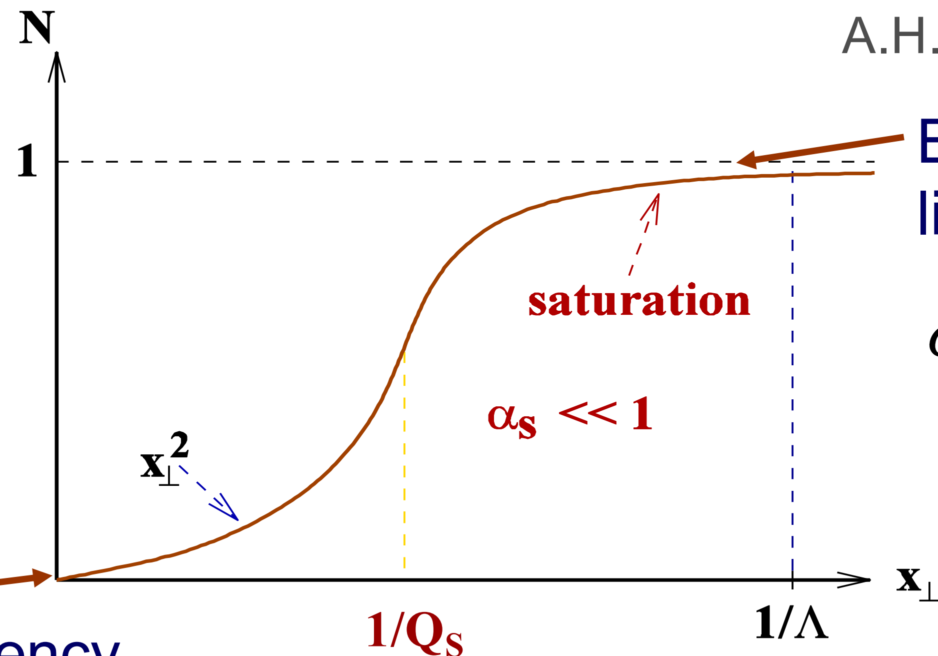
- We see that N=1 is the black disk limit. Hence $N \leq 1$ as we saw above.

DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\,\ln\frac{1}{x_\perp\,\Lambda}\right]$$

A.H. Mueller, '90



Black disk limit,

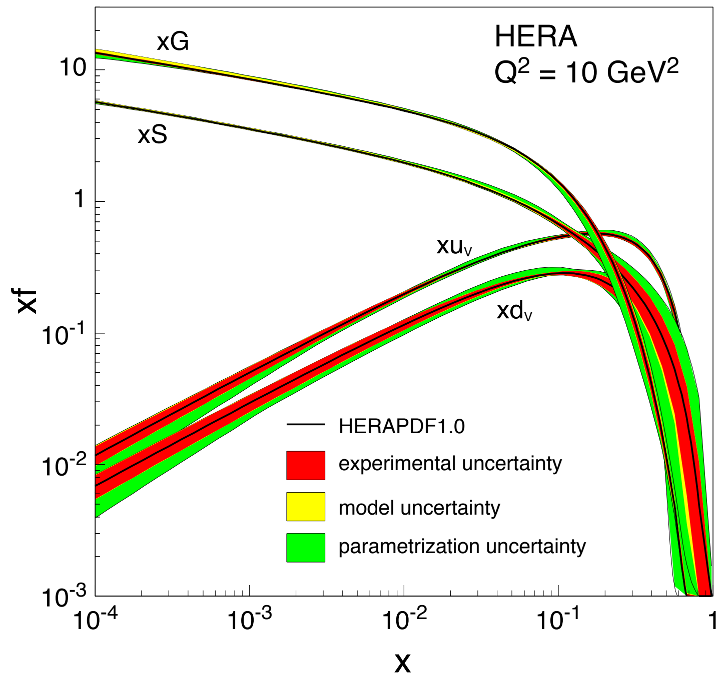
$$\sigma_{tot} < 2\pi R^2$$

Color transparency

B. McLerran-Venugopalan Model

Gluons at Small-x

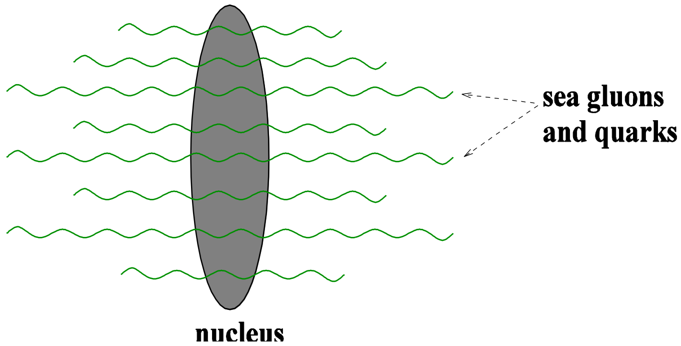
- There is a large number of small-x gluons (and quarks) in a proton:

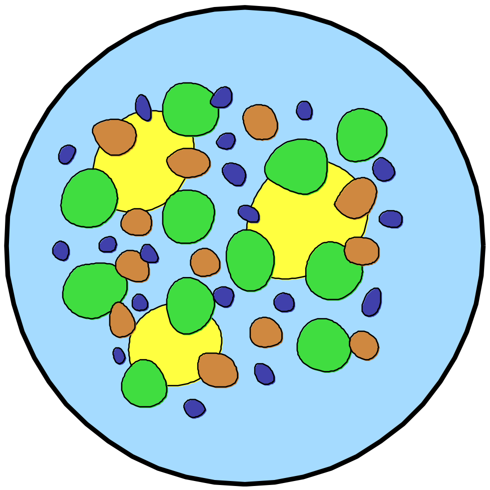


- $G(x, Q^2)$, $q(x, Q^2)$ = gluon and quark number densities (q=u,d, or S for sea).

McLerran-Venugopalan Model

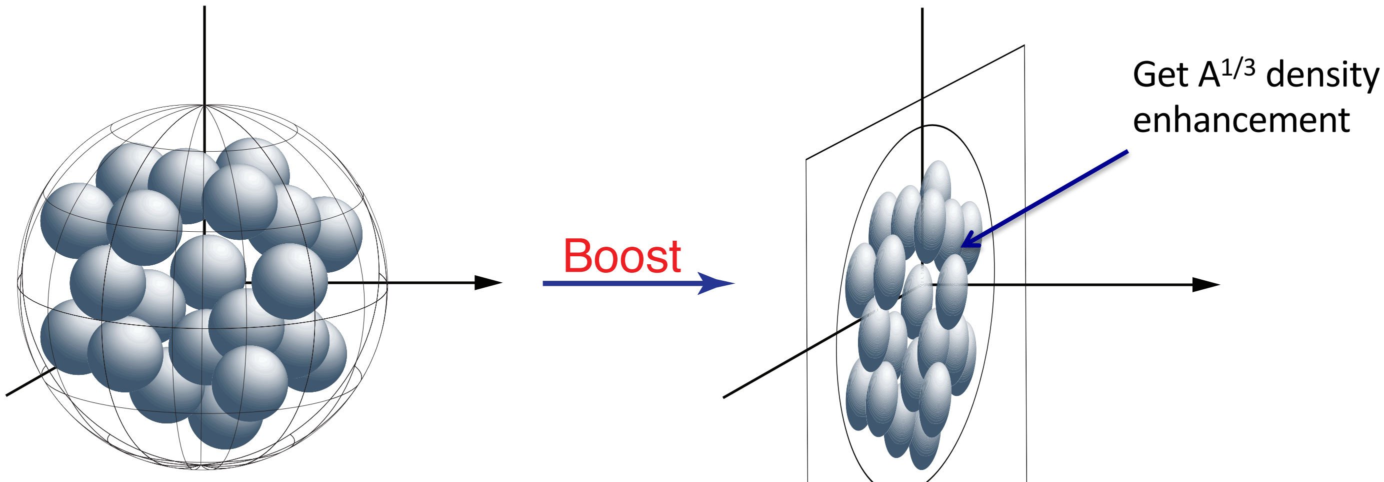
- The wave function of a single nucleus has many small-x quarks and gluons in it.
- In the transverse plane the nucleus is densely packed with gluons and quarks.





Large occupation number ⇨ Classical Field

McLerran-Venugopalan Model

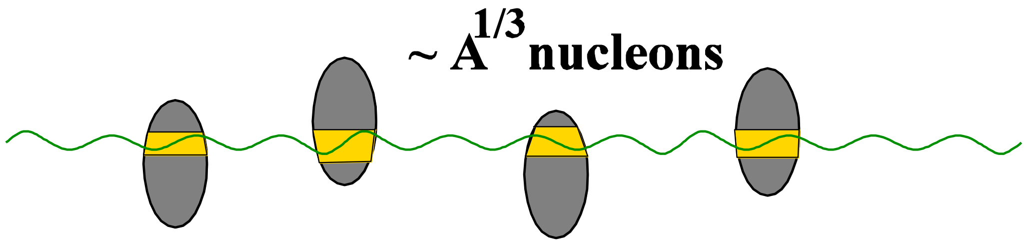


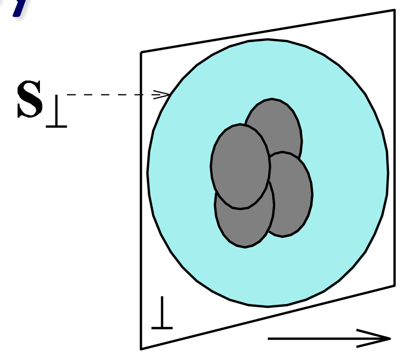
$$M = \rho V = \rho \frac{4}{3}\pi R^3 \sim A \quad \Rightarrow \quad R \sim A^{1/3}$$

$$\boxed{Q_S^2 \sim A^{1/3}}$$

- Large gluon density gives a large momentum scale Q_s (the saturation scale): Q_s^2 ~ # gluons per unit transverse area ~ $A^{1/3}$ (nuclear oomph).
- For $Q_s >> \Lambda_{QCD}$, get a theory at weak coupling $\alpha_s(Q_s^2) \ll 1$ and the leading gluon field is classical.

Color Charge Density





Small-x gluon “sees” the whole nucleus coherently in the longitudinal direction! It “sees” many color charges which form a net effective color charge Q = g (# charges)$^{1/2}$, such that $Q^2 = g^2$ #charges (random walk).

Define color charge density

$$\mu^2 = \frac{Q^2}{S_\perp} = \frac{g^2 \; \# \, charges}{S_\perp} \propto g^2 \, \frac{A}{S_\perp} \propto A^{1/3}$$

McLerran Venugopalan ’93-’94

such that for a large nucleus (A>>1)

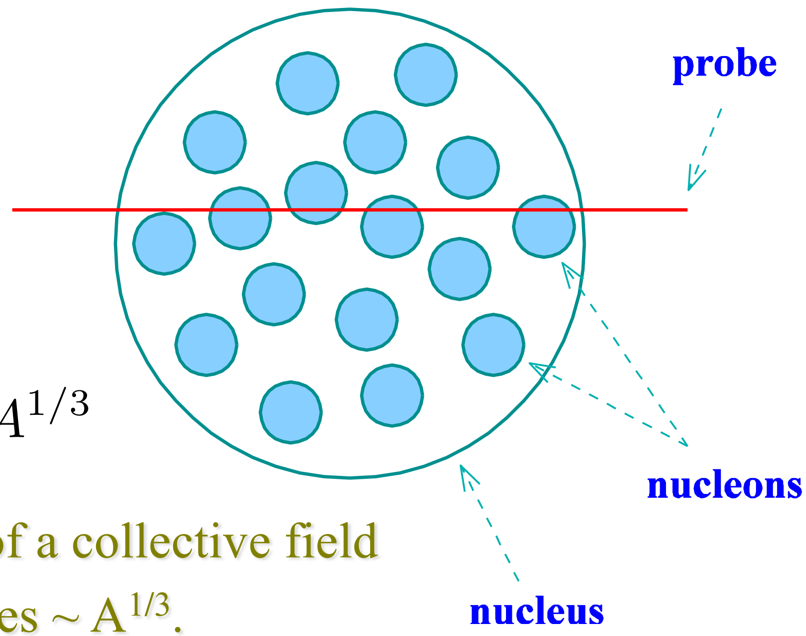
$$\boxed{\mu^2 \propto \Lambda_{QCD}^2 \, A^{1/3} \gg \Lambda_{QCD}^2 \quad \Longrightarrow \quad \alpha_s(\mu^2) \ll 1}$$

Nuclear small-x wave function is perturbative!

$$\mu = Q_s$$

Saturation Scale

To argue that $\boxed{Q_s^2 \sim A^{1/3}}$ let us consider an example of a particle scattering on a nucleus. As it travels through the nucleus it bumps into nucleons. Along a straight line trajectory it encounters $\sim R \sim A^{1/3}$ nucleons, with R the nuclear radius and A the atomic number of the nucleus.



The particle receives $\sim A^{1/3}$ random kicks. Its momentum gets broadened by

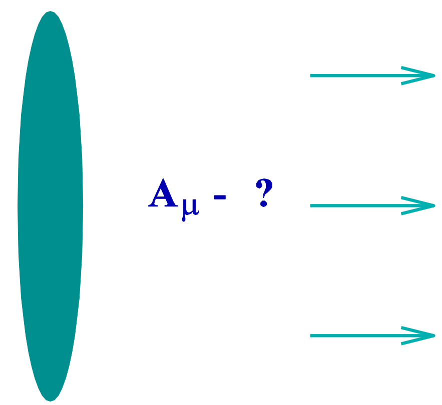
$$\Delta k \sim \sqrt{A^{1/3}} \Rightarrow (\Delta k)^2 \sim A^{1/3}$$

Saturation scale, as a feature of a collective field of the whole nucleus also scales $\sim A^{1/3}$.

McLerran-Venugopalan Model

- To find the classical gluon field A_μ of the nucleus one has to solve the non-linear analogue of Maxwell equations – the Yang-Mills equations, with the nucleus as a source of the color charge:

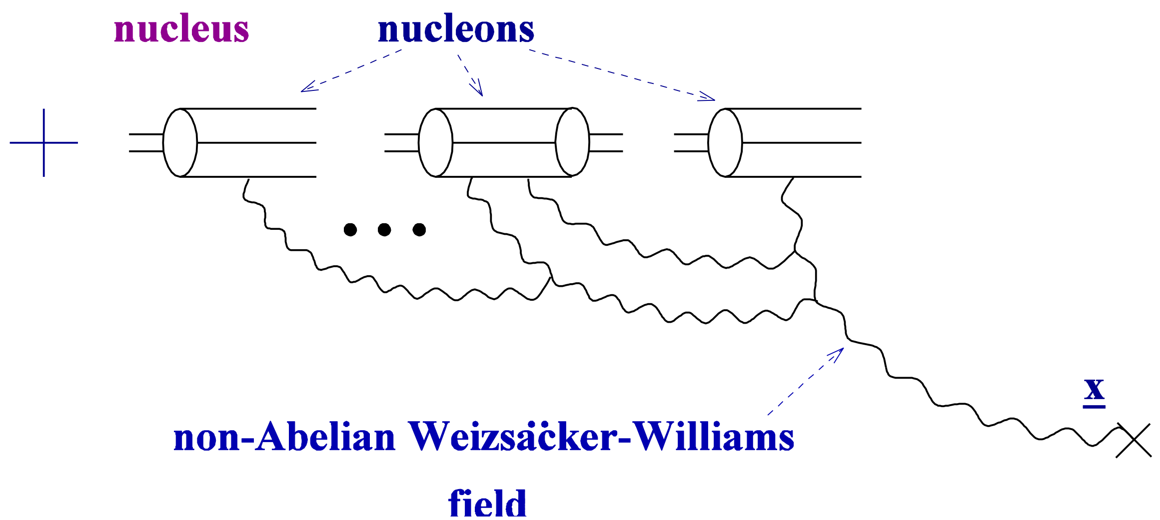
$$\mathcal{D}_\nu \, F^{\mu\nu} = J^\mu$$



nucleus is Lorentz contacted into a pancake

Yu. K. '96; J. Jalilian-Marian et al, '96

Classical Field of a Nucleus



Here's one of the diagrams showing the non-Abelian gluon field of a large nucleus.

The resummation parameter is $\alpha_S^2 A^{1/3}$, corresponding to two gluons per nucleon approximation.

Unpolarized WW Gluon TMD

- One can calculate the unpolarized gluon TMD with, say, the forward-pointing (SIDIS) Wilson line staple

$$f^G(x, k_T^2) = \frac{2}{xP^+(2\pi)^3} \int dx^- d^2x_\perp \, e^{ixP^+x^- - i\vec{k}_T \cdot \vec{x}_\perp} \left\langle P \right| \mathrm{tr} \left[F^{+i}(0) \, \mathcal{U}^{[+]}[0,x] \, F^{+i}(x^-, \vec{x}_\perp) \right] \left| P \right\rangle$$

- In $A^+=0$ gauge, one can choose a sub-gauge eliminating the Wilson line staple (making it 1), and, since $F^{+i} = \partial_- A^i$, one obtains

$$f^G(x, k_T^2) = \frac{2xP^+}{(2\pi)^3} \int dx^- d^2x_\perp \, e^{ixP^+x^- - i\vec{k}_T \cdot \vec{x}_\perp} \left\langle P \right| \mathrm{tr} \left[A^i(0) \, A^i(x^-, \vec{x}_\perp) \right] \left| P \right\rangle$$

- Since the classical (Weizsacker-Williams) A^i field is known exactly from solving the Yang-Mills equations, one can directly calculate the gluon TMD in the classical limit.
- This is the WW gluon TMD.

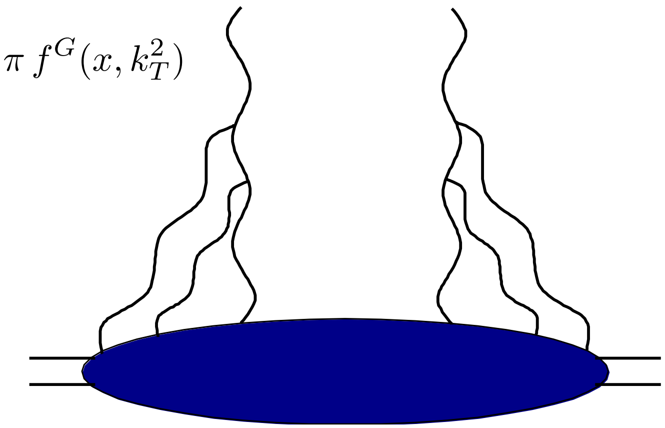
Classical Gluon Field of a Nucleus

Using the obtained classical gluon field one can construct corresponding gluon distribution function (gluon WW TMD):

$$\phi_A(x,k^2) \sim \langle \underline{A}(-k) \cdot \underline{A}(k) \rangle$$

with the field in the $A^+=0$ gauge

$$\phi(x, k_T^2) = x\,\pi\, f^G(x, k_T^2)$$



$$\phi_A(x, k_T^2) = \frac{C_F}{\alpha_s\,\pi} \int \frac{d^2 x_\perp}{x_\perp^2}\, e^{i\,\underline{k}\cdot\underline{x}} \left[1 - \exp\left(- \frac{x_\perp^2\, Q_s^2}{4} \ln \frac{1}{x_\perp\, \Lambda} \right) \right]$$

J. Jalilian-Marian et al, '97; Yu. K. and A. Mueller, '98

⇨ $Q_S=\mu$ is the saturation scale $Q_S^2 \sim A^{1/3}$

⇨ Note that $\phi \sim \langle A_\mu A_\mu \rangle \sim 1/\alpha$ such that $A_\mu \sim 1/g$, which is what one would expect for a classical field.

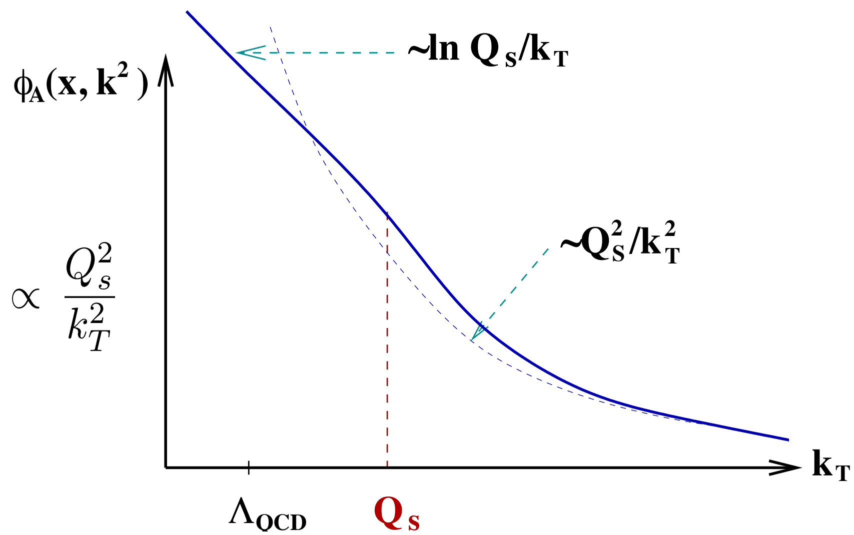
$$\phi_A(x, k_T^2) = \frac{C_F}{\alpha_s \pi} \int \frac{d^2 x_\perp}{x_\perp^2} e^{i \underline{k} \cdot \underline{x}} \left[1 - \exp\left(- \frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda} \right) \right]$$

⇨ In the UV limit of k→∞,

x_T is small and one obtains

$$\phi_A(x, k_T^2) \sim \int d^2 x_\perp \, e^{i \underline{k} \cdot \underline{x}} Q_s^2 \ln \frac{1}{x_\perp \Lambda} \propto \frac{Q_s^2}{k_T^2}$$

which is the usual LO result.



⇨ In the IR limit of small k_T,

x_T is large and we get

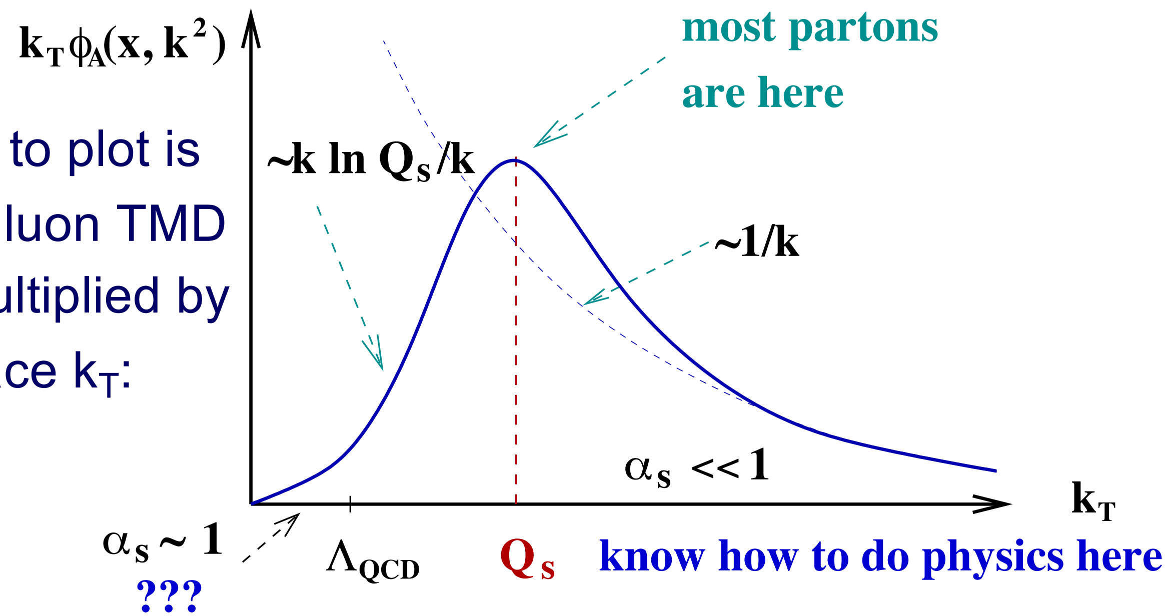
$$\phi_A(x, k_T^2) \approx \frac{C_F}{\alpha_s \pi} \int_{1/Q_s} \frac{d^2 x_\perp}{x_\perp^2} e^{i \underline{k} \cdot \underline{x}} \propto \ln \frac{Q_s}{k_T}$$

SATURATION !

Divergence is regularized.

Classical Gluon Distribution

A good object to plot is the classical gluon TMD distribution multiplied by the phase space k_T:



⇨ Most gluons in the nuclear wave function have transverse momentum of the order of $k_T \sim Q_S$ and $Q_S^2 \sim A^{1/3}$

⇨ We have a small coupling description of the whole wave function in the classical approximation.

Summary

- We applied the quasi-classical small-*x* approach to DIS in the dipole picture, obtaining Glauber-Mueller formula for multiple rescatterings of a dipole in a nucleus.

- We saw that onset of saturation ensures that unitarity (the black disk limit) is not violated. Saturation is a consequence of unitarity!

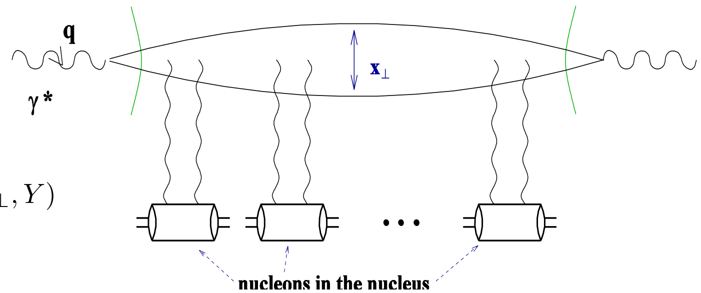
- We have reviewed the McLerran-Venugopalan model for the small-*x* wave function of a large nucleus.

- We saw the onset of gluon saturation and the appearance of a large transverse momentum scale – the saturation scale:

$$Q_s^2 \sim A^{1/3}$$

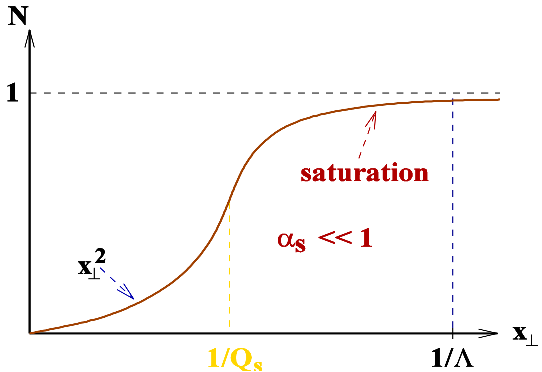
Summary of the last time

- We discussed dipole picture of DIS:



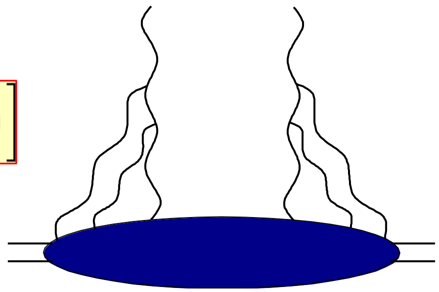
$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\pi} d^2 b_\perp \int_0^1 \frac{dz}{z(1-z)} |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2 \; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$

- We calculated multiple-rescattering of the dipole on a nucleus: $Q_S^2 \sim A^{1/3}$



- We discussed the MV model, in which the gluon field of the proton/nucleus is classical, such that you can calculate the WW gluon TMD directly:

$$\phi_A(x, k_T^2) = \frac{C_F}{\alpha_s \pi} \int \frac{d^2 x_\perp}{x_\perp^2} e^{i \underline{k} \cdot \underline{x}} \left[1 - \exp\left(-\frac{x_\perp^2 Q_s^2}{4} \ln \frac{1}{x_\perp \Lambda}\right)\right]$$

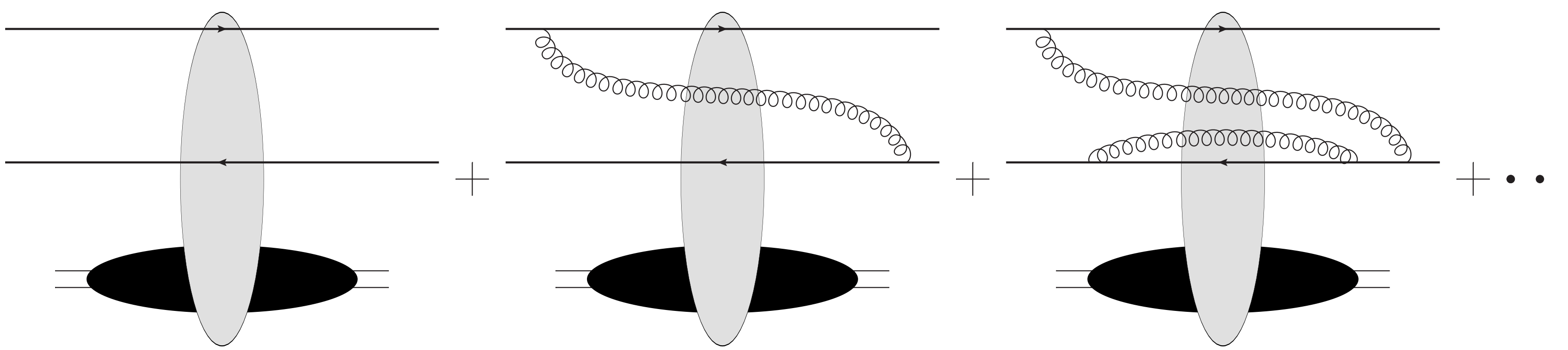


Small-x evolution equations

Small-x Evolution

- Energy dependence comes in through the long-lived *s*-channel gluon corrections (higher Fock states):

$$\alpha_s \ln s \sim \alpha_s \ln \frac{1}{x} \sim 1$$



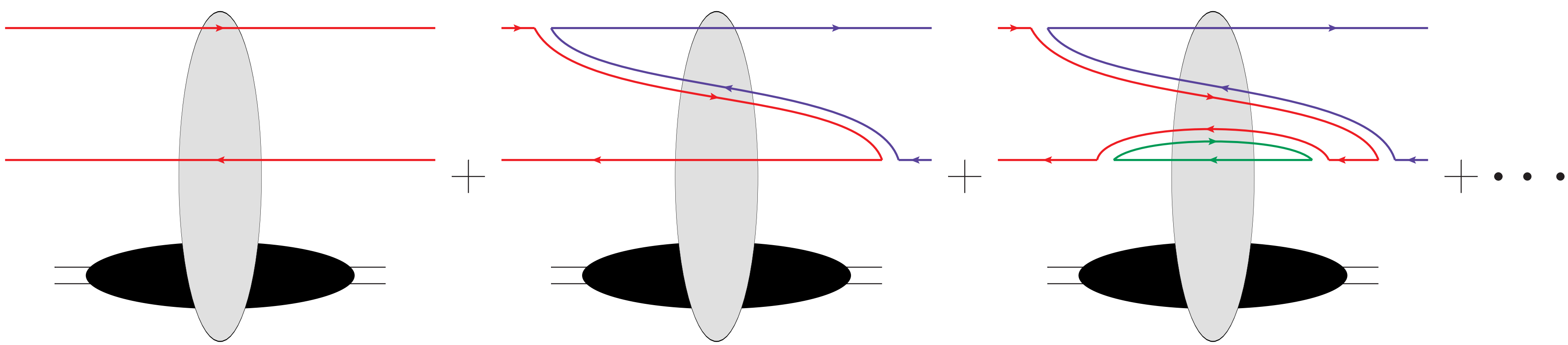
These extra gluons bring in powers of $\alpha_S \ln s$, such that when $\alpha_S \ll 1$ and $\ln s \gg 1$ this parameter is $\alpha_S \ln s \sim 1$ (leading logarithmic approximation, LLA).

Small-x Evolution: Large N_c Limit

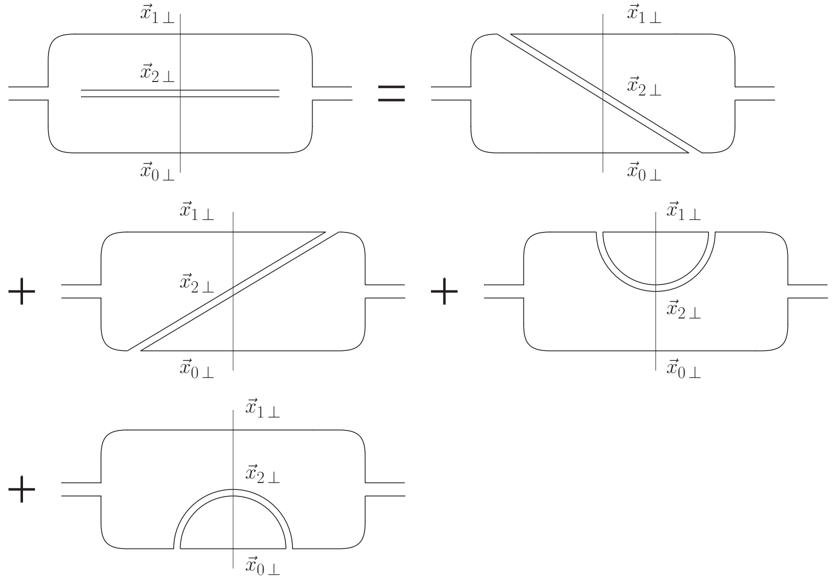
- How do we resum this cascade of gluons?
- The simplification comes from the large-Nc limit, where each gluon becomes a quark-antiquark pair:

$$3 \otimes \bar{3} = 1 \oplus 8 \quad \Rightarrow \quad N_c \otimes \bar{N}_c = 1 \oplus (N_c^2 - 1) \approx N_c^2 - 1$$

- Gluon cascade becomes a dipole cascade (each color outlines a dipole):



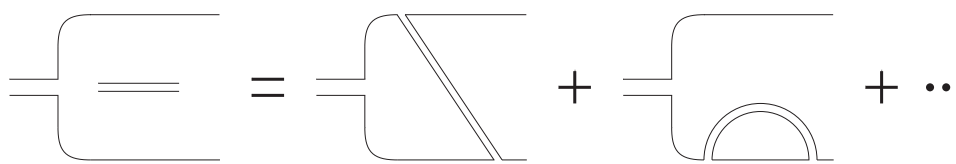
Notation (Large-N_C)



Real emissions in the amplitude squared

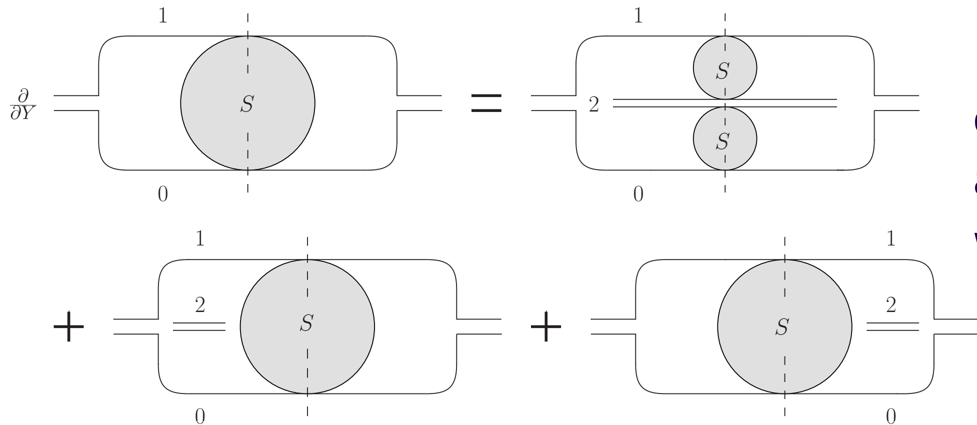
(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



Nonlinear Evolution

To sum up the gluon cascade at large-N_C we write the following equation for the dipole S-matrix:



dashed line = all interactions with the target

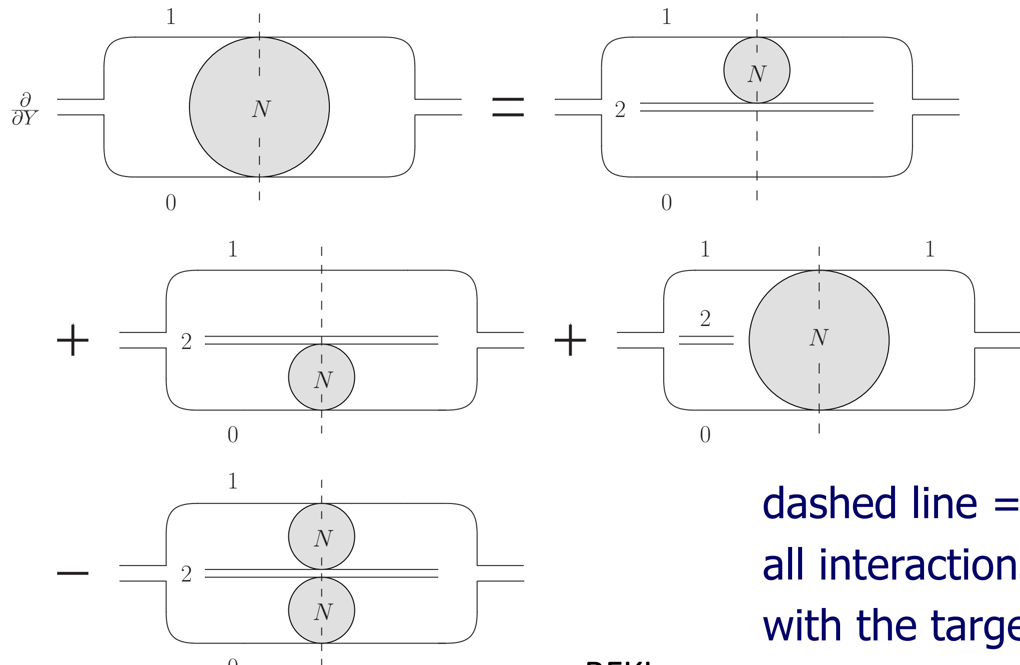
$$Y = \ln \frac{1}{x} \sim \ln s$$

$$\partial_Y S_{\mathbf{x_0},\mathbf{x_1}}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \left[S_{\mathbf{x_0},\mathbf{x_2}}(Y) \, S_{\mathbf{x_2},\mathbf{x_1}}(Y) - S_{\mathbf{x_0},\mathbf{x_1}}(Y) \right]$$

Remembering that S=1 + i T = 1 - N where N = Im(T) we can rewrite this equation in terms of the dipole scattering amplitude N.

Nonlinear evolution at large N_c

As N=1-S we write



dashed line = all interactions with the target

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \, [\overbrace{N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y)}^{\text{BFKL}} - N_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, N_{\mathbf{x}_2,\mathbf{x}_1}(Y)]$$

Balitsky '96, Yu.K. '99; beyond large N_c, JIMWLK evolution, 0.1% correction for the dipole amplitude

Resummation parameter

- BK equation resums powers of

$$\alpha_s \, N_c \, Y$$

- The Glauber-Mueller/McLerran-Venugopalan initial conditions for it resum powers of

$$\alpha_s^2 \, A^{1/3}$$

- Beyond the large-N_c limit: use the JIMWLK functional evolution equation (Iancu, Jalilian-Marian, Kovner, Leonidov, McLerran and Weigert, 1997-2002)

JIMWLK: derivation outline

A.H. Mueller, 2001

- Start by introducing a weight functional, $W_Y[\alpha]$. Here $\alpha=A^+$ is the gluon field of the target proton or nucleus. $\alpha(x^-,\vec{x}) \equiv A^+(x^+=0,x^-,\vec{x})$
- The functional is used to generate expectation values of gluon-field dependent operators in the target state:

$$\langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \; \hat{O}_\alpha \, W_Y[\alpha]$$

- Imagine that we know small-x evolution for some operator O:

$$\partial_Y \, \langle \hat{O}_\alpha \rangle_Y = \langle \mathcal{K}_\alpha \otimes \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \, \left[\mathcal{K}_\alpha \otimes \hat{O}_\alpha \right] \, W_Y[\alpha]$$

- On the other hand, we can differentiate the first equation above,

$$\partial_Y \, \langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \; \hat{O}_\alpha \; \partial_Y \, W_Y[\alpha]$$

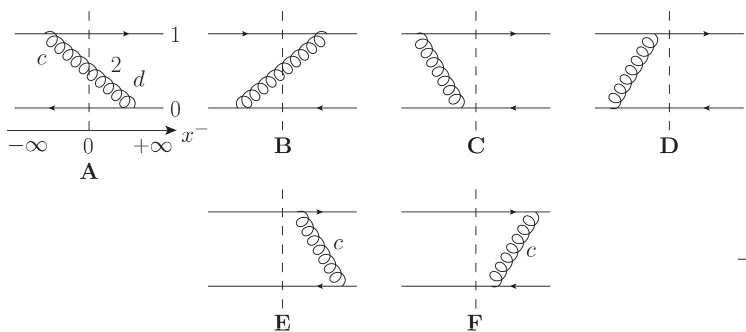
- Comparing the last two equations and integrating by parts in the second to last equation, we will arrive at and equation for the weight functional $W_Y[\alpha]$.

JIMWLK: derivation outline

- As a test operator, take a pair of Wilson lines (not a dipole!):

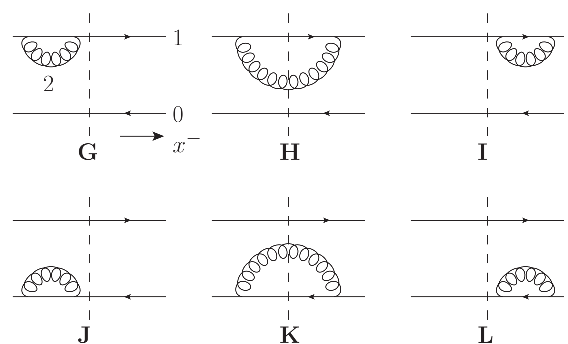
$$\hat{O}_{\vec{x}_{1\perp},\vec{x}_{0\perp}} = V_{\vec{x}_{1\perp}} \otimes V^{\dagger}_{\vec{x}_{0\perp}}$$

- Construct the evolution of this operator by summing the following familiar diagrams:



real

virtual



JIMWLK Equation

- In the end one arrive at the JIMWLK evolution equation (1997-2002):

$$\partial_Y W_Y[\alpha] = \alpha_s \left\{ \frac{1}{2} \int d^2x_\perp \, d^2y_\perp \, \frac{\delta^2}{\delta\alpha^a(x^-, \vec{x}_\perp)\,\delta\alpha^b(y^-, \vec{y}_\perp)} \left[\eta^{ab}_{\vec{x}_\perp \vec{y}_\perp} W_Y[\alpha]\right] \right.$$
$$\left. - \int d^2x_\perp \, \frac{\delta}{\delta\alpha^a(x^-, \vec{x}_\perp)} \left[\nu^a_{\vec{x}_\perp} W_Y[\alpha]\right] \right\}$$

with

$$\eta^{ab}_{\vec{x}_{1\perp}\vec{x}_{0\perp}} = \frac{4}{g^2\,\pi^2} \int d^2x_2 \, \frac{\vec{x}_{21}\cdot\vec{x}_{20}}{x_{21}^2\,x_{20}^2} \left[\mathbf{1} - U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} + U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{0\perp}}\right]^{ab}$$

$$\nu^a_{\vec{x}_{1\perp}} = \frac{i}{g\,\pi^2} \int \frac{d^2x_2}{x_{21}^2} \, \mathrm{Tr}\left[T^a U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}}\right]$$

- Here U is the adjoint Wilson line on a light cone,

$$U_{\vec{x}_\perp} = \mathrm{P}\exp\left\{ i\,g \int\limits_{-\infty}^{\infty} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \vec{x}_\perp) \right\}$$

JIMWLK Equation

- JIMWLK equation can be used to construct any-N_C small-x evolution of any operator made of infinite light-cone Wilson lines (in any representation), such as color-dipole, color-quadrupole, etc., and other operators.

- Since

$$\Box \alpha(x^-, \vec{x}) = \rho(x^-, \vec{x})$$

  JIMWLK evolution can be re-written in terms of the color density ρ in the kernel.

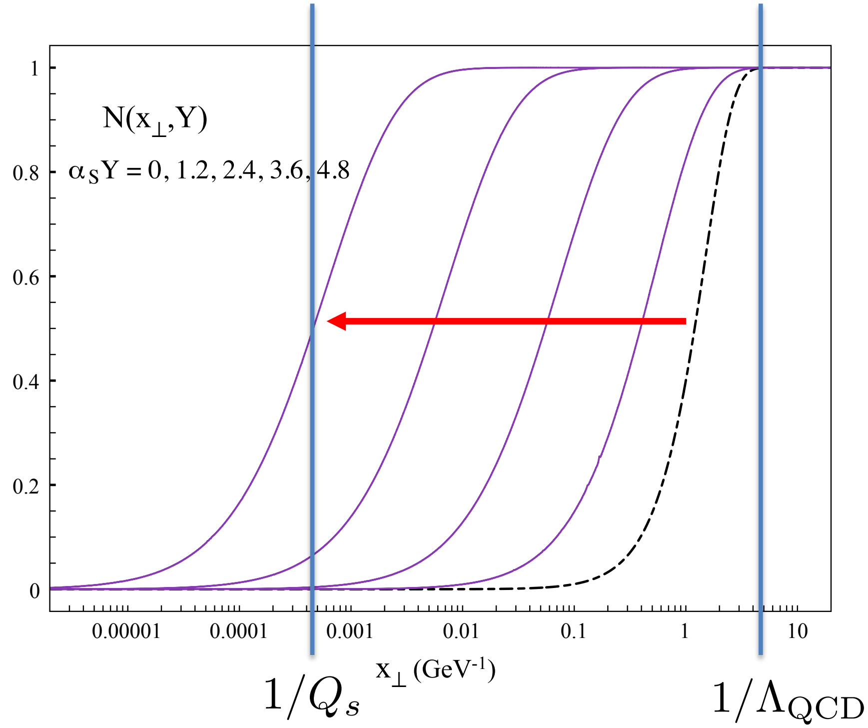
- JIMWLK approach sums up powers of $\alpha_s \, Y$ and $\alpha_s^2 \, A^{1/3}$

Solving JIMWLK

- The JIMWLK equation was solved on the lattice by K. Rummukainen and H. Weigert '04 (and others since).
- For the dipole amplitude $N(x_0,x_1, Y)$, the **relative** corrections to the large-N_C limit BK equation are **< 0.001 !** Not the naïve $1/N_C^2 \sim 0.1$! (For realistic rapidities/energies.)
- The reason for that is dynamical and is largely due to saturation effects suppressing the bulk of the potential $1/N_C^2$ corrections (Yu.K., J. Kuokkanen, K. Rummukainen, H. Weigert, '08).
- There are other objects at small x, quadrupoles, double-trace operators, etc. Some (linear combinations) of them are subleading-N_c, and one has to use JIMWLK to describe their evolution.

Solution of the nonlinear equation

Solution of BK equation



numerical solution
by J. Albacete '03

Energy increases → Q_s increases moving further away from $\Lambda_{\rm QCD}$

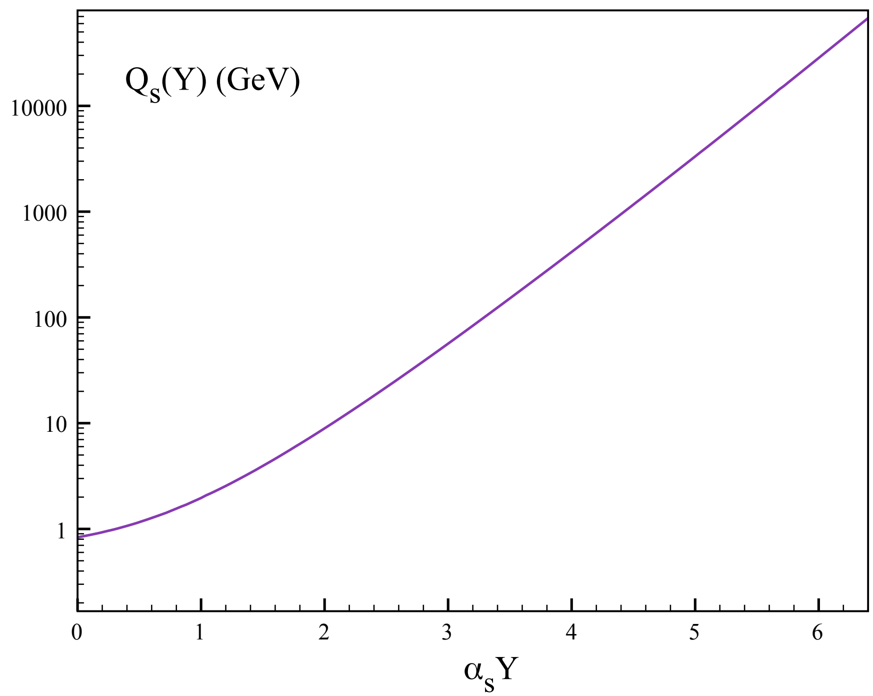
We conclude that

$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

BK solution preserves the black disk limit, N<1 always
(unlike the linear BFKL equation)

$$\sigma^{q\bar{q}A} = 2\int d^2b\, N(x_\perp, b_\perp, Y)$$

Saturation scale



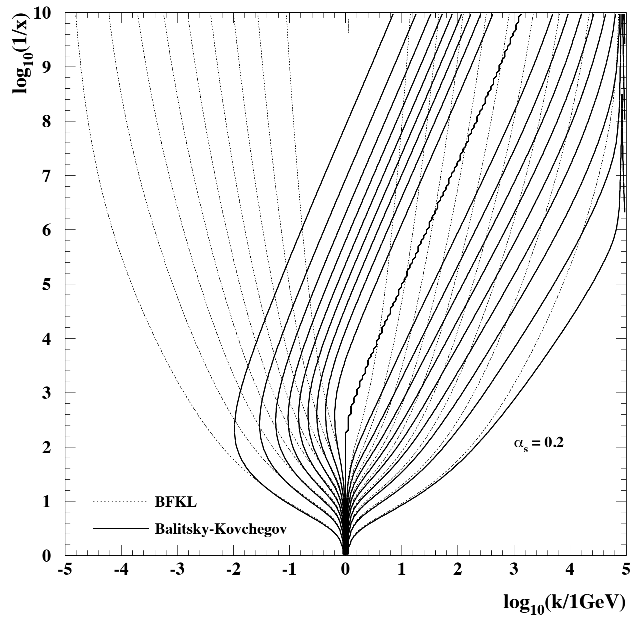
numerical solution by J. Albacete (ca. 2006)

BK Solution

- Preserves the black disk limit, N<1 always.

$$\sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)$$

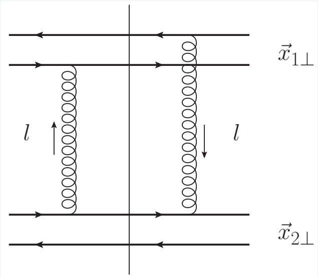
- Avoids the IR problem of BFKL evolution due to the saturation scale screening the IR:



Golec-Biernat, Motyka, Stasto ‘02

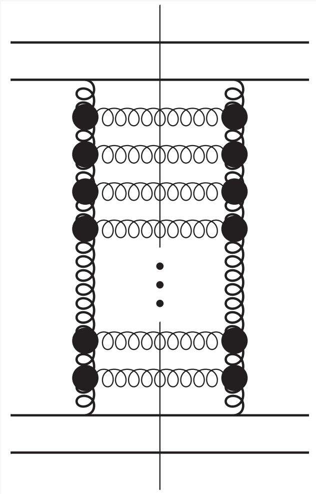
The BFKL Equation

- The Balitsky, Fadin, Kuraev, Lipatov (BFKL) equation was derived in 1977-78.
- One starts with a two-gluon exchange diagram (left) and "dresses" it by radiative corrections.
- The leading high-energy contribution can be drawn as a ladder diagram, with the t-channel gluons being the special "reggeized" gluons and the thick dots representing effective Lipatov vertices.



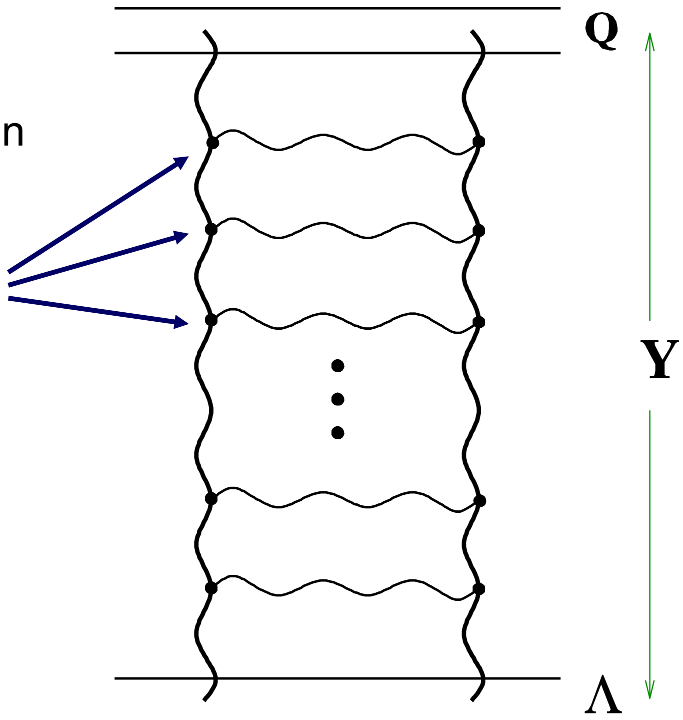
$$\frac{\partial f}{\partial \ln s} = \alpha_s \, K_{BFKL} \otimes f$$

The BFKL equation.
K_{BFKL} is an integral kernel.



BFKL Equation

In the conventional Feynman diagram picture the BFKL equation can be represented by a ladder graph shown here. Each rung of the ladder brings in a power of $\alpha \ln s$.



The resulting dipole amplitude grows as a power of energy

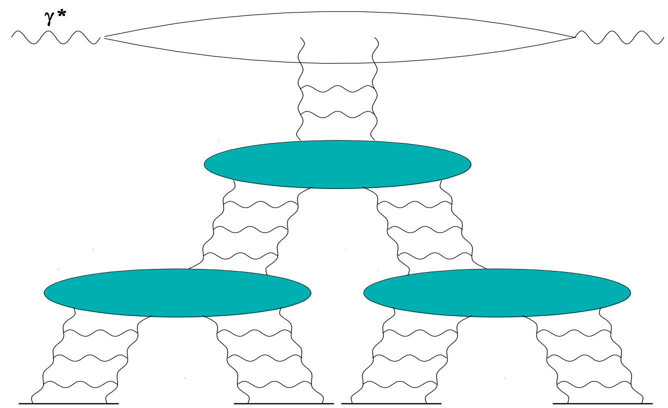
$$N \sim s^{\Delta}$$

violating Froissart unitarity bound

$$\sigma_{tot} \leq const \ln^2 s$$

GLR-MQ Equation

Gribov, Levin and Ryskin (‘81) proposed summing up “fan” diagrams:



Mueller and Qiu (’85) summed “fan” diagrams for large Q^2.

The GLR-MQ equation reads:

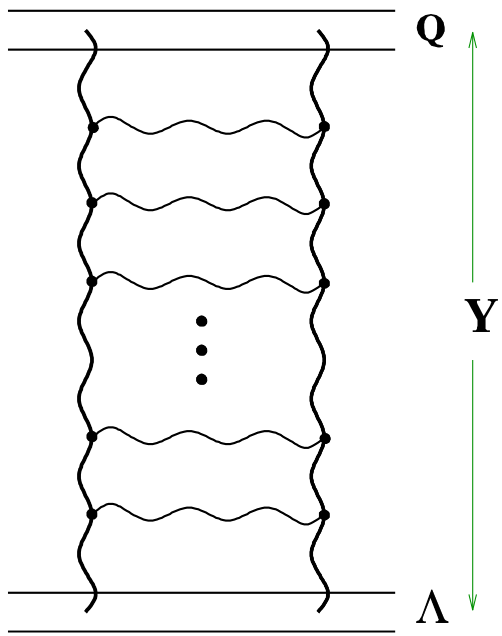
$$\frac{\partial}{\partial \ln 1/x}\,\phi(x,k_T^2) = \alpha_s\,K_{BFKL}\,\otimes\,\phi(x,k_T^2) - \alpha_s\,[\phi(x,k_T^2)]^2$$

GLR-MQ equation has the same principle of recombination as BK and JIMWLK. GLR-MQ equation was thought about as the first nonlinear correction to the linear BFKL evolution. An AGL (Ayala, Gay Ducati, Levin ‘96) equation was suggested to resum higher-order nonlinear corrections.

BK/JIMWLK derivation showed that for the dipole amplitude N (!) there are no more terms in the large-N_C limit and obtained the correct kernel for the non-linear term (compared to GLR suggestion).

Energy Dependence of the Saturation Scale

Single BFKL ladder gives scattering amplitude of the order $$N \sim \frac{\Lambda}{k_T}\, s^{\Delta}$$

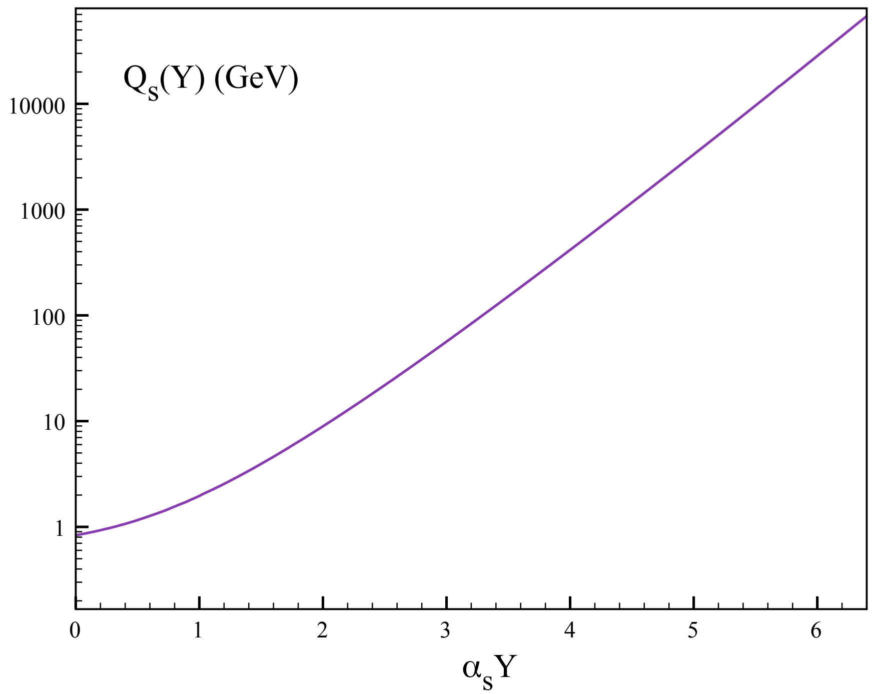


Nonlinear saturation effects become important when N ~ N^2 ⇨ N ~ 1. This happens at $$\boxed{k_T = Q_s \sim \Lambda\, s^{\Delta}}$$

Saturation scale grows with energy!

Typical partons in the wave function have $k_T \sim Q_S$, so that their characteristic size is of the order $r \sim 1/k_T \sim 1/Q_S$.
⇨ Typical parton size decreases with energy!

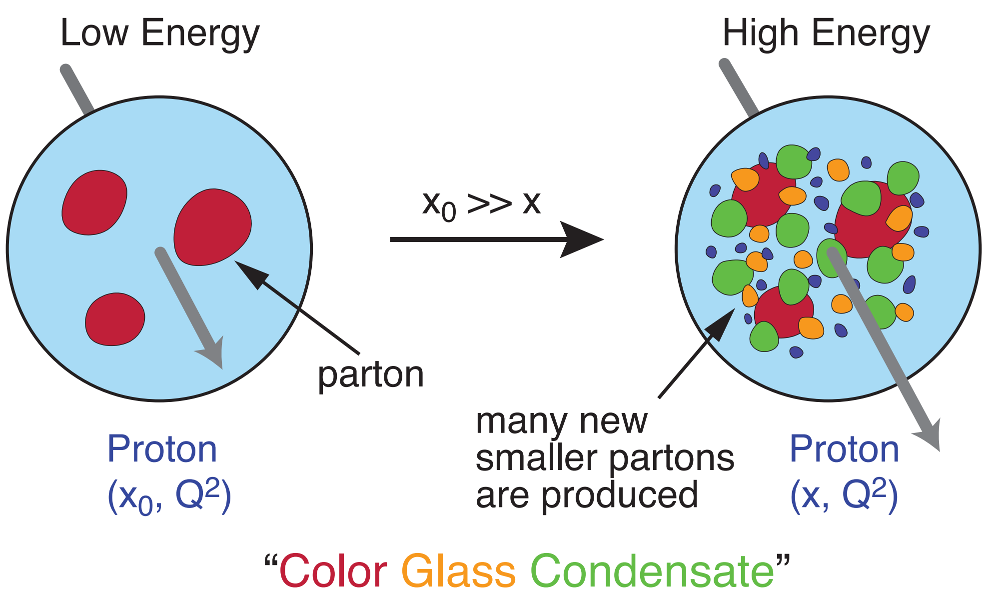
Saturation scale



numerical solution by J. Albacete

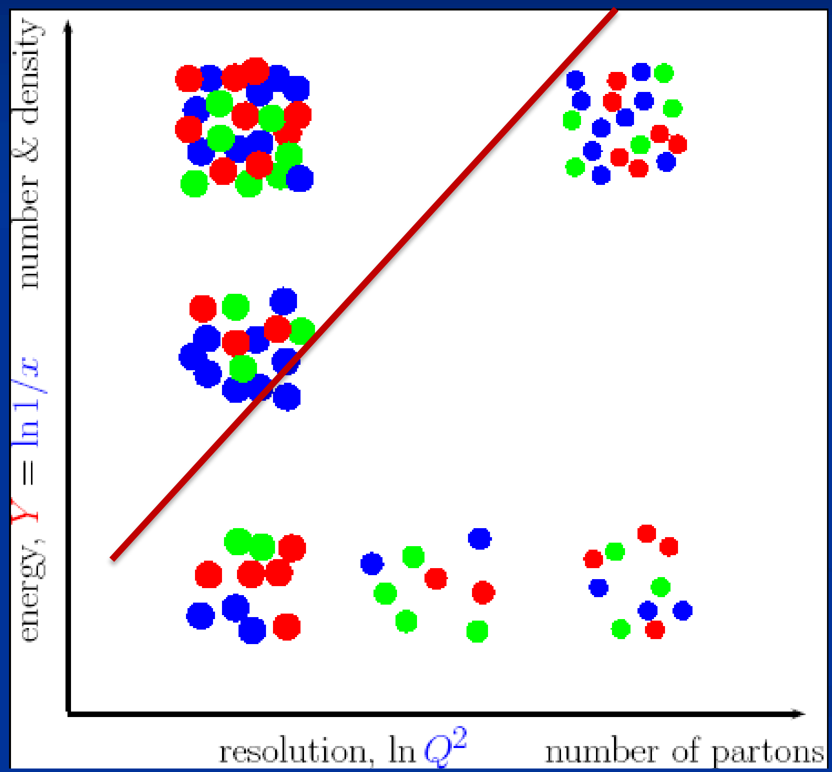
High Density of Gluons

- High number of gluons populates the transverse extent of the proton or nucleus, leading to a very dense saturated wave function known as the Color Glass Condensate (CGC):



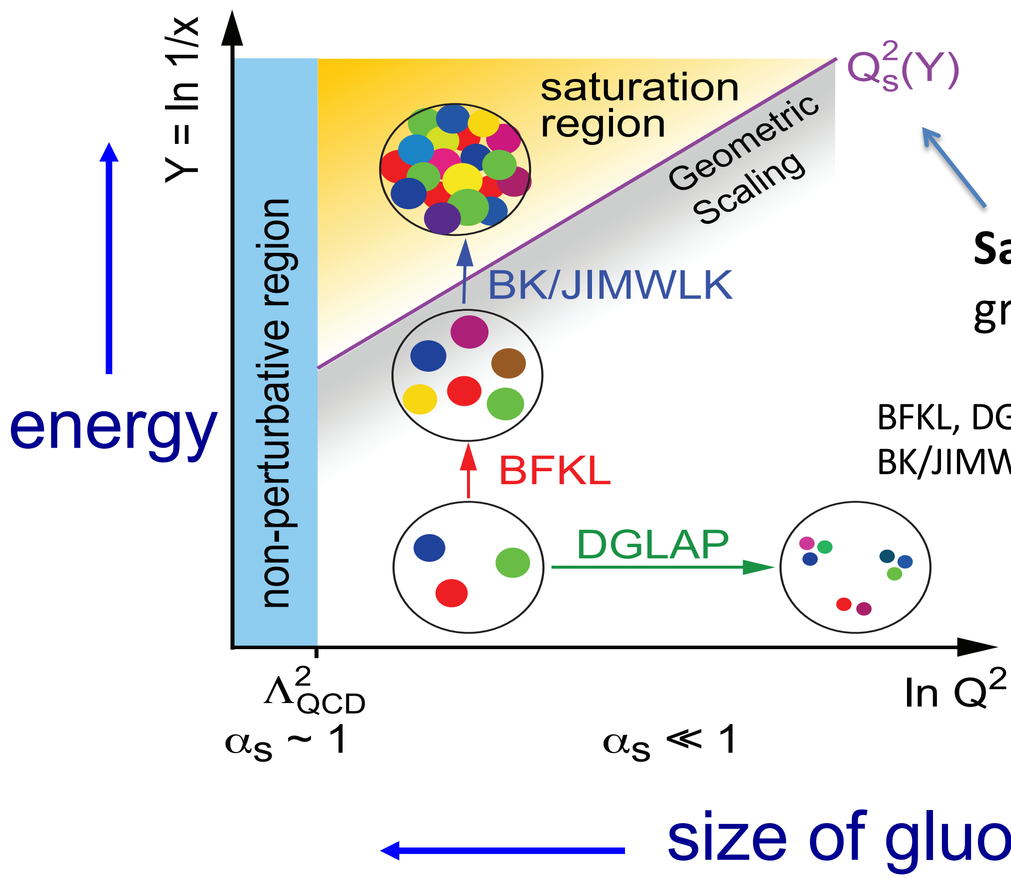
Map of High Energy QCD

energy



size of gluons

Map of High Energy QCD



$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

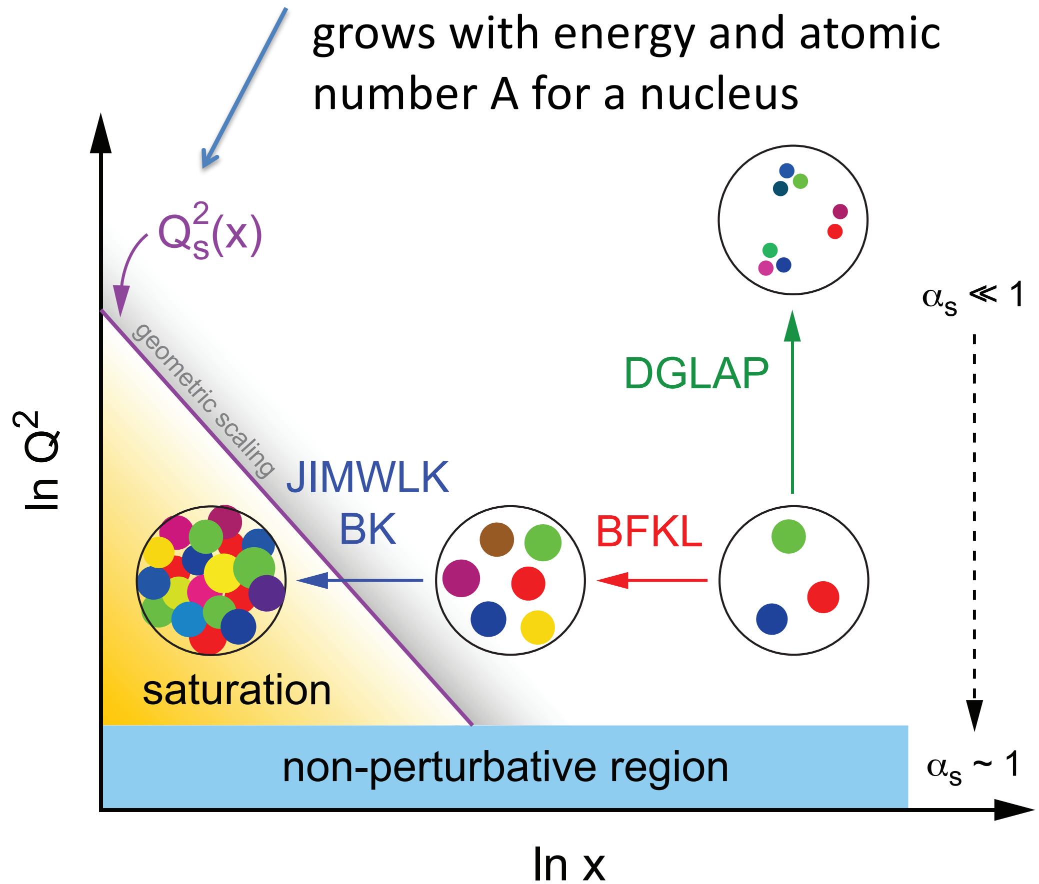
Saturation Scale
grows with energy

BFKL, DGLAP – linear equations
BK/JIMWLK – nonlinear

Map of High Energy QCD

Saturation Scale
grows with energy and atomic number A for a nucleus

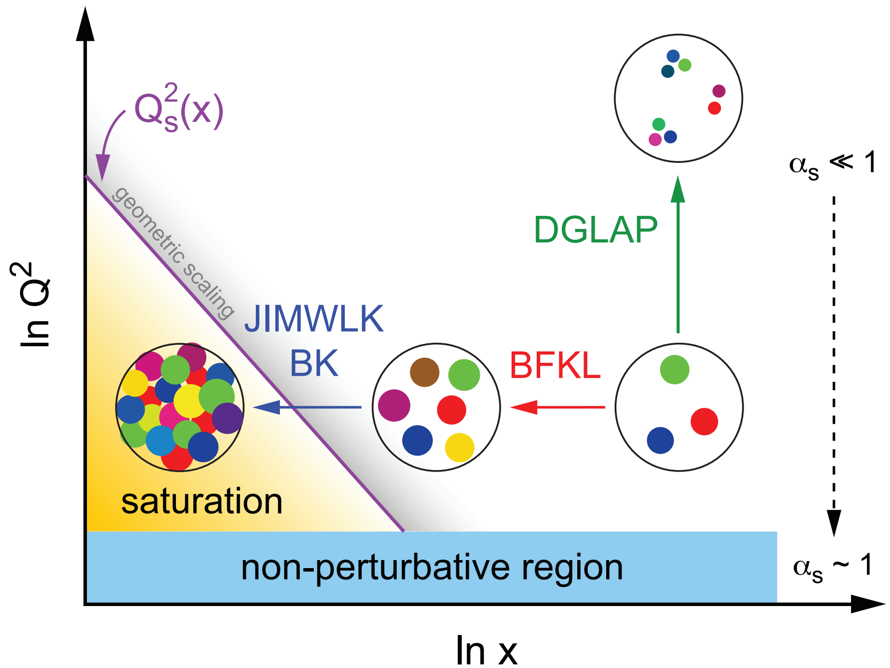
$$Q_s^2 \sim \left(\frac{A}{x}\right)^{1/3}$$

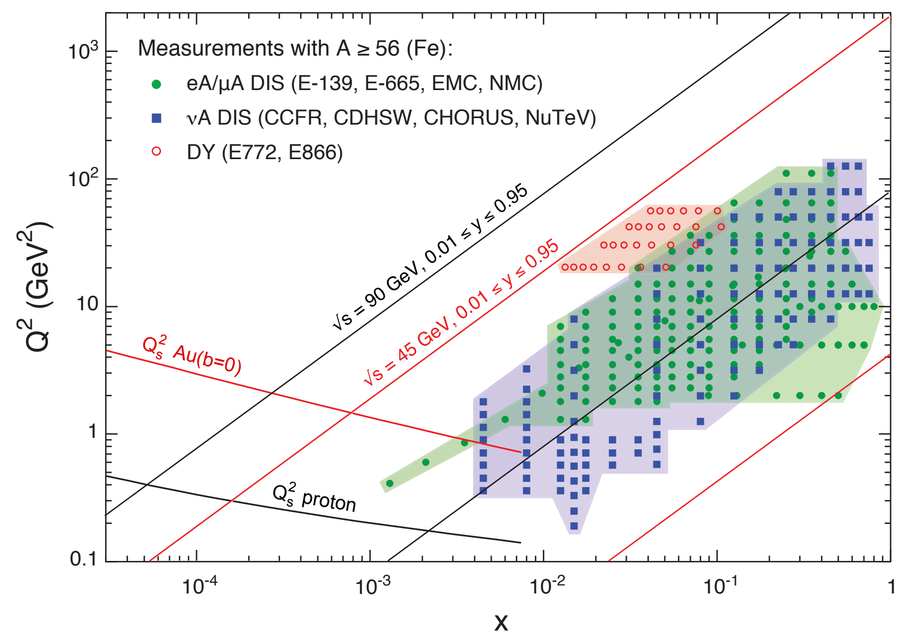


BFKL, DGLAP – linear equations
BK/JIMWLK – nonlinear

Can Saturation be Discovered at EIC?

EIC will have an unprecedented small-x reach for DIS on large nuclear targets, enabling decisive tests of saturation and non-linear evolution:





Plots from the EIC White Paper, '12, '14 (2nd ed).

Geometric Scaling

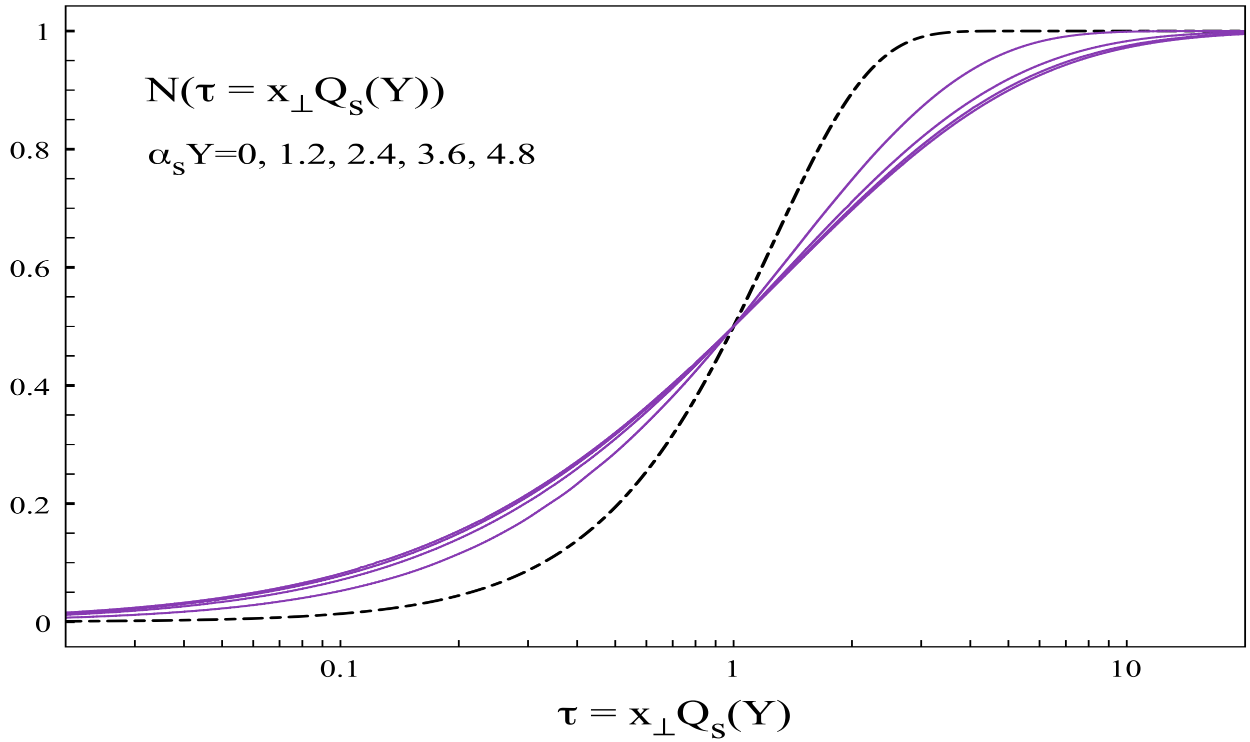
- One of the predictions of the JIMWLK/BK evolution equations is geometric scaling:

  DIS cross section should be a function of one parameter:

$$\sigma_{DIS}(x, Q^2) = \sigma_{DIS}(Q^2 / Q_S^2(x))$$

(Levin, Tuchin '99; Iancu, Itakura, McLerran '02)

Geometric Scaling



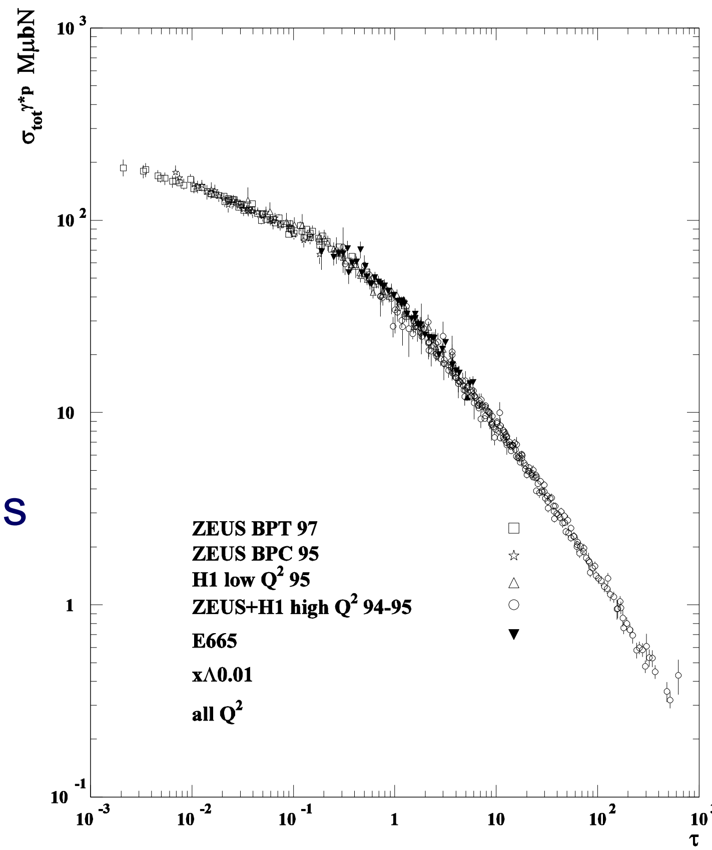
numerical solution by J. Albacete

Geometric Scaling in DIS

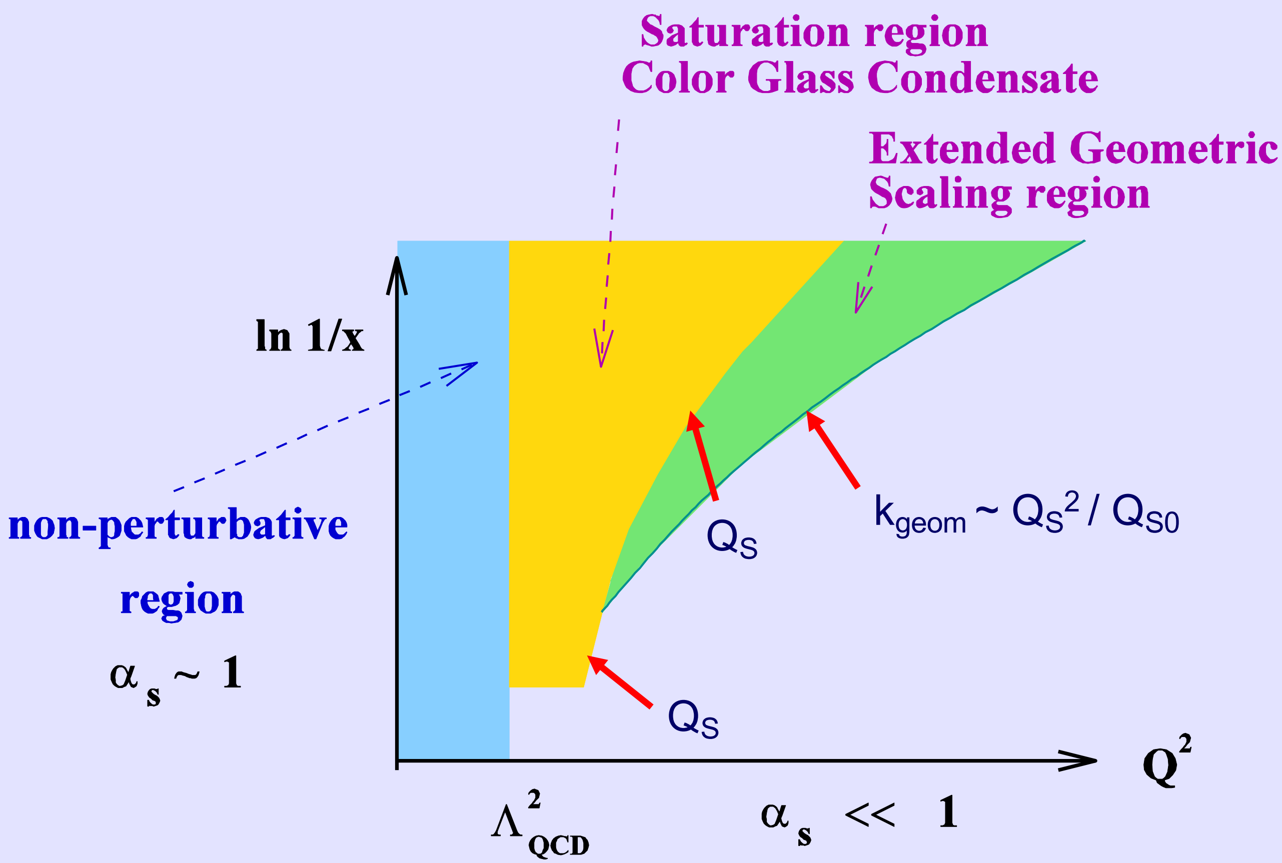
Geometric scaling has been observed in DIS data by Stasto, Golec-Biernat, Kwiecinski in ’00.

Here they plot the total DIS cross section, which is a function of 2 variables - Q^2 and x, as a function of just <u>one</u> variable:

$$\tau = \frac{Q^2}{Q_s^2}$$



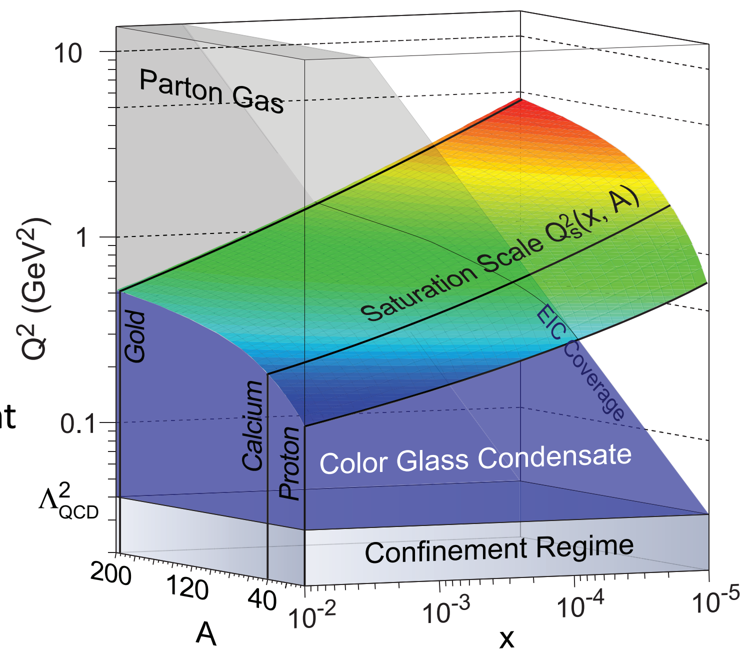
Map of High Energy QCD



Saturation Scale

To summarize, saturation scale is an increasing function of both energy (1/x) and A:

$$Q_s^2 \sim \left(\frac{A}{x}\right)^{1/3}$$



Gold nucleus provides an enhancement by $197^{1/3}$, which is equivalent to doing scattering on a proton at 197 times smaller x / higher s!

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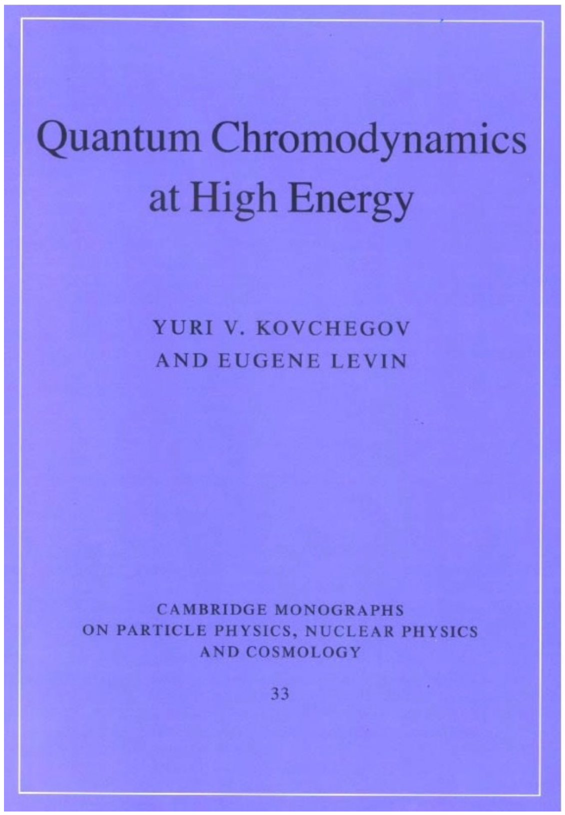

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Summary

- We have constructed nuclear/hadronic wave function in the quasi-classical approximation (MV model), and studied DIS in the same approximation
- We included small-*x* evolution corrections into the DIS process, obtaining nonlinear BK/JIMWLK evolution equations
- We found the saturation scale justifying the whole procedure.

$$\boxed{Q_S^2 \sim A^{1/3} \left(\frac{1}{x}\right)^{\lambda}}$$

- Saturation/CGC physics predicts geometric scaling observed experimentally at HERA.